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The Generalized 3-Connectivity of Exchanged Folded Hypercubes

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Abstract: For $S \subseteq V(G), \kappa_G(S)$ denotes the maximum number k of edge disjoint trees $T_1, T_2, \ldots, T_k$ in G, such that $V(T_i) \cap V(T_j) = S$ for any $i, j \in \{1, 2, \ldots, k\}$ and $i \neq j$. For an integer $2 \leq r \leq |V(G)|$, the generalized r-connectivity of G is defined as $\kappa_r(G) = \min\{\kappa_G(S) | S \subseteq V(G) \text{ and } |S| = r\}$. In fact, $\kappa_2(G)$ is the traditional connectivity of G. Hence, the generalized r-connectivity is an extension of traditional connectivity. The exchanged folded hypercube $EFH(s, t)$, in which $s \geq 1$ and $t \geq 1$ are positive integers, is a variant of the hypercube. In this paper, we find that $\kappa_3(EFH(s, t)) = s + 1$ with $3 \leq s \leq t$.

Keywords: exchanged folded hypercubes; generalized 3-connectivity; fault tolerance; interconnection networks

MSC: 05C40; 68R10

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1. Introduction

An interconnection network is usually modeled as a simple graph $G = (V(G), E(G))$, in which $V(G)$ represents the set of processors and $E(G)$ represents the set of links. For $v \in V(G)$, $N(v)$ is the neighborhood of v in G. $d(v) = |N(v)|$ is the degree of v in G. The minimum degree of G is defined as $\delta(G) = \min\{d(v) | v \in V(G)\}$. For two graphs G_1 and G_2, $G_1 \cong G_2$ means that they are isomorphic. Let $S \subseteq V(G)$. The subgraph of G, whose vertex set is S and whose edge set is the set of those edges of G that have both ends in S, is called the subgraph of G induced by S and is denoted by $G[S]$. We say that $G[S]$ is an induced subgraph of G. $G - S$ means the induced subgraph $G[V(G) \backslash S]$, where $V(G) \backslash S$ represents the vertex set obtained from $V(G)$ by deleting the vertices in S. Let $V \subseteq V(G) \backslash \{v\}$. The (v, V) paths is a family of internally disjoint paths whose starting vertex is v and terminal vertices are distinct in V, which is called a fan from v to V. For other terminologies and notations, please refer to [1].

Connectivity is a basic and important metric in measuring the reliability and fault tolerance of networks. A cut set S of G is a vertex set of G, such that $G - S$ is disconnected or it is only one vertex. $\kappa(G) = \min\{|S| | S \text{ is a cut set of } G\}$, which is the connectivity of G. In [2], Whitney proposed an equivalent concept of connectivity. For each 2-subset $S = \{u, w\}$ of vertices of G, let $\kappa_G(S)$ be the maximum number of internally disjoint paths from u to w in G. Then, $\kappa(G) = \min\{\kappa_G(S) | S \subseteq V(G) \text{ and } |S| = 2\}$. As an extension of connectivity, Chartrand et al. [3]showed the concept of generalized k-connectivity in 1984. Let $S \subseteq V(G)$. A tree T in G is called an S-tree if $S \subseteq V(T)$. The trees $T_1, T_2, \ldots, T_r$ are called internally edge disjoint S-trees if $V(T_i) \cap V(T_j) = S$ and $E(T_i) \cap E(T_j) = \emptyset$ for any distinct integers i, j with $1 \leq i, j \leq r$. $\kappa_G(S)$ refers to the maximum number of internally edge disjoint S-trees. For an integer k with $2 \leq k \leq |V(G)|$, $\kappa_k(G) = \min\{\kappa_G(S) | S \subseteq V(G) \text{ and } |S| = k\}$ is defined as the generalized k-connectivity of G.

