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Translational Regular Variability and the Index Function

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Abstract: This paper deals with translational regular and rapid variations. By using a new method of proving the Galambos–Bojanić-Seneta type theorems, we prove two theorems of this type for translationally regularly varying and translationally rapidly varying functions and sequences, important objects in the asymptotic analysis of divergent processes. Also, we introduce and study the index functions for translationally regularly varying functions and sequences. For example, we prove that the index function of a translationally regularly varying function is also in the same class of functions.

Keywords: translational regular variation; translational rapid variation; index function

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1. Introduction

Classical Karamata theory (see, for instance, [1–3]) was initiated in 1930 in the investigation of the qualitative asymptotic behavior of the Riemann–Stieltjes integral (and, in particular, of Dirichlet and power series). This theory has had interesting and diverse applications in both theoretical and applied mathematics as well as in other sciences. Its use in applied mathematics is more intensive in various disciplines: summability theory, differential and difference equations [4–9], Tauberian theorems [10,11], probability theory [12], selection principles theory, game theory, Ramsey theory, number theory, complex analysis, generalized inverses, machine learning, and so on. Also, the theory has applications in energetic electronics [13] and cosmology [14]. For more details see [15–18].

Karamata's theory of regular variability has functional and sequential aspects which are related to each other through theorems of the Galambos–Bojanić-Seneta types ([19–21]). Also, this theory has a number of modifications. For example, in [22] it was observed that the theory of regular variation can be considered in a more general setting, i.e., in a Banach algebra of operators by taking a sequence of regular elements of that algebra which satisfies (2) below.

A function $f : [a, \infty) \to (0, \infty)$, $a > 0$, is *regularly varying* (in the sense of Karamata) if it is measurable and

$$\lim_{x\to\infty} \frac{f(\lambda x)}{f(x)} = r_f(\lambda) < \infty \tag{1}$$

for each $\lambda > 0$.

The class of all such functions is dented by RV_φ.

A sequence of positive real numbers $\mathbf{c} = (c_n)$ is said to be *regularly varying* (in the sense of Karamata) if

$$\lim_{n\to\infty} \frac{c_{[\lambda n]}}{c_n} = r_{\mathbf{c}}(\lambda) < \infty \tag{2}$$

for each $\lambda > 0$, where for a real number a, $[a]$ denotes the largest integer which is less than or equal to a.

The class of all such sequences is dented by $\mathsf{RV_s}$.

Observe that for a function f in (1) it follows

$$\limsup_{x\to\infty} \frac{f(\lambda x)}{f(x)} = \overline{k}_f(\lambda) < \infty \tag{3}$$

and

$$\liminf_{x\to\infty} \frac{f(\lambda x)}{f(x)} = \underline{k}_f(\lambda) < \infty \tag{4}$$

for each $\lambda > 0$. The function $\overline{k}_f$ is called the *index function of* f, while $\underline{k}_f$ is called the *auxiliary function of* f. Similarly, we have the sequential analogs for (3) and (4).

In the papers [19,20] the following result was proved.

Theorem 1. *For a sequence* $\mathbf{c} = (c_n)$ *of positive real numbers, the following are equivalent;*

(1) $\mathbf{c} \in \mathsf{RV_s}$;
(2) *the function* f *defined by* $f(x) = c_{[x]}$, $x \geq 1$, *belongs to the class* RV_φ.

This theorem is a Galambos–Bojanić-Seneta type result for regularly varying functions and sequences.

For the famous characterization theorem of regularly varying functions for the function f in (1) we have

$$r_f(\lambda) = \lambda^\rho \tag{5}$$

for some $\rho \in \mathbb{R}$ and each $\lambda > 0$.

Also, by [15], for the sequence $\mathbf{c}$ in Theorem 1 it holds

$$r_{\mathbf{c}}(\lambda) = \lambda^\rho \tag{6}$$

for the same $\rho \in \mathbb{R}$ and each $\lambda > 0$.

