## Article

# Initial Coefficient Bounds for Certain New Subclasses of bi-Bazilevič Functions and Exponentially bi-Convex Functions with Bounded Boundary Rotation 

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#### Abstract

The objective of the present article is to introduce new subclasses of bi-Bazilevič functions, bi-quasi-convex functions and $\alpha$-exponentially bi-convex functions involving functions with bounded boundary rotation of order $v$. For the above-said newly defined classes, we obtain first two initial coefficient bounds. In addition, the familiar Fekete-Szegö coefficient inequality is too found for the newly introduced subclasses of bi-univalent functions. Apart from the new findings that are obtained, it also improves the prior estimates that are presented already in the literature.


Keywords: analytic; univalent; Bazilevič functions; quasi-convex functions; exponential-convex functions; bounded boundary rotation; coefficient estimates

MSC: 30C45; 33C50; 30C80

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## 1. Introduction

Signify $\mathcal{A}$ to be the class of all functions of the normalized form

$$
\begin{equation*}
f(z)=z+\sum_{n=2}^{\infty} a_{n} z^{n} \tag{1}
\end{equation*}
$$

normalized by the conditions $f(0)=f^{\prime}(0)-1=0$, that are analytic in $\mathcal{D}=\{z:|z|<1\}$. Furthermore, Let us symbolize by $\mathcal{S}$, the subclass of $\mathcal{A}$, where the functions in $\mathcal{S}$ are analytic as well as univalent in $\mathcal{D}$. Let $f$ be a function analytic and locally univalent in a given simply connected domain. Then, we call $f$ as a function with bounded boundary rotation if its range has bounded boundary rotation. It is to be recalled at this moment that bounded boundary rotation is defined as the total variation of the direction angle of the tangent to the boundary curve underneath the complete circuit. Let $f(z)$ map $\mathcal{D}$ onto a domain $\mathbb{G}$. If $\mathbb{G}$ is a Schlicht domain with a continuously differentiable boundary curve and $\pi \mu(t)$ denotes the angle of the tangent vector at the point $f\left(e^{i t}\right)$ to the boundary curve with respect to the positive real axis, then the boundary rotation of $\mathbb{G}$ is equal to $\pi \int_{0}^{2 \pi}|d \mu(t)|$. If $\mathbb{G}$ does not have a sufficiently smooth boundary curve, the boundary rotation is defined by a limiting process.

Let $k \geq 2$ and $0 \leq v<1$ and $\mathcal{P}_{k}(v)$ be defined as

$$
\mathcal{P}_{k}(v)=\{p: p \text { analytic and normalized with } p(0)=1\}
$$

and such that for $z=r e^{i t} \in \mathcal{D}$, it satisfies $\int_{0}^{2 \pi}\left|\frac{\Re(p(z))-v}{1-v}\right| d \theta \leq k \pi$.
The class $\mathcal{P}_{k}(v)$ was investigated by Padmanabhan and Parvatham [1]; see the work in [2] for recent work on bounded boundary rotation. For $v=0, \mathcal{P}_{k}(v) \equiv \mathcal{P}_{k}$, studied
in detail by Pinchuk [3], which will consist of functions $p(z)$ that are analytic with the normalization $p(0)=1$. Therefore, a function $p \in \mathcal{P}_{k}$ will possess an integral form as

$$
p(z)=\int_{0}^{2 \pi}\left|\frac{1-z e^{i t}}{1+z e^{i t}}\right| d \mu(t) .
$$

Here $\mu$ is a real-valued function with a bounded variation and satisfies

$$
\int_{0}^{2 \pi} d \mu(t)=2 \text { and } \int_{0}^{2 \pi}|d \mu(t)| \leq k, k \geq 2
$$

It is to be noted at this occurrence that $\mathcal{P}_{2}$ is the class of analytic functions with a positive real part in $\mathcal{D}$, prominently known as the Carathéodory function class, and is denoted by $\mathcal{P}$.

For the prominent class $\mathcal{P}_{k}$, the lemma, which was established earlier in [3], is stated now in the following lemma.

Lemma 1. Let $p \in \mathcal{P}_{k}$. Then, there exist functions $p_{1} \in \mathcal{P}$ and $p_{2} \in \mathcal{P}$ such that

$$
p(z)=\left(\frac{k}{4}+\frac{1}{2}\right) p_{1}(z)-\left(\frac{k}{4}-\frac{1}{2}\right) p_{2}(z) .
$$

Two of the subclasses of $\mathcal{S}$ are the starlike functions of order $v$ denoted by $\mathcal{S}^{*}(v)$ and convex functions of order $v, 0 \leq v<1$ denoted by $\mathcal{C}(v)$. Analytic characterizations of the classes $\mathcal{S}^{*}(v)$ and $\mathcal{C}(v)$ are given as below:

$$
\mathcal{S}^{*}(v)=\left\{f \in \mathcal{S}: \Re\left(\frac{z f^{\prime}(z)}{f(z)}\right)>v, 0 \leq v<1\right\}
$$

and

$$
\mathcal{C}(v)=\left\{f \in \mathcal{S}: \Re\left(1+\frac{z f^{\prime \prime}(z)}{f^{\prime}(z)}\right)>v, 0 \leq v<1\right\} .
$$

We also observe that $f \in \mathcal{C}(v) \Leftrightarrow z f^{\prime}(z) \in \mathcal{S}^{*}(v)$. We also have $\mathcal{S}^{*}(v) \subseteq \mathcal{S}^{*}(0) \equiv \mathcal{S}^{*}$, $\mathcal{C}(v) \subseteq \mathcal{C}(0) \equiv \mathcal{C}$ and $\mathcal{C}(v) \subseteq \mathcal{S}^{*}(v)$ for $0 \leq v<1$. The classes $\mathcal{S}^{*}(v)$ and $\mathcal{C}(v)$ were introduced and investigated by Robertson [4] and then were analyzed in [5-7] and also in [8].

Let $\mathcal{U}_{k}(v)$ and $\mathcal{V}_{k}(v)$ represent the class of analytic functions $f(z)$ in $\mathcal{D}$ with $f(0)=0$, $f^{\prime}(0)=1$, satisfying

$$
\frac{z f^{\prime}(z)}{f(z)} \in \mathcal{P}_{k}(v) \text { and } 1+\frac{z f^{\prime \prime}(z)}{f^{\prime}(z)} \in \mathcal{P}_{k}(v), 0 \leq v<1,
$$

respectively. The class $\mathcal{U}_{k}(v)$ extends the class $\mathcal{S}^{*}(v)$ of class of starlike functions of order $v$, introduced and studied by Robertson [4]. For $v=0$, the class $\mathcal{U}_{k}(v)$ reduces to the $\mathcal{U}_{k}(0) \equiv \mathcal{U}_{k}$, the family of all functions of bounded radius rotation. Similarly, for $v=0$, $\mathcal{V}_{k}(v)$ reduces to the class $\mathcal{V}_{k}(0) \equiv \mathcal{V}_{k}$, the family of all analytic functions of bounded boundary rotation investigated in detail by Paatero [9]. If $A_{n}(k)=\max \left|a_{n}\right|, n \geq 2$, it is known (see, for details, Leach [10] and Thomas [11]) that for $k \geq 2$,

$$
A_{2}(k)=\frac{k}{2}, A_{3}(k)=\frac{k^{2}+2}{6}, A_{4}(k)=\frac{k^{3}+8 k}{24}
$$

It is obvious that every univalent function $f$ belonging to the class $\mathcal{S}$ has an inverse $f^{-1}$, given by

$$
\left(f^{-1} \odot f\right)(z)=z(z \in \mathcal{D})
$$

and

$$
\left(f \odot f^{-1}\right)(w)=w\left(|w|<r_{0}(h) ; r_{0}(f) \geq \frac{1}{4}\right) .
$$

One may look into [12] for details. It is pointed out at this moment that for an univalent function $f$ belonging to the class $\mathcal{S}$ and of the form (1), the inverse $f^{-1}$ may have an analytic continuation to $\mathcal{D}$, where

$$
\begin{equation*}
f^{-1}(w)=g(w)=w-a_{2} w^{2}+\left(2 a_{2}^{2}-a_{3}\right) w^{3}-\left(5 a_{2}^{3}-5 a_{2} a_{3}+a_{4}\right) w^{4}+\cdots . \tag{2}
\end{equation*}
$$