In a graph G, an S-tree is also called an S-Steiner tree. Steiner trees have significant applications in computer networks [4]. Internally edge disjoint S-Steiner trees have been applied to VLSI [5]. From the definition of generalized k-connectivity, we can see that the core of generalized k-connectivity is to seek the maximum number of internally edge disjoint S-Steiner trees. The generalized k-connectivity is an extension of traditional connectivity. It can more precisely measure the fault tolerance of networks. To decide whether there exist k internally edge disjoint S-Steiner trees is NP-complete for a graph [6]. The generalized 3-connectivities of augmented cubes, (n,k)-bubble-sort graphs, and generalized hypercubes have been obtained in [7–9], respectively. The generalized 4-connectivities of hypercubes, crossed cubes, exchanged hypercubes, and hierarchical cubic networks have been obtained in [10–13], respectively. On the whole, the generalized k-connectivity is known for a small number of graphs and almost all known results are about $k = 3$ or 4.

The n-dimensional hypercube is denoted by Q_n, whose vertices are the ordered n-tuples of 0's and 1's. Two vertices are adjacent if and only if they differ in exactly one dimension. As variants of hypercubes Q_n, folded hypercubes FQ_n and exchanged hypercubes $EH(s,t)$ were proposed in [14,15], respectively. Based on $EH(s,t)$ and FQ_n, Qi et al. proposed an interconnection network named exchanged folded hypercube $EFH(s,t)$ in [16]. In this work, we will prove $\kappa_3(EFH(s,t)) = s+1$ for $3 \le s \le t$.

2. Definitions and Lemmas

Exchanged hypercubes were defined by Lou et al. [15] as follows. Let $s \ge 1$ and $t \ge 1$ be positive integers. The exchanged hypercubes $EH(s,t)$ are defined as undirected graphs, whose vertex set V is

$$V = \{a_s \cdots a_1 b_t \cdots b_1 c | a_i, b_j, c \in \{0,1\} \text{ for } i \in [1,s], j \in [1,t]\}.$$

For $u, v \in V$, $u[0]$ means the c index of u. $u[i:j]$ is the indexes of u from dimension j to dimension i. $H(u[i:j], v[i:j])$ represents the number of different indexes at the same dimension between $u[i:j]$ and $v[i:j]$.

The edge set consists of three disjoint subsets E_H, E_R and E_L, where

$$E_H = \{(u,v) | u[s+t:1] = v[s+t:1], u[0] \ne v[0]\},$$

$$E_R = \{(u,v) | u[s+t:t+1] = v_2[s+t:t+1], H(u[t:1], v[t:1]) = 1, u[0] = v[0] = 1\},$$

$$E_L = \{(u,v) | u[t:1] = v[t:1], H(u[s+t:t+1], v[s+t:t+1]) = 1, u[0] = v[0] = 0\},$$

Figure 1 shows an example of $EH(1,2)$. Based on the concept of $EH(s,t)$, Qi et al. [16] put in a network called an exchanged folded hypercube $EFH(s,t)$. $EFH(s,t)$ and $EH(s,t)$ have the same vertex set. The edge set of $EFH(s,t)$ consists of E_H, E_R, E_L and E_{comp}, where

$$E_{\text{comp}} = \{(u,v) | H(u[s+t:1], v[s+t:1]) = s+t, u[0] \ne v[0]\}.$$

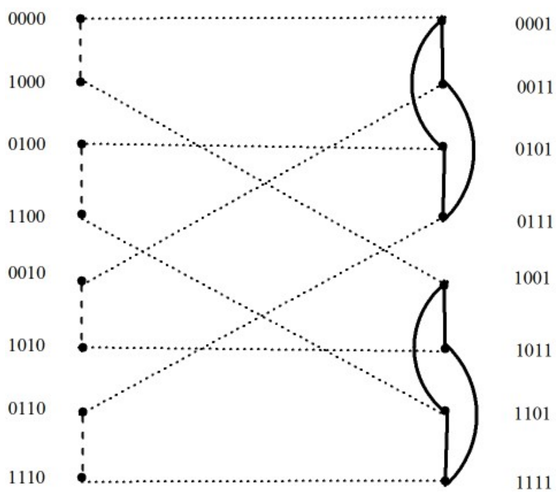

Figure 1. $EH(1,2)$.