In [23] (see also [24]), the index function operator K was introduced by

$$K : f \mapsto r_f = \overline{k}_f \tag{7}$$

for each function $f \in \mathsf{RV}_\varphi$. Evidently,

$$K(K(f)) = K(f),\ f \in \mathsf{RV}_\varphi, \tag{8}$$

i.e., $K(f)$ is a fixed point for the operator K whenever $f \in \mathsf{RV}_\varphi$.

In the paper [23] the authors also considered the operator K^* for sequences $\mathbf{c}$ from $\mathsf{RV_s}$ given by

$$K^* : \mathbf{c} \mapsto r_{\mathbf{c}} = \overline{k}_{\mathbf{c}}. \tag{9}$$

By (8) we have

$$K(K^*(\mathbf{c})) = K^*(\mathbf{c}). \tag{10}$$

In [25] the following class of sequences plays an important role in selection principles theory and infinite topological games. (Several results in selection principles theory and game theory related to the classical regular (and rapid) variation are significantly improved in the context of translational regular (and rapid) variation.)

A sequence $\mathbf{c} = (c_n)$ of positive real numbers belongs to the class $\mathsf{Tr}(\mathsf{RV_s})$ of *translationally regularly varying sequences* if for each $\lambda \in \mathbb{R}$,

$$r^T_{\mathbf{c}}(\lambda) := \lim_{n\to\infty} \frac{c_{[\lambda+n]}}{c_n} < \infty. \tag{11}$$

The following characterization result is from ([25] Theorem 3.6].

Theorem 2. *If a sequence* $(c_n) \in \mathsf{Tr}(\mathsf{RV_s})$, *then* $r^T_{\mathbf{c}}(\lambda) = e^{\rho[\lambda]}$, *for some* $\rho \in \mathbb{R}$ *and each* $\lambda \in \mathbb{R}$.

The number ρ from the previous theorem is called the *index of variability of* (c_n).

By $\mathsf{Tr}(\mathsf{RV}_{\mathsf{s},\rho})$ we denote the family of all sequences in $\mathsf{Tr}(\mathsf{RV}_{\mathsf{s}})$ of index ρ.

In [26], the class $\mathsf{Tr}(\mathsf{RV}_{\varphi})$ of *translationally regularly varying functions* was introduced and studied.

A measurable function $f : [a, \infty) \to (0, \infty)$, $a \in \mathbb{R}$ fixed, is translationally regularly varying if the following asymptotic condition holds for each $\lambda \in \mathbb{R}$:

$$r_f^T(\lambda) := \lim_{x\to\infty} \frac{f(x+\lambda)}{f(x)} < \infty. \tag{12}$$

We also consider the classes $\mathsf{Tr}(\mathsf{R}_{\varphi,\infty})$ of translationally rapidly varying functions and $\mathsf{Tr}(\mathsf{R}_{\mathsf{s},\infty})$ of translationally rapidly varying sequences [25].

A measurable function $f : [a, \infty) \to (0, \infty)$, $a > 0$, is said to be *translationally rapidly varying* if for each $\lambda \in \mathbb{R}$

$$\lim_{x\to\infty} \frac{f(\lambda+x)}{f(x)} = \infty. \tag{13}$$

A sequence $\mathbf{c} = (c_n)$ is in the class $\mathsf{Tr}(\mathsf{R}_{\mathsf{s},\infty})$ of the *translationally rapidly varying sequences* if for each $\lambda \geq 1$, the following holds:

$$\lim_{n\to\infty} \frac{c_{[\lambda+n]}}{c_n} = \infty. \tag{14}$$

The importance of translationally rapidly varying functions and sequences can be found in the papers [27,28].

The goal of our study is to extend and generalize the known results concerning Galambs–Bojanić-Seneta type theorems and to complement the existing results about the index function. We first prove two theorems of Galambos–Bojanić-Seneta type for translationally regularly and rapidly varying functions and sequences. The proofs of these theorems are quite different from the proofs of the corresponding theorems for classical regular and rapid variations and contain new methods and ideas. Then, we consider the index functions for the classes of translationally regularly varying functions and sequences.

2. Main Results

For translational regular variability of functions and sequences the following Galambos–Bojanić-Seneta type result is true.