Let $\mathcal{A}_{\sigma}$ denote the family of functions of the form (1) defined on $\mathcal{D}$, for which the function $f \in \mathcal{A}$ and its inverse $f^{-1} \equiv g$ with Taylor series expansion as in (2), are univalent in $\mathcal{D}$. An univalent function $f \in \mathcal{S}$ is known as bi-univalent in $\mathcal{D}$ if there exists another univalent function $g \in \mathcal{S}$ where $g(z)$ has an univalent extension of $f^{-1}$ to $\mathcal{D}$. Let $\sigma$ be the class consisting of all bi-univalent functions in $\mathcal{D}$. If

$$
f_{1}=\frac{z}{1-z}, f_{2}(z)=\frac{1}{2} \log \left(\frac{1+z}{1-z}\right) \text { and } f_{3}(z)=-\log (1-z)
$$

then it is to be noted that the functions $f_{1}(z), f_{2}(z)$ and $f_{3}(z)$ are in the class $\sigma$, and it is also a bit of surprise to make a note that the familiar Koebe function $\frac{z}{(1-z)^{2}}$ is not in the family of bi-univalent functions. Lewin [13] was the first one who investigated the family of bi-univalent functions $\sigma$ and obtained a non-sharp bound $\left|a_{2}\right|<1.51$. Moreover, Brannan and Clunie [14] and Brannan and Taha [15] focused on certain subclasses of the bi-univalent function class $\sigma$ and obtained the bounds for their initial coefficients. The analysis of bi-univalent functions gained attention as fine as push, primarily because of the exploration by Srivastava et al. [16]. Brannan and Taha [15] defined the classes $\mathcal{S}_{\sigma}^{*}(v)$ and $\mathcal{C}_{\sigma}(v)$ of bi-starlike functions of order $v$ and bi-convex functions of order $v$. The bounds on $\left|a_{n}\right|(n=2,3)$ for the classes $\mathcal{S}_{\sigma}^{*}(v)$ and $\mathcal{C}_{\sigma}(v)$ (for details, see [15]) were established, and were also identified as non-sharp ones.

Let $\beta>0$ and $0 \leq v<1$. A function $f \in \mathcal{A}$ represented in (1) is known as in the family of Bazilevič functions of order $v$ and type $\beta$ denoted by $\mathcal{B}(\beta, v)$ if there exist a function $\phi(z) \in \mathcal{S}^{*}$ such that

$$
\Re\left(\frac{z f^{\prime}(z) f^{\beta-1}(z)}{\phi^{\beta}(z)}\right)>v, \quad z \in \mathcal{D} .
$$

When $\phi(z)=z$, we will denote the class $\mathcal{B}(\beta, v)$ as the subclass $\mathcal{B}_{1}(\beta, v)$. For various choices of the parameters, we have $\mathcal{B}(0,0) \equiv \mathcal{B}_{1}(0,0) \equiv \mathcal{S}^{*}, \mathcal{B}(0, v) \equiv \mathcal{B}_{1}(0, v) \equiv \mathcal{S}^{*}(v)$, and that $\mathcal{B}_{1}(1, v)$ is the subclass of $\mathcal{A}$ consisting of functions for which $\Re\left(f^{\prime}(z)\right)>v$. This is familiarly called as the class of functions where derivatives of the functions have positive real parts of order $v$. When $v=0$, the class $\mathcal{B}(\beta, 0)$ was studied by Singh [17] and also by Obradović [18,19]. One may also look up a recent work of Aouf et al. [20] for results on the bi-Bazilevič functions.

A function $f \in \mathcal{S}$ in the open unit disk $\mathcal{D}$ is known as exponentially convex if $e^{f(z)}$ maps $\mathcal{D}$ onto a convex domain ([21], Theorem 1). Let $\alpha$ be a nonzero complex number. Then a function $f \in \mathcal{S}$ is known as $\alpha$-exponentially convex if the following condition is satisfied:

$$
f \text { is } \alpha \text {-exponentially convex } \Longleftrightarrow \Re\left(1+\frac{z f^{\prime \prime}(z)}{f^{\prime}(z)}+\alpha z f^{\prime}(z)\right)>0 \quad z \in \mathcal{D} .
$$

The family of all $\alpha$-exponentially convex functions are denoted by $\mathcal{E}(\alpha)$.
Let $0 \leq v<1$. A function $f \in \mathcal{A}$ of the form (1) with a nonzero derivative on on $\mathcal{D}$ is said to be in the class of the close-to-convex function of order $v$ if there exists a function $\phi \in \mathcal{S}^{*}$ such that

$$
\Re\left(\frac{z f^{\prime}(z)}{\phi(z)}\right)>v .
$$

The family of all close-to-convex functions of order $v$ are denoted by $\mathcal{K}(v)$. Further details on $\mathcal{K}(v)$ or $\mathcal{K}$ function class may be found in the interesting works done in [22,23] (see [24] also).

Let $0 \leq v<1$. A function $f \in \mathcal{A}$ of the form (1) with non zero derivative on $\mathcal{D}$ is said to be in the family of the quasi-convex function of order $v$ if there exists a function $\chi \in \mathcal{C}$ such that

$$
\Re\left(\frac{\left(z f^{\prime}(z)\right)^{\prime}}{\chi^{\prime}(z)}\right)>v
$$

The family of all quasi-convex functions of order $v$ are denoted by $\mathcal{Q}^{*}(v)$. Note that every quasi-convex function is close-to-convex. A function is

$$
f \in \mathcal{Q}^{*}(v) \Leftrightarrow z f^{\prime} \in \mathcal{K}(v)
$$

For details on quasi-convex functions, one may see the work of [25].

## 2. Preliminaries and Lemmas

For

$$
\begin{equation*}
\varphi(z)=z+g_{2} z^{2}+g_{3} z^{3}+g_{4} z^{4}+\cdots \tag{3}
\end{equation*}
$$

one may obtain

$$
\begin{equation*}
\psi(w)=w-g_{2} w^{2}+\left(2 g_{2}^{2}-g_{3}\right) w^{3}-\left(5 g_{2}^{3}-5 g_{2} g_{3}+g_{4}\right) w^{4}+\cdots \tag{4}
\end{equation*}
$$

where $\varphi^{-1}(w)=\psi(w)$. Also, for

$$
\begin{equation*}
\chi(z)=z+c_{2} z^{2}+c_{3} z^{3}+c_{4} z^{4}+\cdots, \tag{5}
\end{equation*}
$$

one may obtain

$$
\begin{equation*}
\xi(w)=w-c_{2} w^{2}+\left(2 c_{2}^{2}-c_{3}\right) w^{3}-\left(5 c_{2}^{3}-5 c_{2} c_{3}+c_{4}\right) w^{4}+\cdots, \tag{6}
\end{equation*}
$$

where $\chi^{-1}(w)=\xi(w)$.
In order to prove our main theorems, we need few lemmas and stated now.
Lemma 2 ([2,26]). Let $\Omega(z)=1+\Omega_{1} z+\Omega_{2} z^{2}+\Omega_{3} z^{3}+\cdots=1+\sum_{n=1}^{\infty} \Omega_{n} z^{n}, z \in \mathcal{D}$ be such that $\Omega(z) \in \mathcal{P}_{k}(v)$. Then,

$$
\begin{equation*}
\left|\Omega_{n}\right| \leq k(1-v), \forall n \in \mathbb{N}=\{1,2,3, \cdots\} . \tag{7}
\end{equation*}
$$

Lemma 3 ([12], Theorem 2.14, p. 44). If $h(z)=z+h_{2} z^{2}+\cdots=z+\sum_{n=2}^{\infty} h_{n} z^{n}, z \in \mathcal{D}$ where $h \in \mathcal{S}^{*}$, then

$$
\begin{equation*}
\left|h_{n}\right| \leq n, \forall n \geq 2 . \tag{8}
\end{equation*}
$$

Strict inequality holds for all $n$ unless $f$ is a rotation of the Koebe function $\frac{z}{(1-z)^{2}}$.
Lemma 4 ([27]). $h(z)=z+h_{2} z^{2}+\cdots=z+\sum_{n=2}^{\infty} h_{n} z^{n}, z \in \mathcal{D}$ where $h \in \mathcal{S}^{*}$, then for $\delta \in \mathbb{R}$,

$$
\left|h_{3}-\delta h_{2}^{2}\right| \leq \begin{cases}3-4 \delta & \text { for } \delta \leq \frac{1}{2}  \tag{9}\\ 1 & \text { for } \frac{1}{2} \leq \delta \leq 1 \\ 4 \delta-3 & \text { for } \delta \geq 1\end{cases}
$$