The edges in E_{comp} are called complementary edges of $EFH(s,t)$. From the two definitions, we know that $EFH(s,t)$ can be obtained from $EH(s,t)$ by adding extra 2^{s+t} edges. Figure 2 is an example of $EFH(1,2)$. From the definition, we can see that $|V(EFH(s,t))|=2^{s+t+1}$. For each vertex $v \in V(EFH(s,t))$, $d(v) = s+2$ or $t+2$. For simplicity, we always use EFH instead of $EFH(s,t)$. The following results are useful.

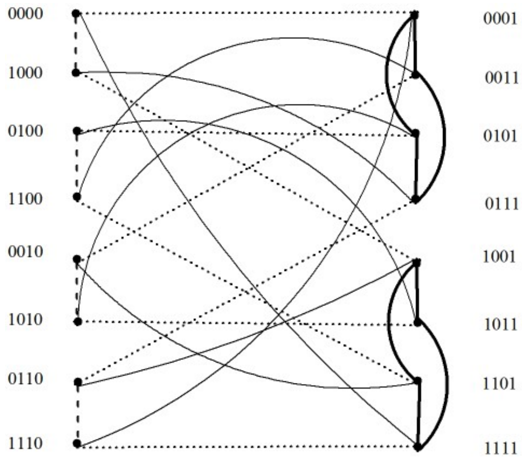

Figure 2. $EFH(1,2)$

Lemma 1. *([16]) $EFH(t,s) \cong EFH(s,t)$.*

From the lemma, we always assume $s \leq t$ from now on. Then, $\delta(EFH(s,t)) = s+2$.

Lemma 2. *([1]) $\kappa(Q_n) = n$ for $n \geq 2$.*

Lemma 3. *([17]) $\kappa_3(Q_n) = n-1$ for $n \geq 2$.*

Lemma 4. *([18]) If there are two adjacent vertices of degree $\delta(G)$ in graph G, then $\kappa_k(G) \leq \delta(G) - 1$ for $3 \leq k \leq |V(G)|$.*

Lemma 5. *([1]) Let G be a k-connected graph, and let u and v be a pair of distinct vertices in G. Then, there exist k internally disjoint paths in G connecting u and v.*

Lemma 6. *(Fan lemma [1]) For a k-connected graph G, let $u \in V(G)$, and suppose $U \subseteq V(G) \backslash \{u\}$ and $|U| \geq k$. Then, there exists a k-fan in G from u to U, that is, there exists a family of k internally disjoint (u,U) paths whose terminal vertices are distinct in U.*

In this work, we will prove the following result.

Theorem 1. *$\kappa_3(EFH(s,t)) = s+1$ for $3 \leq s \leq t$.*

3. Proof of Theorem 1

We partition $EFH(s,t)$ into two subgraphs L, R and edges between them, in which for $u \in V(L)$ and $v \in V(R)$, $u[0] = 0$ and $v[0] = 1$.

In $V(L)$, each collection of 2^s vertices u, with $u[t:1]$ being identical, forms Q_s via the edges in E_L. We use L_i to denote these Q_s for $i = 1,2,\ldots,2^t$. Similarly, in $V(R)$, each collection of 2^t vertices v, with $v[s+t:t+1]$ being identical, forms Q_t via the edges in E_R. We use R_j to denote these Q_t for $j = 1,2,\ldots,2^s$.

Each vertex $x \in V(L)$ has two neighbors in $V(R)$. One is x' with $xx' \in E_H$. It is called the hypercube neighbor of x. The other is $\bar{x}$ with $x\bar{x} \in E_{\text{comp}}$. It is called the complement neighbor of x. x' and $\bar{x}$ are called outside neighbors of x. Similarly, for $y \in V(R)$, y' and $\bar{y}$, the outside neighbors of y, are called the hypercube neighbor and the complement neighbor of y, respectively.

In the following, for each vertex x in a graph, we use x' and $\bar{x}$ to denote the hypercube neighbor and the complement neighbor of x, respectively.

Lemma 7. *For Q_n and $EFH(s,t)$, the following results hold.*

1. Each $L_i \cong Q_s$, $R_j \cong Q_t$ and $|V(L_i)| = 2^s$, $|V(R_j)| = 2^t$ for $i = 1,2,\ldots,2^t, j = 1,2,\ldots,2^s$.