Theorem 3. *For a sequence $\mathbf{c} = (c_n)$ of positive real numbers, the following are equivalent:*

(1) $\mathbf{c} \in \mathsf{Tr}(\mathsf{RV}_{\mathsf{s}})$;

(2) *the function $f : [1, \infty) \to (0, \infty)$ defined by $f(x) = c_{[x]}$ is in the class $\mathsf{Tr}(\mathsf{RV}_{\varphi})$.*

Proof. (1) $\Rightarrow$ (2) Let $\mathbf{c} = (c_n) \in \mathsf{Tr}(\mathsf{RV}_{\mathsf{s}})$. Then, by Theorem 2, for each $\lambda \in \mathbb{R}$ and some $\rho \in \mathbb{R}$ it holds

$$r_{\mathbf{c}}^T(\lambda) = \lim_{n\to\infty} \frac{c_{[\lambda+n]}}{c_n} = (r_{\mathbf{c}}^T(1))^{[\lambda]} = e^{\rho[\lambda]}, \tag{15}$$

where $\rho = \ln(r_{\mathbf{c}}^T(1))$ and $0 < r_{\mathbf{c}}^T(\lambda) < \infty$ form each $\lambda \in \mathbb{R}$. We consider four cases for ρ and prove that in each of these cases we obtain $r_f^T(\lambda) = e^{\rho[\lambda]}$, $\lambda \in \mathbb{R}$.

(i) For $\rho = 0$, we have $r_{\mathbf{c}}^T(1) = 1 = r_{\mathbf{c}}^T(\lambda)$ for each $\lambda \in \mathbb{R}$.

Consider the function $f : [1, \infty) \to (0, \infty)$ defined by $f(x) = c_{[x]}$. Then for each $\lambda \in \mathbb{R}$

$$r_f^T(\lambda) = \lim_{x\to\infty} \frac{c_{[\lambda+x]}}{c_{[x]}} = \lim_{x\to\infty} \left(\frac{c_{[\lambda+x]}}{c_{[\lambda+[x]]}} \cdot \frac{c_{[\lambda+[x]]}}{c_{[x]}} \right) = \lim_{x\to\infty} \left(\frac{c_{[\lambda+x]}}{c_{[\lambda+[x]]}} \right) \cdot 1. \tag{16}$$

For an arbitrary and fixed $\lambda \in \mathbb{R}$ it is true

$$0 \leq [\lambda + x] - [\lambda + [x]] < (\lambda + x) - (\lambda + [x] - 1) = x - [x] + 1 < x - (x - 1) + 1 = 2. \quad (17)$$

Since $[\lambda + x]$ and $[\lambda + [x]]$ are integers for $x \geq 1$ and $\lambda \in \mathbb{R}$, for the same λ and x we have $[\lambda + x] = [\lambda + [x]]$ or $[\lambda + x] = [\lambda + [x]] + 1$. Therefore,

$$\lim_{x \to \infty} \frac{c_{[\lambda + x]}}{c_{[\lambda + [x]]}} = 1, \quad (18)$$

and thus, in this case, $r_f^T(\lambda) = 1 = e^{\rho[\lambda]}$ for each $\lambda \in \mathbb{R}$.

(ii) If $\rho > 0$, then $r_{\mathbf{c}}^T(1) = e^{\rho} = \alpha > 1$. It follows that there is a sufficiently large $i \in \mathbb{N}$ such that the sequence $\mathbf{c}$ is strictly increasing beginning with c_i. Let $\mathbb{R}^* = \mathbb{R} \setminus \mathbb{Z}_-$, where $\mathbb{Z}_-$ is the set of negative integers. Then for $\lambda \in \mathbb{R}^*$ we have

$$\begin{aligned} \underline{k}_f^T(\lambda) &= \liminf_{x \to \infty} \frac{c_{[\lambda + x]}}{c_{[x]}} = \liminf_{x \to \infty} \frac{c_{[\lambda + \frac{x}{[x]} \cdot [x]]}}{c_{[x]}} \\ &\geq \liminf_{x \to \infty} \frac{c_{[\lambda + [x]]}}{c_{[x]}} = \lim_{n \to \infty} \frac{c_{[\lambda + n]}}{c_n} = e^{\rho[\lambda]}. \end{aligned}$$