Lemma 5 ([15], Corollary, p. 45). If $s(z)=z+s_{2} z^{2}+\cdots=z+\sum_{n=2}^{\infty} s_{n} z^{n}, z \in \mathcal{D}$, where $s(z)$ is a convex function, then

$$
\left|s_{n}\right| \leq 1, \forall n \geq 2
$$

Strict inequality holds for all $n$ unless $f$ is a rotation of the function $l$ defined by $l(z)=\frac{z}{1-z}$.
Lemma 6 ([28]). If $s(z)=z+s_{2} z^{2}+\cdots=z+\sum_{n=2}^{\infty} s_{n} z^{n}, z \in \mathcal{D}$ is a bi-convex function, then for $\delta \in \mathbb{R}$,

$$
\left|s_{3}-\delta s_{2}^{2}\right| \leq \begin{cases}1-\delta & \text { for } \delta<\frac{2}{3}  \tag{10}\\ \frac{1}{3} & \text { for } \frac{2}{3} \leq \delta \leq \frac{4}{3} \\ \delta-1 & \text { for } \delta>\frac{4}{3}\end{cases}
$$

Lemma 7 ([29]). If $s(z)=z+\sum_{n=2}^{\infty} s_{n} z^{n}, z \in \mathcal{D}$, where $s(z)$ is a convex function, then for $\delta \in \mathbb{R}$,

$$
\left|s_{3}-\delta s_{2}^{2}\right| \leq \begin{cases}1-\delta & \text { for } \delta<\frac{2}{3}  \tag{11}\\ 1 & \text { for } \frac{2}{3} \leq \delta \leq \frac{4}{3} \\ \delta-1 & \text { for } \delta>\frac{4}{3}\end{cases}
$$

In the present exploration, we introduced novel subclasses, namely the class of bi-Bazilevič functions, bi-quasi-convex functions and $\alpha$-exponentially bi-convex functions associated with bounded boundary rotation. For the new subclasses of functions that are being introduced, the authors obtain the first two initial coefficient bounds. Additionally, for the newly defined subclasses of bi-univalent functions, the famous Fekete-Szegö coefficient bounds are also found.
3. Coefficient Bounds for $\mathcal{B}_{\sigma}^{\beta}(k, v)$

In this section, we introduce a new class of bi-Bazilevič functions with bounded boundary rotation of order $v$ and type $\beta$ of bi-univalent functions.

Definition 1. Let $0 \leq v<1, k \in[2,4]$ and $\beta>0$. Additionally, let $f \in \sigma$ given by (1) be such that $f^{\prime}(z) \neq 0$ on $\mathcal{D}$. Then, $f$ is known as a bi-Bazilevič function with bounded boundary rotation of order $v$ if there exist two functions $\varphi \in \mathcal{S}^{*}$ and $\psi \in \mathcal{S}^{*}$ such that the following conditions hold good:

$$
\frac{z f^{\prime}(z) f^{\beta-1}(z)}{\varphi^{\beta}(z)} \in \mathcal{P}_{k}(v)
$$

and

$$
\frac{w g^{\prime}(w) g^{\beta-1}(w)}{\psi^{\beta}(w)} \in \mathcal{P}_{k}(v)
$$

with $g$ being the analytic continuation of $f^{-1}$ to the open unit disk $\mathcal{D}$. The family of all bi-Bazilevič functions with bounded boundary rotation of order $v$ and type $\beta$ is denoted by $\mathcal{B}_{\sigma}^{\beta}(k, v)$.

## Remark 1.

(i) When $\beta=1$, we have $\mathcal{B}_{\sigma}^{\beta}(k, v) \equiv \mathcal{B}_{\sigma}^{1}(k, v) \equiv \mathcal{K}_{\sigma}(k, v)$, the family consisting of bi-close-to-convex functions with bounded boundary rotation of order $v$.
(ii) When $\beta=1$ and $v=0$, one may obtain $\mathcal{B}_{\sigma}^{\beta}(k, v) \equiv \mathcal{B}_{\sigma}^{1}(k, 0) \equiv \mathcal{K}_{\sigma}(k)$, the family consisting of bi-close-to-convex functions with bounded boundary rotation.
(iii) When $k=2$, we have $\mathcal{B}_{\sigma}^{\beta}(k, v) \equiv \mathcal{B}_{\sigma}^{\beta}(2, v) \equiv \mathcal{B}_{\sigma}^{\beta}(v)$, the family consisting of bi-Bazilevič functions of order $v$ and type $\beta$.
(iv) If $k=2$ and $v=0$, we have $\mathcal{B}_{\sigma}^{\beta}(k, v) \equiv \mathcal{B}_{\sigma}^{\beta}(2,0) \equiv \mathcal{B}_{\sigma}(\beta)$, the family consisting of bi-Bazilevič functions of type $\beta$.
(v) By selecting the value of $k=2$ and $\beta=1$, we have $\mathcal{B}_{\sigma}^{\beta}(k, v) \equiv \mathcal{B}_{\sigma}^{1}(2, v) \equiv \mathcal{K}_{\sigma}(v)$, the family of bi-close-to-convex functions of order $\nu$.

Now, we attain the first two initial coefficient estimates and $\left|a_{3}-\delta a_{2}^{2}\right|$ for the new class $\mathcal{B}_{\sigma}^{\beta}(k, v)$.

Theorem 1. Let $0 \leq v<1, k \in[2,4]$ and $\beta>0$. If the function $f \in \sigma$ given by (1) belong to the class $\mathcal{B}_{\sigma}^{\beta}(k, v)$, then

$$
\begin{gather*}
\left|a_{2}\right| \leq \sqrt{\frac{2 k(1-v)(2 \beta+1)+4 \beta(1+\beta)}{(1+\beta)(2+\beta)}}  \tag{12}\\
\left|a_{3}\right| \leq \frac{3 \beta(1+\beta)+2 k(1-v)(2 \beta+1)}{(1+\beta)(2+\beta)} \tag{13}
\end{gather*}
$$

Further, if $\delta \in \mathbb{R}$, then

$$
\left|a_{3}-\delta a_{2}^{2}\right| \leq \begin{cases}\frac{\beta(1+\beta)(3-4 \delta)+4 \beta k(1-v)(1-\delta)+2 k(1-v)(1-\delta)}{(1+\beta)(2+\beta)} & \text { for } \delta<\frac{1-\beta}{2}  \tag{14}\\ \frac{\beta(1+\beta)(3-4 \delta)+4 \beta k(1-v)(1-\delta)+(1+\beta) k(1-v)}{(1+\beta)(2+\beta)} & \text { for } \frac{1-\beta}{2} \leq \delta<\frac{1}{2} \\ \frac{\beta(1+\beta)+4 \beta k(1-v)(1-\delta)+(1+\beta) k(1-v)}{(1+\beta)(2+\beta)} & \text { for } \frac{1}{2} \leq \delta<1 \\ \frac{\beta(1+\beta)(4 \delta-3)+4 \beta k(1-v)(\delta-1)+(1+\beta) k(1-v)}{(1+\beta)(2+\beta)} & \text { for } 1 \leq \delta<\frac{\beta+3}{2} \\ \frac{\beta(1+\beta)(4 \delta-3)+4 \beta k(1-v)(\delta-1)+2 k(1-v)(\delta-1)}{(1+\beta)(2+\beta)} & \text { for } \delta \geq \frac{\beta+3}{2}\end{cases}
$$

Proof. Let $g, \varphi$ and $\psi$ be represented, respectively, in the form (2)-(4). As the function $f \in \mathcal{B}_{\sigma}^{\beta}(k, v)$, there exist functions $p \in \mathcal{P}_{k}(v)$ and $q \in \mathcal{P}_{k}(v)$ that are analytic with

$$
p(z)=1+\sum_{n=1}^{\infty} p_{n} z^{n}=1+p_{1} z+p_{2} z^{2}+p_{3} z^{3} \ldots
$$

and

$$
q(z)=1+q_{1} z+q_{2} z^{2}+\cdots
$$

satisfying

$$
\begin{equation*}
\frac{z f^{\prime}(z) f^{\beta-1}(z)}{\varphi^{\beta}(z)}=p(z) \tag{15}
\end{equation*}
$$

and

$$
\begin{equation*}
\frac{w g^{\prime}(w) g^{\beta-1}(w)}{\psi^{\beta}(w)}=q(w) \tag{16}
\end{equation*}
$$