2. There are no edges between any two distinct L_i and L_k for $i,k \in \{1,2,\ldots,2^t\}$. Similarly, there are no edges between any two distinct R_j and R_h for $j,h \in \{1,2,\ldots,2^s\}$.

3. For each vertex $x \in V(L)$, x' and $\bar{x}$ belong to distinct $V(R_j)$ and $V(R_h)$, where $j,h \in \{1,2,\ldots,2^s\}$. Similarly, for each vertex $w \in V(R)$, w' and $\bar{w}$ belong to distinct $V(L_i)$ and $V(L_k)$, where $i,k \in \{1,2,\ldots,2^t\}$.

4. For two distinct vertices $x,y \in V(L_i)$ with $i \in \{1,2,\ldots,2^t\}$, x' and y' lie in distinct $V(R_j)$ and $V(R_h)$, where $j,h \in \{1,2,\ldots,2^s\}$, $\bar{x}$ and $\bar{y}$ lie in distinct $V(R_i)$ and $V(R_k)$, where $i,k \in \{1,2,\ldots,2^s\}$. Similar results hold for two distinct vertices $u,v \in V(R_k)$ for $k \in \{1,2,\ldots,2^s\}$.

5. For two distinct vertices $x,y \in V(L_i)$ with $i \in \{1,2,\ldots,2^t\}$, if $x',\bar{y} \in V(R_j)$ for some $j \in \{1,2,\ldots,2^s\}$, then $\bar{x},y' \in V(R_k)$ for some $k \in \{1,2,\ldots,2^s\}$ with $k \neq j$. A similar result holds for two distinct vertices $u,v \in V(R_k)$ for $k \in \{1,2,\ldots,2^s\}$.

Proof. The first and second results are obvious. For two distinct vertices $x,y \in V(L_i)$ with $i \in \{1,2,\ldots,2^t\}$, there exists at least one index m for which x and y differ. Let $x = a_s \cdots a_m \cdots a_1 b_t \cdots b_1 0$, $y = a'_s \cdots \bar{a}_m \cdots a'_1 b_t \cdots b_1 0$ in same $V(L_i)$ with some $m \in \{1,2,\ldots,s\}$. Then, $x' = a_s \cdots a_m \cdots a_1 b_t \cdots b_1 1$, $\bar{x} = \bar{a}_s \cdots \bar{a}_m \ldots \bar{a}_1 \bar{b}_t \cdots \bar{b}_1 1$, $y' = a'_s \cdots \bar{a}_m \cdots a'_1 b_t \cdots b_1 1$. $\bar{y} = \bar{a'}_s \cdots a_m \cdots \bar{a'}_1 \bar{b}_t \cdots \bar{b}_1 1$, where $\bar{a}_i = 1 - a_i, \bar{a'}_i = 1 - a'_i, \bar{b}_j = 1 - b_j$ (Figure 3).

x' and $\bar{x}$ belong to distinct $V(R_j)$ and $V(R_h)$ where $j,h \in \{1,2,\ldots,2^s\}$ since $a_i \neq \bar{a}_i$ for $i = 1,2,\ldots,s$. Similarly, we can prove that, for any vertex $w \in V(R)$, w' and $\bar{w}$ belong to distinct $V(L_i)$ and $V(L_k)$, where $i,k \in \{1,2,\ldots,2^t\}$. Hence, the third result holds.

Since $a_m \neq \bar{a}_m$ for some $m \in \{1,2,\ldots,s\}$, x' and y' lie in different $V(R_j)$ and $V(R_h)$, where $j,h \in \{1,2,\ldots,2^s\}$, $\bar{x}$ and $\bar{y}$ lie in different $V(R_i)$ and $V(R_k)$, where $i,k \in \{1,2,\ldots,2^s\}$. We can prove that similar results for any distinct vertices $u,v \in V(R_k)$ for $k \in \{1,2,\ldots,2^s\}$. Hence, the fourth result holds.