Also, we have

$$\begin{aligned} \overline{k}_f^T(\lambda) &= \limsup_{x \to \infty} \frac{c_{[\lambda + x]}}{c_{[x]}} = \limsup_{x \to \infty} \frac{c_{[\lambda + \frac{x}{[x]} \cdot [x]]}}{c_{[x]}} \\ &\leq \limsup_{x \to \infty} \frac{c_{[\lambda + \delta + [x]]}}{c_{[x]}} = \lim_{n \to \infty} \frac{c_{[\lambda + \delta + n]}}{c_n} = e^{\rho[\lambda + \delta]} \end{aligned}$$

for each $\delta > 0$.

As the function $[\cdot]$ is right continuous on $\mathbb{R}^*$, one obtains

$$\overline{k}_f^T(\lambda) = \lim_{\delta \to 0} e^{\rho[\lambda + \delta]} = e^{\rho[\lambda]}.$$

Therefore, for $\rho > 0$, $r_f^T(\lambda) = \overline{k}_f^T(\lambda) = \underline{k}_f^T(\lambda) = e^{\rho[\lambda]}$ for $\lambda \in \mathbb{R}^*$. (We suppose that $\rho < \infty$.)

(iii) For $\rho < 0$ we have $r_{\mathbf{c}}^T(1) = e^{\rho} = \beta$, where $0 < \beta < 1$ (we assume $\rho > -\infty$). There is $j \in \mathbb{N}$ such that the sequence $\mathbf{c}$ is strictly decreasing beginning from c_j. One concludes that for $\lambda \in \mathbb{R}^*$

$$\begin{aligned} \underline{k}_f^T(\lambda) &= \liminf_{x \to \infty} \frac{c_{[\lambda + x]}}{c_{[x]}} = \liminf_{x \to \infty} \frac{c_{[\lambda + \frac{x}{[x]} \cdot [x]]}}{c_{[x]}} \\ &\geq \liminf_{x \to \infty} \frac{c_{[\lambda + \delta + [x]]}}{c_{[x]}} = \lim_{n \to \infty} \frac{c_{[\lambda + \delta + n]}}{c_n} = e^{\rho[\lambda + \delta]} \end{aligned}$$

and

$$\begin{aligned} \overline{k}_f^T(\lambda) &= \limsup_{x \to \infty} \frac{c_{[\lambda + x]}}{c_{[x]}} = \limsup_{x \to \infty} \frac{c_{[\lambda + \frac{x}{[x]} \cdot [x]]}}{c_{[x]}} \\ &\leq \limsup_{x \to \infty} \frac{c_{[\lambda + [x]]}}{c_{[x]}} = \lim_{n \to \infty} \frac{c_{[\lambda + n]}}{c_n} = e^{\rho[\lambda]} \end{aligned}$$

for each $\delta > 0$.

Using again the fact that the integer function $[\cdot]$ is right continuous on $\mathbb{R}^*$ we obtain

$$r_f^T(\lambda) = \overline{k}_f^T(\lambda) = \underline{k}_f^T(\lambda) = e^{\rho[\lambda]}$$

for $\lambda \in \mathbb{R}^*$.

(iv) Let $\rho \in \mathbb{Z}_-$. So, there is $k \in \mathbb{N}$ such that $\lambda = -k$. For such λ and ρ we have

$$\begin{aligned} r_f^T(\lambda) &= \lim_{x\to\infty} \frac{c_{[-k+x]}}{c_{[x]}} = \frac{1}{\lim_{x\to\infty} \frac{c_{[x]}}{c_{[x-k]}}} \\ &= \frac{1}{\lim_{x\to\infty} \frac{c_{[k+x-k]}}{c_{[x-k]}}} = \left(\lim_{x\to\infty} \frac{c_{k+[x-k]}}{c_{[x-k]}} \right)^{-1} \\ &= \left(e^{\rho k}\right)^{-1} = e^{-k\rho} = e^{\rho[\lambda]}. \end{aligned}$$

From (i)–(iv) we conclude $f \in \mathsf{Tr}(\mathsf{RV}_\varphi)$ which completes the proof of (1) $\Rightarrow$ (2).