From (15) and (16), we attain

$$
\begin{equation*}
(1+\beta) a_{2}=\beta g_{2}+p_{1} \tag{17}
\end{equation*}
$$

$$
\begin{gather*}
(2+\beta) a_{3}=\beta g_{3}+\beta g_{2} a_{2}+p_{2}+\beta g_{2} p_{1}+a_{2} p_{1}-\frac{\beta(\beta+3)}{2} a_{2}^{2}+\frac{\beta(\beta-1)}{2} g_{2}^{2}  \tag{18}\\
-(1+\beta) a_{2}=-\beta g_{2}+q_{1} \tag{19}
\end{gather*}
$$

and
$-(2+\beta) a_{3}=-\beta g_{3}+\beta g_{2} a_{2}+q_{2}-\beta g_{2} q_{1}-a_{2} q_{1}-\frac{8+\beta-\beta^{2}}{2} a_{2}^{2}+\frac{\beta(\beta+3)}{2} g_{2}^{2}$.
Then, from (17) and (19), we obtain $p_{1}=-q_{1}$. Adding (18), (20), and by using relation $p_{1}=-q_{1}$, we obtain

$$
\begin{equation*}
a_{2}^{2}=\frac{2 \beta g_{2} p_{1}+\beta(1+\beta) g_{2}^{2}+p_{2}+q_{2}}{(1+\beta)(2+\beta)} \tag{21}
\end{equation*}
$$

Now, by triangle inequality and by using Lemmas 2 and 3 in (21), we obtain

$$
\begin{equation*}
\left|a_{2}\right|^{2} \leq \frac{2 k(1-v)(2 \beta+1)+4 \beta(1+\beta)}{(1+\beta)(2+\beta)} . \tag{22}
\end{equation*}
$$

Upon simplification of Equation (22) gives (12). Again from (18), (20) and by using relation $p_{1}=-q_{1}$ and (21), we obtain

$$
\begin{equation*}
a_{3}=\frac{2 \beta(1+\beta) g_{3}+4 \beta g_{2} p_{1}+p_{2}(\beta+3)+q_{2}(1-\beta)}{2(1+\beta)(2+\beta)} \tag{23}
\end{equation*}
$$

Now, by triangle inequality and by using Lemma 2 and Lemma 3 in (23), which gives (13), for any $\delta \in \mathbb{R}$ and by Equations (21) and (23), we have

$$
\begin{equation*}
a_{3}-\delta a_{2}^{2}=\frac{2 \beta(1+\beta)\left[g_{3}-\delta g_{2}^{2}\right]+4 \beta g_{2} p_{1}[1-\delta]+p_{2}[\beta+3-2 \delta]+q_{2}[1-\beta-2 \delta]}{2(1+\beta)(2+\beta)} . \tag{24}
\end{equation*}
$$

Now, by triangle inequality and by using Lemma 2 and Lemma 3 in (24), we obtain

$$
\begin{equation*}
\left|a_{3}-\delta a_{2}^{2}\right| \leq \frac{2 \beta(1+\beta)\left|g_{3}-\delta g_{2}^{2}\right|+8 \beta k(1-v)|1-\delta|+k(1-v)[|\beta+3-2 \delta|+|1-\beta-2 \delta|]}{2(1+\beta)(2+\beta)} \tag{25}
\end{equation*}
$$

By applying Lemma 4 in (25), we obtain (14). The proof of Theorem 1 is now completed.

By selecting the value of $\beta$ as $\beta=1$, Theorem 1 reduces to the next coefficient bounds for the class $\mathcal{K}_{\sigma}(k, v)$, and is given now below as a corollary.

Corollary 1. Let $0 \leq v<1$ and $k \in[2,4]$. If $f \in \sigma$ given by (1) be in the class $\mathcal{K}_{\sigma}(k, v)$, then

$$
\begin{gathered}
\left|a_{2}\right| \leq \sqrt{\frac{4+3 k(1-v)}{3}} \\
\left|a_{3}\right| \leq 1+k(1-v)
\end{gathered}
$$

Further, if $\delta \in \mathbb{R}$, then

$$
\left\lvert\, \begin{array}{ll}
\frac{1}{3}[(3-4 \delta)+3 k(1-v)(1-\delta)] & \text { for } \delta<0 \\
a_{3}-\delta a_{2}^{2} \mid \leq\{ & \text { for } 0 \leq \delta<\frac{1}{2} \\
\frac{1}{3}[(3-4 \delta)+k(1-v)(3-2 \delta) \\
\frac{1}{3}[1+k(1-v)(3-2 \delta)] & \text { for } \frac{1}{2} \leq \delta<1 \\
\frac{1}{3}[(4 \delta-3)+k(1-v)(2 \delta-1)] & \text { for } 1 \leq \delta<2 \\
\frac{1}{3}[(4 \delta-3)+3 k(1-v)(\delta-1)] & \text { for } \delta \geq 2
\end{array}\right.
$$

Remark 2. Corollary 1 verifies the coefficient bounds of $\left|a_{2}\right|,\left|a_{3}\right|$ and $\left|a_{3}-\mu a_{2}^{2}\right|$, attained by Prathviraj et al. [30].

By making a selection for $k$ as $k=2$, Theorem 1 reduces to the following coefficient estimates for the class $\mathcal{B}_{\sigma}^{\beta}(v)$, and is given now below as a corollary.

Corollary 2. Let $0 \leq v<1$ and $\beta \geq 0$. If the function $f \in \sigma$ given by (1) be in the class $\mathcal{B}_{\sigma}^{\beta}(v)$, then

$$
\begin{gathered}
\left|a_{2}\right| \leq \sqrt{\frac{4(1-v)(2 \beta+1)+4 \beta(1+\beta)}{(1+\beta)(2+\beta)}} \\
\left|a_{3}\right| \leq \frac{3 \beta(1+\beta)+4(1-v)(2 \beta+1)}{(1+\beta)(2+\beta)}
\end{gathered}
$$

Further, if $\delta \in \mathbb{R}$, then

$$
\left|a_{3}-\delta a_{2}^{2}\right| \leq \begin{cases}\frac{\beta(1+\beta)(3-4 \delta)+8 \beta(1-v)(1-\delta)+4(1-v)(1-\delta)}{(1+\beta)(2+\beta)} & \text { for } \delta<\frac{1-\beta}{2}, \\ \frac{\beta(1+\beta)(3-4 \delta)+8 \beta(1-v)(1-\delta)+2(1+\beta)(1-v)}{(1+\beta)(2+\beta)} & \text { for } \frac{1-\beta}{2} \leq \delta<\frac{1}{2}, \\ \frac{\beta(1+\beta)+8 \beta(1-v)(1-\delta)+2(1+\beta)(1-v)}{(1+\beta)(2+\beta)} & \text { for } \frac{1}{2} \leq \delta<1, \\ \frac{\beta(1+\beta)(4 \delta-3)+8 \beta(1-v)(\delta-1)+2(1+\beta)(1-v)}{(1+\beta)(2+\beta)} & \text { for } 1 \leq \delta<\frac{\beta+3}{2}, \\ \frac{\beta(1+\beta)(4 \delta-3)+8 \beta(1-v)(\delta-1)+4(1-v)(\delta-1)}{(1+\beta)(2+\beta)} & \text { for } \delta \geq \frac{\beta+3}{2} .\end{cases}
$$

Let us make an assumption for $\phi(z)$ as $\phi(z)=z$. For this choice of $\phi$, let us denote the class $\mathcal{B}_{\sigma}^{\beta}(k, v)$ by $\mathcal{B}_{\sigma}^{\beta}[k, v]$. In fact, the class $\mathcal{B}_{\sigma}^{\beta}[k, v]$ will be consisting of functions of the form (1) with $f \in \sigma$, and satisfying the conditions

$$
\left(\frac{z}{f(z)}\right)^{1-\beta} f^{\prime}(z) \in \mathcal{P}_{k}(v)
$$

and

$$
\left(\frac{w}{g(w)}\right)^{1-\beta} g^{\prime}(z) \in \mathcal{P}_{k}(v)
$$

with $g$ being the analytic continuation of $f^{-1}$ to the open unit disk $\mathcal{D}$. For attaining the bounds for the class $\mathcal{B}_{\sigma}^{\beta}[k, v]$, the computation that may be akin to Theorem 1 has to be worked again. However, it is affirmed at this instant as a theorem without the details concerned.