If $x',\bar{y} \in V(R_j)$ for some $j \in \{1,2,\ldots,2^s\}$, then $a_j = \bar{a'}_j$ for $j = 1,\ldots,m-1$, $m+1,\ldots,s$. Hence, $\bar{a}_j = a'_j$ for $j = 1,\ldots,m-1,m+1,\ldots,s$. This implies that $\bar{x},y' \in V(R_k)$ for some $k \in \{1,2,\ldots,2^s\}$ with $k \neq j$. We can prove that a similar result for any distinct vertices $u,v \in V(R_k)$ for $k \in \{1,2,\ldots,2^s\}$. Hence, the fifth result holds. □

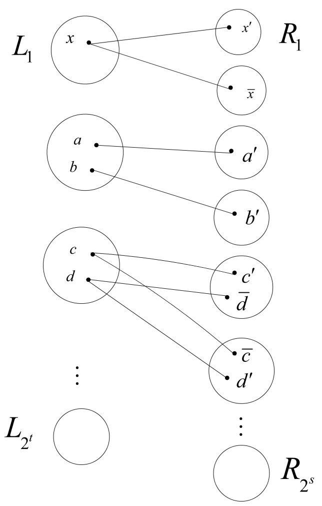

Figure 3. A partitioned sketch of $EFH(s,t)$.

Proof of Theorem 1. By Lemma 7, for any vertex $u \in V(L_1)$, $d(u) = s+2$. Since $\delta(EFH(s,t)) = s+2$, $\kappa_3(EFH(s,t)) \leq s+1$ by Lemma 4. In the following, we will prove $\kappa_3(EFH(s,t)) \geq s+1$. Take any three distinct vertices x, y, and z in EFH and let $S = \{x, y, z\}$. If we can prove that there are $s+1$ internally edge disjoint S-trees in EFH, we are done.

Case 1. $x, y, z \in V(L_i)$ for some $i \in \{1, 2, \ldots, 2^t\}$.

Without loss of generality, let $x, y, z \in V(L_1)$. By Lemma 3, there exist $s-1$ internally edge disjoint S-trees $T_1, T_2, \ldots, T_{s-1}$ in L_1. Without loss of generality, suppose $x' \in V(R_1), y' \in V(R_2)$, and $z' \in V(R_3)$ by Lemma 7(4).

If $\{\bar{x}, \bar{y}, \bar{z}\} \cap (V(R_1) \cup V(R_2) \cup V(R_3)) = \emptyset$, we can assume $\bar{x} \in V(R_4), \bar{y} \in V(R_5), \bar{z} \in V(R_6)$. By Lemma 7(4), $EFH[V(R_1) \cup V(R_2) \cup V(R_3) \cup V(L_2)]$ is connected. Hence, there exists a tree $\overline{T}_s$ containing x', y', and z' in $EFH[V(R_1) \cup V(R_2) \cup V(R_3) \cup V(L_2)]$. Take $T_s = \overline{T}_s \cup xx' \cup yy' \cup zz'$. Since $EFH[V(R_4) \cup V(R_5) \cup V(R_6) \cup V(L_3)]$ is connected, there exists a tree $\overline{T}_{s+1}$ containing $\bar{x}, \bar{y}$, and $\bar{z}$ in $EFH[V(R_4) \cup V(R_5) \cup V(R_6) \cup V(L_3)]$. Take $T_{s+1} = \overline{T}_{s+1} \cup x\bar{x} \cup y\bar{y} \cup z\bar{z}$. Then, $T_1, T_2, \ldots, T_{s+1}$ are $s+1$ internally edge disjoint S-trees. Thus, $\kappa_3(EFH) \geq s+1$.