(2) $\Rightarrow$ (1) is trivial. □

Example 1. *Consider the sequence* $\mathbf{c} = (c_n)$ *defined by*

$$c_n = e^n \sin(1/n), \quad n \in \mathbb{N}.$$

By a direct calculation one obtains

$$\lim_{n\to\infty} \frac{c_{[\lambda+n]}}{c_n} = \lim_{n\to\infty} \frac{e^{[\lambda+n]} \sin(1/[\lambda+n])}{e^n \sin(1/n)} = e^{[\lambda]}$$

for all $\lambda \in \mathbb{R}$*, which means that* $\mathbf{c} \in \mathsf{Tr}(\mathsf{RV}_\mathsf{s})$ *(of index of variability 1). By the above theorem, the function* f*,* $f(x) = c_{[x]}$*,* $x \geq 1$*, belongs to the class* $\mathsf{Tr}(\mathsf{RV}_\varphi)$*.*

Similarly to the previous theorem one can prove the following result of Galambos–Bojanić-Seneta type for translationally rapidly varying functions and translationally rapidly varying sequences.

Theorem 4. *A sequence* $\mathbf{c} = (c_n)$ *of positive real numbers belongs to the class* $\mathsf{Tr}(\mathsf{R}_{\mathsf{s},\infty})$ *if and only if the function* f *defined by* $f(x) = c_{[x]}$*,* $x \geq 1$*, belongs to the class* $\mathsf{Tr}(\mathsf{R}_{\varphi,\infty})$*.*

Proof. ($\Rightarrow$) Let $\mathbf{c} \in \mathsf{Tr}(\mathsf{R}_{\mathsf{s},\infty})$. Then $\lim_{n\to\infty} \frac{c_{[\lambda+n]}}{c_n} = \infty$ for each $\lambda \geq 1$. Therefore, for sufficiently large $n \in \mathbb{N}$ it is satisfied $c_{n+1} \geq 2c_n$ because $\lim_{n\to\infty} \frac{c_{n+1}}{c_n} = \infty$. Thus from sufficiently large n, the sequence $\mathbf{c}$ is increasing. Also, for $x \geq 1$, $x \geq [x]$. Hence, for each $\lambda \geq 1$ we have

$$\lim_{x\to\infty} \frac{c_{[\lambda+x]}}{c_{[x]}} \geq \lim_{x\to\infty} \frac{c_{[\lambda+[x]]}}{c_{[x]}} = \infty,$$

i.e., the function $f(x) = c_{[x]}$, $x \geq 1$, belongs to the class $\mathsf{Tr}(\mathsf{R}_{\varphi,\infty})$.

($\Leftarrow$) It is evident. □

Example 2. *Let the sequence* $\mathbf{c} = (c_n)$ *be given by*

$$c_n = \begin{cases} 1, & n = 1, \\ e^{n^2} \ln(\ln(n)), & n \geq 2. \end{cases}$$

It is not hard to conclude

$$\lim_{n\to\infty} \frac{c_{[\lambda+n]}}{c_n} = \infty,$$

i.e., $\mathbf{c} \in \mathsf{Tr}(\mathsf{R}_{\mathsf{s},\infty})$*. This implies, by Theorem 4, that the function* $c_{[x]} \in \mathsf{Tr}(\mathsf{R}_{\varphi,\infty})$*.*

Now, we consider the behavior of the index function of translationally regularly varying sequences.