Theorem 2. Let $0 \leq v<1, k \in[2,4]$ and $\beta \geq 0$. A function $f \in \sigma$ given by (1) be in the class $\mathcal{B}_{\sigma}^{\beta}[k, v]$, then

$$
\begin{gather*}
\left|a_{2}\right| \leq \sqrt{\frac{2 k(1-v)}{(1+\beta)(2+\beta)}}  \tag{26}\\
\left|a_{3}\right| \leq \frac{2 k(1-v)}{(1+\beta)(2+\beta)} \tag{27}
\end{gather*}
$$

Further, if $\delta \in \mathbb{R}$, then

$$
\left|a_{3}-\delta a_{2}^{2}\right| \leq \begin{cases}\frac{2 k(1-v)(1-\delta)}{(1+\beta)(2+\beta)} & \text { for } \delta<\frac{1-\beta}{2}  \tag{28}\\ \frac{k(1-v)}{\beta+2} & \text { for } \frac{1-\beta}{2} \leq \delta<\frac{3+\beta}{2} \\ \frac{2 k(1-v)(\delta-1)}{(1+\beta)(2+\beta)} & \text { for } \delta \geq \frac{3+\beta}{2}\end{cases}
$$

## Remark 3.

(i) By making a choice for $\beta$ as $\beta=1$ in Theorem 2, we have the class $\mathcal{B}_{\sigma}^{\beta}[k, v] \equiv \mathcal{N}_{\sigma}(k, v)$, consisting of all functions $f \in \sigma$ of the form (1) and satisfying the conditions

$$
f^{\prime}(z) \in \mathcal{P}_{k}(v)
$$

and

$$
g^{\prime}(w) \in \mathcal{P}_{k}(v)
$$

(ii) By making a choice of $\beta$ as $\beta=1$ and $k=2$ in Theorem 2 , we have the class $\mathcal{B}_{\sigma}^{\beta}[k, v] \equiv \mathcal{H}_{\sigma}(v)$, consisting of all functions $f \in \sigma$ of the form (1) and satisfying the conditions

$$
\Re\left(f^{\prime}(z)\right)>v
$$

and

$$
\Re\left(g^{\prime}(w)\right)>v
$$

By making a selection for $\beta$ as $\beta=1$, Theorem 2 reduces to the following coefficient estimates for the class $\mathcal{N}_{\sigma}(k, v)$, and is given now below as a corollary.

Corollary 3. Let $0 \leq v<1$ and $k \in[2,4]$. A function $f \in \sigma$ given by (1) is said to be in the class $\mathcal{N}_{\sigma}(k, v)$, then

$$
\begin{gathered}
\left|a_{2}\right| \leq \sqrt{\frac{k(1-v)}{3}} \\
\left|a_{3}\right| \leq \frac{k(1-v)}{3}
\end{gathered}
$$

Further, if $\delta \in \mathbb{R}$, then

$$
\left|a_{3}-\delta a_{2}^{2}\right| \leq \begin{cases}\frac{k(1-v)(1-\delta)}{3} & \text { for } \delta<\frac{1-\beta}{2} \\ \frac{k(1-v)}{3} & \text { for } \frac{1-\beta}{2} \leq \delta<\frac{3+\beta}{2} \\ \frac{k(1-v)(\delta-1)}{3} & \text { for } \delta \geq \frac{3+\beta}{2}\end{cases}
$$

By making a selection for $\beta$ and $k$ as $\beta=1$ and $k=2$, Theorem 2 reduces to the following coefficient estimates for the class $\mathcal{H}_{\sigma}(v)$, and is given now below as a corollary.

Corollary 4. Let $0 \leq v<1$. A function $f \in \sigma$ given by (1) is said to be in the class $\mathcal{H}_{\sigma}(v)$, then

$$
\left|a_{2}\right| \leq \sqrt{\frac{2(1-v)}{3}}
$$

$$
\left|a_{3}\right| \leq \frac{2(1-v)}{3}
$$

Further, if $\delta \in \mathbb{R}$, then

$$
\left|a_{3}-\delta a_{2}^{2}\right| \leq \begin{cases}\frac{2(1-v)(1-\delta)}{3} & \text { for } \delta<0 \\ \frac{2(1-v)}{3} & \text { for } 0 \leq \delta<2 \\ \frac{2(1-v)(\delta-1)}{3} & \text { for } \delta \geq 2\end{cases}
$$

Definition 2. Let $0 \leq v<1, k \in[2,4]$ and $\eta>0$. A function $f \in \sigma$ given by ( 1 ) is said to be in the class $\mathcal{T}_{\sigma}^{\eta}(k, v)$ if the following conditions holds good:

$$
\left(\frac{f(z)}{z}\right)^{\eta} \in \mathcal{P}_{k}(v)
$$

and

$$
\left(\frac{g(w)}{w}\right)^{\eta} \in \mathcal{P}_{k}(v)
$$

Here, $g$ is the analytic continuation of $f^{-1}$ to the open unit disk $\mathcal{D}$.
Theorem 3. Let $0 \leq v<1, k \in[2,4]$ and $\eta>0$. If $f$ given by (1) is in the class $\mathcal{T}_{\sigma}^{\eta}(k, v)$, then

$$
\begin{gather*}
\left|a_{2}\right| \leq \sqrt{\frac{2 k(1-v)}{\eta(\eta+1)}}  \tag{29}\\
\quad\left|a_{3}\right| \leq \frac{2 k(1-v)}{\eta(\eta+1)} \tag{30}
\end{gather*}
$$

and

$$
\left|a_{3}-\delta a_{2}^{2}\right| \leq \begin{cases}\frac{2 k(1-v)(1-\delta)}{\eta(\eta+1)} & \text { for } \delta<\frac{1-\eta}{2}  \tag{31}\\ \frac{k(1-v)}{\eta} & \text { for } \frac{1-\eta}{2} \leq \delta<\frac{\eta+3}{2} \\ \frac{2 k(1-v)(\delta-1)}{\eta(\eta+1)} & \text { for } \delta \geq \frac{\eta+3}{2}\end{cases}
$$

Proof. Let $g$ be given in the form (2). Since $f \in \mathcal{T}_{\sigma}^{\eta}(k, v)$, there exist functions $p, q \in \mathcal{P}_{k}(v)$ that are analytic with

$$
p(z)=1+\sum_{n=1}^{\infty} p_{n} z^{n}=1+p_{1} z+p_{2} z^{2}+\cdots
$$

and

$$
q(z)=1+q_{1} z+q_{2} z^{2}+\cdots
$$

satisfying

$$
\begin{equation*}
\left(\frac{f(z)}{z}\right)^{\eta}=p(z) \tag{32}
\end{equation*}
$$

and

$$
\begin{equation*}
\left(\frac{g(w)}{w}\right)^{\eta}=q(w) \tag{33}
\end{equation*}
$$

Hence, from (32) and (33), we obtain

$$
\begin{gather*}
\eta a_{2}=p_{1}  \tag{34}\\
\eta a_{3}+\frac{\eta(\eta-1)}{2} a_{2}^{2}=p_{2}  \tag{35}\\
-\eta a_{2}=q_{1} \tag{36}
\end{gather*}
$$

and

$$
\begin{equation*}
\eta\left(2 a_{2}^{2}-a_{3}\right)+\frac{\eta(\eta-1)}{2} a_{2}^{2}=q_{2} \tag{37}
\end{equation*}
$$

Then, from (34) and (36), we obtain $p_{1}+q_{1}=0$. Adding (35) and (37), we obtain

$$
\begin{equation*}
a_{2}^{2}=\frac{p_{2}+q_{2}}{\eta(\eta+1)} . \tag{38}
\end{equation*}
$$

By using triangle inequality and Lemma 2 in (38), we obtain

$$
\begin{equation*}
\left|a_{2}\right|^{2} \leq \frac{2 k(1-v)}{\eta(\eta+1)} . \tag{39}
\end{equation*}
$$

Hence, (39) gives (29). Now, again from (35), (37) and (38), we get

$$
\begin{equation*}
a_{3}=\frac{p_{2}[\eta+3]+q_{2}[1-\eta]}{2 \eta(\eta+1)} \tag{40}
\end{equation*}
$$

By using triangle inequality and Lemma 2 in (40) which gives (30), for any $\delta \in \mathbb{R}$ and by Equation (38) and (40), we have

$$
\begin{equation*}
a_{3}-\delta a_{2}^{2}=\frac{p_{2}[\eta+3-2 \delta]+q_{2}[1-\eta-2 \delta]}{2 \eta(\eta+1)} . \tag{41}
\end{equation*}
$$

By using triangle inequality and Lemma 2 in (41), we obtain

$$
\begin{equation*}
\left|a_{3}-\delta a_{2}^{2}\right| \leq \frac{k(1-v)[|\eta+3-2 \delta|+|1-\eta-2 \delta|]}{2 \eta(\eta+1)} \tag{42}
\end{equation*}
$$

Upon simplification of Equation (42) gives (31). The proof of Theorem 3 is now completed.

Remark 4. For $\eta=1$, Theorem 3 verifies the coefficient bounds of $\left|a_{2}\right|$ and $\left|a_{3}\right|$, attained by Prathviraj et al. [30].