If $\{\bar{x}, \bar{y}, \bar{z}\} \cap (V(R_1) \cup V(R_2) \cup V(R_3)) \neq \emptyset$, without loss of generality, noting that $\bar{x} \notin V(R_1)$ by Lemma 7(3), let $\bar{x} \in V(R_2)$. By Lemma 7(5), $\bar{y} \in V(R_1)$. By Lemma 7(3)(4), we can let $\bar{z} \in V(R_4)$. Since $EFH[V(R_1) \cup V(R_3) \cup V(L_2)]$ is connected, there exists a tree $\overline{T}_s$ containing $x', \bar{y}$, and z' in $EFH[V(R_1) \cup V(R_3) \cup V(L_2)]$. Take $T_s = \overline{T}_s \cup xx' \cup y\bar{y} \cup zz'$. Since $EFH[V(R_2) \cup V(R_4) \cup V(L_3)]$ is connected, there exists a tree $\overline{T}_{s+1}$ containing $\bar{x}, y'$, and $\bar{z}$ in $EFH[V(R_2) \cup V(R_4) \cup V(L_3)]$. Take $T_{s+1} = \overline{T}_{s+1} \cup x\bar{x} \cup yy' \cup z\bar{z}$. Then, $T_1, T_2, \ldots, T_{s+1}$ are $s+1$ internally edge disjoint S-trees. Thus, $\kappa_3(EFH) \geq s+1$.

By symmetry and $t \geq s$, if $x, y, z \in V(R_j)$ for some $j \in \{1, 2, \ldots, 2^s\}$, we can also obtain $\kappa_3(EFH) \geq s+1$.

Case 2. $x, y \in V(L_i)$ for some $i \in \{1, 2, \ldots, 2^t\}$. $z \in V(L_j)$ for some $j \in \{1, 2, \ldots, 2^t\}$ and $i \neq j$ or $z \in V(R_k)$ for some $k \in \{1, 2, \ldots, 2^s\}$.

Without loss of generality, we let $x, y \in V(L_1)$. By Lemmas 2 and 5, there exist s internally disjoint paths $P_1, P_2, \ldots, P_s$ from x to y in L_1. Let $x_i \in V(P_i)$, such that $xx_i \in E(P_i)$ for $i = 1, 2, \ldots, s$. In the following, we will show that for any two distinct vertices x_i and x_j with $i, j \in \{1, 2, \ldots, s\}$, $x', x_i', x_j', \bar{x}, \bar{x}_i, \bar{x}_j$ lie in distinct $V(R_k)$ for $k \in \{1, 2, \ldots, 2^s\}$. Without loss of generality, let $x = a_s \cdots a_2 a_1 b_t \cdots b_1 0$, $x_i = a_s \cdots a_2 \bar{a}_1 b_t \cdots b_1 0$, and $x_j = a_s \cdots \bar{a}_2 a_1 b_t \cdots b_1 0$. Then, $x' = a_s \cdots a_2 a_1 b_t \cdots b_1 1$, $\bar{x} = \bar{a}_s \cdots \bar{a}_2 \bar{a}_1 \bar{b}_t \cdots \bar{b}_1 1$, $x_i' = a_s \cdots a_2 \bar{a}_1 b_t \cdots b_1 1$, $\bar{x}_i = \bar{a}_s \cdots \bar{a}_2 a_1 \bar{b}_t \cdots \bar{b}_1 1$, $x_j' = a_s \cdots \bar{a}_2 a_1 b_t \cdots b_1 1$, $\bar{x}_j = \bar{a}_s \cdots a_2 \bar{a}_1 \bar{b}_t \cdots \bar{b}_1 1$. By $s \geq 3$ and the definition of R_k, we can show that $x', x_i', x_j', \bar{x}, \bar{x}_i, \bar{x}_j$ lie in different $V(R_k)$ for $k \in \{1, 2, \ldots, 2^s\}$, where $i, j \in \{1, 2, \ldots, s\}$ and $i \neq j$. This implies that $x', x_1', x_2', \ldots, x_s', \bar{x}, \bar{x}_1, \bar{x}_2, \ldots, \bar{x}_s$ lie in distinct $V(R_k)$ for $k \in \{1, 2, \ldots, 2^s\}$.

Subcase 2.1. $z \in V(R_k)$ for some $k \in \{1, 2, \ldots, 2^s\}$.