Theorem 5. *If a sequence* $\mathbf{c} = (c_n)$ *belongs to the class* $\mathsf{Tr}(\mathsf{RV_s})$*, then the index function* $r^T_{\mathbf{c}}(\lambda)$ *belongs to the class* $\mathsf{Tr}(\mathsf{RV}_\varphi)$ *for* $\lambda \geq a$*,* $a > 0$*.*

Proof. By Theorem 2, $r^T_{\mathbf{c}}(\lambda) = e^{\rho[\lambda]}$, $\lambda \in \mathbb{R}$. For $\alpha \in \mathbb{R}$ we have $[\lambda] + [\alpha] \leq [\lambda + \alpha] \leq [\lambda] + [\alpha] + 1$. Thus, for $\rho > 0$ we have

$$r^T_{r^T_{\mathbf{c}}}(\alpha) = \lim_{\lambda \to \infty} \frac{r^T_{\mathbf{c}}(\lambda + \alpha)}{r^T_{\mathbf{c}}(\lambda)} = \lim_{\lambda \to \infty} \frac{e^{\rho[\lambda+\alpha]}}{e^{\rho[\lambda]}}$$

which implies

$$\lim_{\lambda \to \infty} \frac{e^{\rho([\lambda]+[\alpha])}}{e^{\rho[\lambda]}} \leq r^T_{r^T_{\mathbf{c}}}(\alpha) \leq \lim_{\lambda \to \infty} \frac{e^{\rho([\lambda]+[\alpha]+1)}}{e^{\rho[\lambda]}}$$

i.e.,

$$e^{\rho[\alpha]} \leq r^T_{r^T_{\mathbf{c}}}(\alpha) \leq e^{\rho[\alpha+1]}.$$

Similarly, for $\rho < 0$ one obtains

$$e^{\rho([\alpha]+1)} \leq r^T_{r^T_{\mathbf{c}}}(\alpha) \leq e^{\rho[\alpha]}.$$

In other words, there is some $\sigma \in \mathbb{R}$ such that $r^T_{r_{\mathbf{c}^T}}(\alpha) = e^{\sigma[\alpha]}$. Therefore, $r^T_{r^T_{\mathbf{c}}}(\alpha) \in Tr(RV_\varphi)$ holds for $\rho \in \mathbb{R} \setminus \{0\}$. Evidently, it is also true for $\rho = 0$. □

In the following theorem, we consider and prove an important property of the index function $r^T_f(\lambda)$ for $\lambda \geq a$, $a > 0$.

Theorem 6. *For a function* $f \in \mathsf{Tr}(\mathsf{RV}_\varphi)$*, the index function* $r^T_f(\lambda)$*,* $\lambda \geq a$*,* $a > 1$*, belongs to the class* $\mathsf{Tr}(\mathsf{RV}_\varphi)$*.*

Proof. Since $f \in \mathsf{Tr}(\mathsf{RV}_\varphi)$ we have

$$\lim_{x \to \infty} \frac{f(\lambda + x)}{f(x)} = r^T_f(\lambda) < \infty$$

for each $\lambda \in R$.

For all $\alpha \in \mathbb{R}$ we have

$$\frac{r^T_f(\lambda + \alpha)}{r^T_f(\lambda)} = r^T_f(\alpha) \tag{19}$$

so that

$$r^T_{r^T_f}(\alpha) = \lim_{\lambda \to \infty} \frac{r^T_f(\lambda + \alpha)}{r^T_f(\lambda)} = r^T_f(\alpha) \tag{20}$$

because $r^T_f(\lambda) > 0$ for $\lambda > 0$. This means that the function $r^T_f(\lambda)$, $\lambda \geq a$, $a > 0$, belongs to the class $\mathsf{Tr}(\mathsf{RV}_\varphi)$. □

Remark 1. *(a) Let a sequence* $\mathbf{c} = (c_n)$ *belong to the class* $\mathsf{Tr}(\mathsf{RV_s})$*. By Theorem 2,* $r^T_{\mathbf{c}}(\lambda) = e^{\rho[\lambda]}$*, for each* $\lambda \in \mathbb{R}$ *and some* $\rho \in \mathbb{R}$*. Therefore, for the function* $f(x) = c_{[x]}$*,* $x \geq 1$*, it holds*

$$r^T_f(\lambda) = \lim_{x \to \infty} \frac{c_{[\lambda+x]}}{c_{[x]}} = \lim_{n \to \infty} \frac{c_{[\lambda+n]}}{c_n} = r^T_{\mathbf{c}}(\lambda) = e^{\rho[\lambda]}$$

for some $\rho \in \mathbb{R}$ *and each* $\lambda \in \mathbb{R}$*. Besides,* $r^T_{\mathbf{c}}(\lambda) = r^T_{\mathbf{c}}([\lambda])$*.*