## 4. Coefficient Bounds for $\mathcal{Q}_{\sigma}^{*}(k, v)$

In this section, we introduce and obtain the initial bounds for the family of bi-quasi-convex with bounded boundary rotation of order $v$, which we define now.

Definition 3. Let $0 \leq v<1$ and $k \in[2,4]$. Let the function of the form (1) belong to the class $\sigma$ such that $f^{\prime}(z) \neq 0$ on $\mathcal{D}$. Then, $f$ is known as bi-quasi-convex with bounded boundary rotation of order $v$ if there exist functions $\chi \in \mathcal{C}$ and $\xi \in \mathcal{C}$ satisfying

$$
\frac{\left(z f^{\prime}(z)\right)^{\prime}}{\chi^{\prime}(z)} \in \mathcal{P}_{k}(v)
$$

and

$$
\frac{\left(w g^{\prime}(w)\right)^{\prime}}{\xi^{\prime}(w)} \in \mathcal{P}_{k}(v)
$$

Here, $g$ is the analytic continuation of $f^{-1}$ to $\mathcal{D}$. Let $\mathcal{Q}_{\sigma}^{*}(k, v)$ denote the family of all bi-quasi-convex functions with bounded boundary rotation of order $v$.

## Remark 5.

(i) For the choice of $k=2$, we get $\mathcal{Q}_{\sigma}^{*}(k, v) \equiv \mathcal{Q}_{\sigma}^{*}(2, v) \equiv \mathcal{Q}_{\sigma}^{*}(v)$, the family of bi-quasi-convex functions of order $v$.
(ii) For $k=2$ and $v=0$, we get $\mathcal{Q}_{\sigma}^{*}(k, v) \equiv \mathcal{Q}_{\sigma}^{*}(2,0) \equiv \mathcal{Q}_{\sigma}^{*}$, the family of bi-quasi-convex functions.

Next, we attain the initial coefficient bounds and the bound $\left|a_{3}-\delta a_{2}^{2}\right|$ for the class $\mathcal{Q}_{\sigma}^{*}(k, v)$.

Theorem 4. Let $0 \leq v<1$ and $k \in[2,4]$ and let $f$ given by (1) be in the class $\mathcal{Q}_{\sigma}^{*}(k, v)$. Then

$$
\begin{gather*}
\left|a_{2}\right| \leq \sqrt{\frac{1+k(1-v)}{3}}  \tag{43}\\
\left|a_{3}\right| \leq \frac{1+k(1-v)}{3} \tag{44}
\end{gather*}
$$

Further, if $\delta \in \mathbb{R}$, then

$$
\left|a_{3}-\delta a_{2}^{2}\right| \leq \begin{cases}\frac{(1+M)(1-\delta)}{3} & \text { for } \delta<0  \tag{45}\\ \frac{3(1-\delta)+M(3-2 \delta)}{9} & \text { for } 0 \leq \delta<\frac{2}{3} \\ \frac{1+M(3-2 \delta)}{9} & \text { for } \frac{2}{3} \leq \delta<1, \\ \frac{1+M(2 \delta-1)}{9} & \text { for } 1 \leq \delta \leq \frac{4}{3} \\ \frac{3(\delta-1)+M(2 \delta-1)}{9} & \text { for } \frac{4}{3}<\delta<2 \\ \frac{(1+M)(\delta-1)}{3} & \text { for } \delta \geq 2\end{cases}
$$

where

$$
M \leq k(1-v)
$$

Proof. Let us consider the functions $g, \chi$ and $\xi$, which are represented as in Equations (2), (5) and (6). Since $f \in \mathcal{Q}_{\sigma}^{*}(k, v)$, there exist functions $p, q \in \mathcal{P}_{k}(v)$ that are analytic with

$$
p(z)=1+\sum_{n=1}^{\infty} p_{n} z^{n}=1+p_{1} z+p_{2} z^{2}+\cdots
$$

and

$$
q(z)=1+q_{1} z+q_{2} z^{2}+\cdots
$$

satisfying

$$
\begin{equation*}
\frac{\left(z f^{\prime}(z)\right)^{\prime}}{\chi^{\prime}(z)}=p(z) \tag{46}
\end{equation*}
$$

and

$$
\begin{equation*}
\frac{\left(w g^{\prime}(w)\right)^{\prime}}{\xi^{\prime}(w)}=q(w) \tag{47}
\end{equation*}
$$

From (46) and (47), we obtain

$$
\begin{gather*}
4 a_{2}=2 c_{2}+p_{1}  \tag{48}\\
9 a_{3}=3 c_{3}+2 c_{2} p_{1}+p_{2}  \tag{49}\\
-4 a_{2}=-2 c_{2}+p_{1} \tag{50}
\end{gather*}
$$

and

$$
\begin{equation*}
18 a_{2}^{2}-9 a_{3}=6 c_{2}^{2}-3 c_{3}-2 c_{2} q_{1}+q_{2} . \tag{51}
\end{equation*}
$$

Then, from (48) and (50), we obtain $p_{1}+q_{1}=0$. Adding (49) and (51), we obtain

$$
\begin{equation*}
18 a_{2}^{2}=6 c_{2}^{2}+4 c_{2} p_{1}+p_{2}+q_{2} \tag{52}
\end{equation*}
$$

By applying triangle inequality now and using Lemmas 2 and 5 in (52), we obtain

$$
\begin{equation*}
\left|a_{2}^{2}\right| \leq \frac{1+k(1-v)}{3} \tag{53}
\end{equation*}
$$

Upon simplification of Equation (53) gives (43). Now, again from (49) and (51), we obtain

$$
\begin{equation*}
9 a_{3}=3 c_{3}+2 c_{2} p_{1}+p_{2} . \tag{54}
\end{equation*}
$$

By applying triangle inequality now and using Lemma 2 and Lemma 5 in (54), we obtain

$$
\begin{equation*}
3\left|a_{3}\right| \leq 1+k(1-v) \tag{55}
\end{equation*}
$$

Upon simplification of Equation (55) gives (44). For any $\delta \in \mathbb{R}$ and by Equations (53) and (55), we have

$$
\begin{equation*}
a_{3}-\delta a_{2}^{2}=\frac{6\left[c_{3}-\delta c_{2}^{2}\right]+4 c_{2} p_{1}[1-\delta]+p_{2}[2-\delta]-\delta q_{2}}{18} \tag{56}
\end{equation*}
$$

By applying triangle inequality now and using Lemma 2 in (56), we obtain

$$
\begin{equation*}
\left|a_{3}-\delta a_{2}^{2}\right| \leq \frac{6\left|c_{3}-\delta c_{2}^{2}\right|+4 k(1-v)|1-\delta|+k(1-v)[|2-\delta|+|\delta|]}{18} \tag{57}
\end{equation*}
$$

Now, by using Lemma 6 in (57), we obtain (57). This completes the proof of Theorem 4.

By making a selection for $k$ as $k=2$, Theorem 4 gives the coefficient estimates for the class $\mathcal{Q}_{\sigma}^{*}(v)$, and is declared now as a corollary as below.

Corollary 5. Let $0 \leq v<1$. If a function $f \in \sigma$ of the form (1) belongs to the class $\mathcal{Q}_{\sigma}^{*}(v)$, then we have

$$
\begin{gathered}
\left|a_{2}\right| \leq \sqrt{\frac{1+2(1-v)}{3}} \\
\left|a_{3}\right| \leq \frac{1+2(1-v)}{3}
\end{gathered}
$$

Further, if $\delta \in \mathbb{R}$, then

$$
\left|a_{3}-\delta a_{2}^{2}\right| \leq \begin{cases}\frac{(1+N)(1-\delta)}{3} & \text { for } \delta<0 \\ \frac{3(1-\delta)+N(3-2 \delta)}{9} & \text { for } 0 \leq \delta<\frac{2}{3} \\ \frac{1+N(3-2 \delta)}{9} & \text { for } \frac{2}{3} \leq \delta<1 \\ \frac{1+N(2 \delta-1)}{9} & \text { for } 1 \leq \delta \leq \frac{4}{3} \\ \frac{3(\delta-1)+N(2 \delta-1)}{9} & \text { for } \frac{4}{3}<\delta<2 \\ \frac{(1+N)(\delta-1)}{3} & \text { for } \delta \geq 2\end{cases}
$$

where

$$
N \leq 2(1-v)
$$

For the special choices of $v=0$ and $k=2$, Theorem 4 will reduce to the following coefficient estimates for the class $\mathcal{Q}_{\sigma}^{*}$, and is detailed below as a corollary.

Corollary 6. If a function $f \in \sigma$ of the form (1) belongs to the class $\mathcal{Q}_{\sigma}^{*}$, then

$$
\begin{aligned}
& \left|a_{2}\right| \leq 1, \\
& \left|a_{3}\right| \leq 1 .
\end{aligned}
$$

Further, if $\delta \in \mathbb{R}$, then

$$
\left|a_{3}-\delta a_{2}^{2}\right| \leq \begin{cases}1-\delta & \text { for } \delta<0 \\ \frac{9-7 \delta}{9} & \text { for } 0 \leq \delta<\frac{2}{3} \\ \frac{7-4 \delta}{9} & \text { for } \frac{2}{3} \leq \delta<1 \\ \frac{4 \delta-1}{9} & \text { for } 1 \leq \delta \leq \frac{4}{3} \\ \frac{7 \delta-5}{9} & \text { for } \frac{4}{3}<\delta<2 \\ \delta-1 & \text { for } \delta \geq 2\end{cases}
$$

Remark 6. It can be visible at this point that for the family of bi-quasi-convex functions, the first two initial coefficient bounds are same as for a quasi-convex function. Since the coefficients are unpredictable, it may be interesting to verify whether all coefficients of bi-quasi-convex functions behave in the same way as the first two coefficients.

Let us make an assumption now as $\chi(z)=z$. Moreover, for the above assumption, let us denote the class $\mathcal{Q}_{\sigma}^{*}(k, v)$ by $\mathcal{F}_{\sigma}[k, v]$. In fact, the class $\mathcal{F}_{\sigma}[k, v]$ will be consisting of functions of the $f \in \sigma$ of the form (1) and satisfying the conditions

$$
f^{\prime}(z)+z f^{\prime \prime}(z) \in \mathcal{P}_{k}(v)
$$

and

$$
g^{\prime}(w)+w g^{\prime \prime}(w) \in \mathcal{P}_{k}(v)
$$

with $g$ being the analytic continuation of $f^{-1}$ to the open unit $\operatorname{disk} \mathcal{D}$.
However, for attaining the bounds for the class $\mathcal{F}_{\sigma}[k, v]$, the computation that may be akin to Theorem 4 has to be worked again. However, it is affirmed at this instant as a theorem without the details concerned.

Theorem 5. Let $0 \leq v<1, k \in[2,4]$. If a function $f \in \sigma$ given by (1) is in the class $\mathcal{F}_{\sigma}[k, v]$, then

$$
\begin{gathered}
\left|a_{2}\right| \leq \frac{\sqrt{k(1-v)}}{3}, \\
\left|a_{3}\right| \leq \frac{k(1-v)}{9}
\end{gathered}
$$

and

$$
\left|a_{3}-\delta a_{2}^{2}\right| \leq \begin{cases}\frac{k(1-v)(1-\delta)}{9} & \text { for } \delta<0 \\ \frac{k(1-v)}{9} & \text { for } 0 \leq \delta<2 \\ \frac{k(1-v)(\delta-1)}{9} & \text { for } \delta \geq 2\end{cases}
$$

By selecting the value of $k$ as $k=2$ in Theorem 5 , we have the class $\mathcal{F}_{\sigma}[2, v] \equiv \mathcal{F}_{\sigma}[v]$, which consists of all functions of the form (1) belonging to the class $f \in \sigma$ and satisfying the conditions

$$
\Re\left(f^{\prime}(z)+z f^{\prime \prime}(z)\right)>v
$$

and

$$
\Re\left(g^{\prime}(w)+w g^{\prime \prime}(w)\right)>v .
$$

The following corollary that is stated now gives the coefficient estimates for the class $\mathcal{F}_{\sigma}[v]$, and is as below.

Corollary 7. Let $0 \leq v<1$. A function $f \in \sigma$ given by (1) be in the class $\mathcal{F}_{\sigma}[v]$, then

$$
\begin{gathered}
\left|a_{2}\right| \leq \frac{\sqrt{2(1-v)}}{3} \\
\left|a_{3}\right| \leq \frac{2(1-v)}{9}
\end{gathered}
$$

and

$$
\left|a_{3}-\delta a_{2}^{2}\right| \leq \begin{cases}\frac{2(1-v)(1-\delta)}{9} & \text { for } \delta<0 \\ \frac{2(1-v)}{9} & \text { for } 0 \leq \delta<2 \\ \frac{2(1-v)(\delta-1)}{9} & \text { for } \delta \geq 2\end{cases}
$$

Remark 7. Instead of applying Lemma 6, if we use Lemma 7, the inequality (45) becomes

$$
\left|a_{3}-\delta a_{2}^{2}\right| \leq \begin{cases}\frac{(1+M)(1-\delta)}{3} & \text { for } \delta<0,  \tag{58}\\ \frac{3(1-\delta)+M(3-2 \delta)}{9} & \text { for } 0 \leq \delta<\frac{2}{3}, \\ \frac{3+M(3-2 \delta)}{9} & \text { for } \frac{2}{3} \leq \delta<1, \\ \frac{3+M(2 \delta-1)}{9} & \text { for } 1 \leq \delta \leq \frac{4}{3}, \\ \frac{3(\delta-1)+M(2 \delta-1)}{9} & \text { for } \frac{4}{3}<\delta<2, \\ \frac{(1+M)(\delta-1)}{3} & \text { for } \delta \geq 2,\end{cases}
$$

where

$$
M \leq k(1-v)
$$

Example 1. Let the function $f$ be given by

$$
f(z)=z+\frac{z^{3}}{6}, \quad \chi(z)=z-\frac{z^{3}}{9}
$$

Then, we have

$$
g(w)=w-\frac{w^{3}}{6}, \quad \xi(w)=w+\frac{w^{3}}{9}
$$

These functions belong to the class $\mathcal{Q}_{\sigma}^{*}(k, v)$.
5. Coefficient Bounds for $\mathcal{E}_{\sigma}^{\alpha}(k, v)$

In this section, we introduce and obtain the initial bounds for the family of $\alpha$ -exponentially-bi-convex functions with bounded boundary rotation of order $v$, which we define now.

Definition 4. Let $0 \leq v<1, k \in[2,4]$ and $\alpha \in \mathbb{C} \backslash\{0\}$. Let $f \in \sigma$ be of the form (1) such that $f^{\prime}(z) \neq 0$ on $\mathcal{D}$. Then, $f$ is known as $\alpha$-exponentially-bi-convex function with bounded boundary rotation of order $v$ if the following conditions holds good:

$$
1+\frac{z f^{\prime \prime}(z)}{f^{\prime}(z)}+\alpha z f^{\prime}(z) \in \mathcal{P}_{k}(v)
$$

and

$$
1+\frac{w g^{\prime \prime}(w)}{g^{\prime}(w)}+\alpha w g^{\prime}(w) \in \mathcal{P}_{k}(v)
$$

Here, $g$ is the analytic continuation of $f^{-1}$ to the open unit disk $\mathcal{D}$. We denote the family of all $\alpha$-exponentially-bi-convex functions with bounded boundary rotation of order $v$ by $\mathcal{E}_{\sigma}^{\alpha}(k, v)$.

## Remark 8.

(i) If $\alpha=1, \mathcal{E}_{\sigma}^{\alpha}(k, v)$ reduces to $\mathcal{E}_{\sigma}^{\alpha}(k, v) \equiv \mathcal{E}_{\sigma}^{1}(k, v) \equiv \mathcal{E}_{\sigma}(k, v)$, the family of exponentially-bi-convex functions with bounded boundary rotation of order $v$.
(ii) If $\alpha=1$ and $v=0, \mathcal{E}_{\sigma}^{\alpha}(k, v)$ reduces to $\mathcal{E}_{\sigma}^{\alpha}(k, v) \equiv \mathcal{E}_{\sigma}^{1}(k, 0) \equiv \mathcal{E}_{\sigma}(k)$, the family of exponentially-bi-convex functions with bounded boundary rotation.
(iii) If $k=2, \mathcal{E}_{\sigma}^{\alpha}(k, v)$ reduces to $\mathcal{E}_{\sigma}^{\alpha}(k, v) \equiv \mathcal{E}_{\sigma}^{\alpha}(2, v) \equiv \mathcal{E}_{\sigma}^{\alpha}(v)$, the family consisting of $\alpha$-exponentially-bi-convex functions of order $v$.
(iv) When $\alpha=1$ and $k=2, \mathcal{E}_{\sigma}^{\alpha}(k, v)$ reduces to $\mathcal{E}_{\sigma}^{\alpha}(k, v) \equiv \mathcal{E}_{\sigma}^{1}(2, v) \equiv \mathcal{E}_{\sigma}(v)$, the family of exponentially-bi-convex functions of order $v$.