(b) The equality $r_f^T(-\lambda)\cdot r_f^T(\lambda)=1$ *is satisfied for each* $\lambda\in\mathbb{R}$. *Indeed,*

$$r_f^T(-\lambda)=\lim_{x\to\infty}\frac{f(-\lambda+x)}{f(x)}=\frac{1}{\lim_{x\to\infty}\frac{f(\lambda+x-\lambda)}{f(x-\lambda)}}=\frac{1}{r_f^T(\lambda)}.$$

(c) $r_f^T(0)=1$.
(d) Since $r_f^T(\lambda)>0$ *for each* $\lambda\in\mathbb{R}$, *from (b) we have* $r_f^T(-\lambda)>0$ *for each* $\lambda\in\mathbb{R}$.
(e) $r_{\mathbf{c}}^T(-\lambda)\cdot r_{\mathbf{c}}^T(\lambda)=1$, $r_{\mathbf{c}}^T(0)=1$ *and* $r_{\mathbf{c}}^T(\lambda)>0$, *hence* $r_{\mathbf{c}}^T(-\lambda)>0$ *for any* $\mathbf{c}\in\mathsf{Tr}(\mathsf{RV_s})$.

Consider the operator $K^T:\mathsf{Tr}(\mathsf{RV}_\varphi)\to\mathsf{Tr}(\mathsf{RV}_\varphi)$ defined by

$$K^T(f)=r_f^T(\lambda),\ \ \lambda\in\mathbb{R} \tag{21}$$

analogously to the operator K in (7). Then

$$K^T(K^T(f))(\lambda)=r_f^T(\lambda), \tag{22}$$

i.e., r_f^T is a fixed point for the operator K^T.

Similarly, define the operator $K^{*T}:\mathsf{Tr}(\mathsf{RV_s})\to\mathsf{Tr}(\mathsf{RV}_\varphi)$ by

$$K^{*T}(\mathbf{c})=r_{\mathbf{c}}^T(\lambda),\ \ \lambda\in\mathbb{R}.$$

Then

$$K^T(K^{*T}(\mathbf{c}))(\lambda)=r_{\mathbf{c}}^T(\lambda),\ \ \lambda\in\mathbb{R}$$

which means that $K^{*T}(\mathbf{c})(\lambda)=r_{\mathbf{c}}^T(\lambda)$, $\lambda\in\mathbb{R}$, is a fixed point for the operator K^T.

3. Conclusions

In this paper, we proved two theorems of the Galambos–Bojanić-Seneta type which gives a connection between translationally regularly (respectively, rapidly) varying functions and sequences. Such theorems play an important role in asymptotic analysis related to regular and rapid variation. We also introduced the index function for translationally regularly varying functions and sequences and proved that these index functions are fixed points for the operator K^T assigning to each translationally regularly varying function its index function. We hope that this study will be interesting for readers working in this field. We plan to investigate some other functions similar to the index function.

Our future work should include a representation result for translationally regularly varying functions in the spirit of the famous Karamata's representation theorem for regularly varying functions. Further, we want to investigate the properties of the Landau–Hurwicz sequence $\mathbf{w}(\mathbf{c})$ of a given sequence $\mathbf{c}$ belonging to some classes of sequences relevant to the theory of regular (and rapid) variation. For a sequence $\mathbf{c}=(c_n)$ of positive real numbers, the sequence $\mathbf{w}(\mathbf{c})=(w_n(\mathbf{c}))$ defined by

$$w_n(\mathbf{c})=\sup\{|c_m-c_k|:m\geq n,k\geq n\},\quad n\in\mathbb{N}$$

is called the *Landau–Hurwicz sequence* of $\mathbf{c}$.

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