Next, we attain the initial coefficient bounds and $\left|a_{3}-\delta a_{2}^{2}\right|$ for the class $\mathcal{E}_{\sigma}^{\alpha}(k, v)$.
Theorem 6. Let $0 \leq v<1, k \in[2,4]$ and $\alpha \in \mathbb{C} \backslash\{0\}$. Let $f$ given by (1) be in the class $\mathcal{E}_{\sigma}^{\alpha}(k, v)$, then

$$
\begin{gather*}
\left|a_{2}\right| \leq \sqrt{\frac{k(1-v)}{2}}  \tag{59}\\
\left|a_{3}\right| \leq \frac{|\alpha|^{2}+|\alpha| k(1-v)+3 k(1-v)}{6} \tag{60}
\end{gather*}
$$

Further, if $\delta \in \mathbb{R}$, then

$$
\left|a_{3}-\delta a_{2}^{2}\right| \leq \begin{cases}\frac{|\alpha|^{2}+|\alpha| k(1-v)+3 k(1-v)(1-\delta)}{6} & \text { for } \delta<\frac{2}{3}  \tag{61}\\ \frac{|\alpha|^{2}+k(1-v)(|\alpha|+1)}{6} & \text { for } \frac{2}{3} \leq \delta<\frac{3}{4} \\ \frac{|\alpha|^{2}+|\alpha| k(1-v)+3 k(1-v)(\delta-1)}{6} & \text { for } \delta \geq \frac{3}{4}\end{cases}
$$

Proof. Since $f \in \mathcal{E}_{\sigma}^{\alpha}(k, v)$, there exist functions $p, q \in \mathcal{P}_{k}(v)$ that are analytic with

$$
\begin{gathered}
p(z)=1+p_{1} z+p_{2} z^{2}+\cdots \\
q(z)=1+q_{1} z+q_{2} z^{2}+\cdots
\end{gathered}
$$

and satisfying

$$
\begin{equation*}
1+\frac{z f^{\prime \prime}(z)}{f^{\prime}(z)}+\alpha z f^{\prime}(z)=p(z) \tag{62}
\end{equation*}
$$

and

$$
\begin{equation*}
1+\frac{w g^{\prime \prime}(w)}{g^{\prime}(w)}+\alpha w g^{\prime}(w)=q(w) \tag{63}
\end{equation*}
$$

From (62) and (63), we attain

$$
\begin{gather*}
2 a_{2}+\alpha=p_{1}  \tag{64}\\
6 a_{3}+4 \alpha a_{2}=2 a_{2} p_{1}+p_{2}  \tag{65}\\
-2 a_{2}+\alpha=q_{1} \tag{66}
\end{gather*}
$$

and

$$
\begin{equation*}
12 a_{2}^{2}-6 a_{3}-4 \alpha a_{2}=-2 a_{2} q_{1}+q_{2} \tag{67}
\end{equation*}
$$

Then, from (64) and (66), we obtain $p_{1}+q_{1}=2 \alpha$. Adding (65) and (67), we obtain

$$
\begin{equation*}
12 a_{2}^{2}=2 a_{2}\left(p_{1}-q_{1}\right)+p_{2}+q_{2} \tag{68}
\end{equation*}
$$

Now, by using (64) and (66) in (68), we obtain

$$
\begin{equation*}
4 a_{2}^{2}=p_{2}+q_{2} \tag{69}
\end{equation*}
$$

Hence, by using triangle inequality and Lemma 2 in (69), we obtain

$$
\begin{equation*}
2\left|a_{2}\right|^{2} \leq k(1-v) \tag{70}
\end{equation*}
$$

Upon simplification of Equation (70) gives (59). Again from (65), (67) and by using (69), we obtain

$$
\begin{equation*}
6 a_{3}=\alpha^{2}-\alpha p_{1}+2 p_{2}+q_{2} . \tag{71}
\end{equation*}
$$

Hence, by using triangle inequality and Lemma 2 in (71), we obtain

$$
\begin{equation*}
6\left|a_{3}\right| \leq|\alpha|^{2}+|\alpha| k(1-v)+3 k(1-v) . \tag{72}
\end{equation*}
$$

Upon simplification of Equation (72) gives (60). For any $\delta \in \mathbb{R}$ and by Equations (69) and (71), we have

$$
\begin{equation*}
a_{3}-\delta a_{2}^{2}=\frac{2 \alpha^{2}-2 \alpha p_{1}+p_{2}(4-3 \delta)+q_{2}(2-3 \delta)}{12} \tag{73}
\end{equation*}
$$

Hence, by using triangle inequality and Lemma 2 in (73), we obtain

$$
\begin{equation*}
\left|a_{3}-\delta a_{2}^{2}\right| \leq \frac{2|\alpha|^{2}+2|\alpha| k(1-v)+k(1-v)[|4-3 \delta|+|2-3 \delta|]}{12} . \tag{74}
\end{equation*}
$$

Upon simplification of Equation (74) at once implies (61). The proof of Theorem 6 is thus completed.

Remark 9. It is interesting to observe that the coefficient bound of $\left|a_{2}\right|$ is independent of $\alpha$.
For the special choice of $\alpha=1$, Theorem 6 gives the following coefficient estimates for the class $\mathcal{E}_{\sigma}(k, v)$ and is stated as a corollary below.

Corollary 8. Let $0 \leq v<1$ and $k \in[2,4]$. Let $f$ given by (1) be in the class $\mathcal{E}_{\sigma}(k, v)$. Then,

$$
\begin{aligned}
& \left|a_{2}\right| \leq \sqrt{\frac{(1-v) k}{2}} \\
& \left|a_{3}\right| \leq \frac{1+4 k(1-v)}{6}
\end{aligned}
$$

and

$$
\left|a_{3}-\delta a_{2}^{2}\right| \leq \begin{cases}\frac{1+k(1-v)(1+3(1-\delta))}{6} & \text { for } \delta<\frac{2}{3} \\ \frac{1+2 k(1-v)}{6} & \text { for } \frac{2}{3} \leq \delta<\frac{3}{4} \\ \frac{1+k(1-v)(1+3(\delta-1))}{6} & \text { for } \delta \geq \frac{3}{4}\end{cases}
$$

For the special selection of $k=2$, Theorem 6 reduces to the next coefficient bounds for the class $\mathcal{E}_{\sigma}^{\alpha}(v)$, and is stated as a corollary as follows.

Corollary 9. Let $0 \leq v<1$ and $\alpha \in \mathbb{C} \backslash\{0\}$. Let $f \in \mathcal{E}_{\sigma}^{\alpha}(v)$ be given as in (1). Then, we have the estimates

$$
\begin{gathered}
\left|a_{2}\right| \leq \sqrt{1-v} \\
\left|a_{3}\right| \leq \frac{|\alpha|^{2}+|\alpha| 2(1-v)+6(1-v)}{6}
\end{gathered}
$$

and

$$
\left|a_{3}-\delta a_{2}^{2}\right| \leq \begin{cases}\frac{|\alpha|^{2}+|\alpha| 2(1-v)+6(1-v)(1-\delta)}{6} & \text { for } \delta<\frac{2}{3}, \\ \frac{|\alpha|^{2}+2(1-v)(|\alpha|+1)}{6} & \text { for } \frac{2}{3} \leq \delta<\frac{3}{4}, \\ \frac{|\alpha|^{2}+|\alpha| 2(1-v)+6(1-v)(\delta-1)}{6} & \text { for } \delta \geq \frac{3}{4}\end{cases}
$$

## 6. Concluding Remarks and Observations

In this investigation, the authors have introduced four new subclasses of $\sigma$, the class of bi-univalent functions of order $v$ with bounded boundary rotation. The first two initial upper bounds $\left|a_{2}\right|$ and $\left|a_{3}\right|$ for the Taylor-Maclaurin's coefficients for the classes $\mathcal{B}_{\sigma}^{\beta}(k, v)$, $\mathcal{Q}_{\sigma}^{*}(k, v)$ and $\mathcal{E}_{\sigma}^{\alpha}(k, v)$ are established. Looking at the initial coefficients, it is indeed easy to see that there is an unpredictability in the nature of coefficients and one cannot predict the next coefficients from the existing one. Also, Fekete-Szegö coefficient bounds for the classes $\mathcal{B}_{\sigma}^{\beta}(k, v), \mathcal{Q}_{\sigma}^{*}(k, v)$ and $\mathcal{E}_{\sigma}^{\alpha}(k, v)$ are moreover established. Motivating observations on the foremost consequences as well as improvements of the previous bounds were also specified. For the selection of $v=0$, interested researchers can also obtain additional consequences and corollaries, and those details are omitted here.

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