Let $z \in V(R_1)$. We know that $\{x', x_1', x_2', \ldots, x_s'\} \cap V(R_1) = \emptyset$ or $\{\bar{x}, \bar{x}_1, \bar{x}_2, \ldots, \bar{x}_s\} \cap V(R_1) = \emptyset$. Without loss of generality, let $\{\bar{x}, \bar{x}_1, \bar{x}_2, \ldots, \bar{x}_s\} \cap V(R_1) = \emptyset$. Suppose $\bar{x} \in V(R_4)$ and $\bar{x}_i \in V(R_{i+4})$ for $i = 1, 2, \ldots, s$.

Subcase 2.1.1. $y = x_i$ for some $i \in \{1, 2, \ldots, s\}$.

Without loss of generality, let $y = x_s$. Then, $y' \notin V(R_{i+4})$ for $i = 0, 1, 2, \ldots, s$ by the above discussion. We can let $y' \in V(R_1)$ or $y' \in V(R_2)$.

First, we consider $y' \in V(R_2)$ (Figure 4). By Lemma 7(3), $z' \notin V(L_1)$ or $\bar{z} \notin V(L_1)$. Without loss of generality, let $\bar{z} \notin V(L_1)$. Suppose $\bar{z} \in V(L_2)$. Take s vertices $z_1, z_2, \ldots, z_s$

in $V(R_1)$, such that $\bar{z}_i \in V(L_{i+4})$ for $i = 1, 2, \ldots, s$. Let $Z = \{z_1, z_2, \ldots, z_s\}$. By Lemma 6, there exist s internally disjoint paths $M_1, M_2, \ldots, M_s$ from z to Z in R_1. Let M_i be the path from z to z_i for $i = 1, 2, \ldots, s$. Since $EFH[V(L_{i+4}) \cup V(R_{i+4})]$ is connected, there exists a tree $\overline{T}_i$ containing $\bar{x}_i$ and $\bar{z}_i$ in $EFH[V(L_{i+4}) \cup V(R_{i+4})]$ for $i = 1, 2, \ldots, s$. Take $T_i = \overline{T}_i \cup P_i \cup M_i \cup x_i\bar{x}_i \cup z_i\bar{z}_i$ for $i = 1, 2, \ldots, s$. Since $EFH[V(L_2) \cup V(R_2) \cup V(R_4)]$ is connected, there exists a tree $\overline{T}_{s+1}$ containing $\bar{z}, y'$, and $\bar{x}$ in $EFH[V(L_2) \cup V(R_2) \cup V(R_4)]$. Take $T_{s+1} = \overline{T}_{s+1} \cup x\bar{x} \cup z\bar{z} \cup yy'$. Then, $T_1, T_2, \ldots, T_{s+1}$ are $s+1$ internally edge disjoint S-trees. Thus, $\kappa_3(EFH) \geq s+1$.

Now, we consider $y' \in V(R_1)$.

If $y' = z$, then $\bar{z} \notin V(L_1)$. Let $\bar{z} \in V(L_2)$. Taking $T_1, T_2, \ldots, T_s$ to be the same as above, since $EFH[V(L_2) \cup V(R_4)]$ is connected, there exists a tree $\overline{T}_{s+1}$ containing $\bar{z}$ and $\bar{x}$ in $EFH[V(L_2) \cup V(R_4)]$. Take $T_{s+1} = \overline{T}_{s+1} \cup x\bar{x} \cup yz\bar{z}$. Then, $T_1, T_2, \ldots, T_{s+1}$ are $s+1$ internally edge disjoint S-trees. Thus, $\kappa_3(EFH) \geq s+1$.

Let $y' \neq z$ (Figure 5). By Lemma 7(4), $z' \notin V(L_1)$. Suppose $z' \in V(L_2)$. Take $s-1$ vertices $z_1, z_2, \ldots, z_{s-1}$ in $V(R_1)$, such that $\bar{z}_i \in V(L_{i+4})$ for $i = 1, 2, \ldots, s-1$. Let $Z = \{z_1, z_2, \ldots, z_{s-1}, y'\}$. By Lemma 6, there exist s internally disjoint paths $M_1, M_2, \ldots, M_s$ from z to Z in R_1. Let M_i be the path from z to z_i for $i = 1, 2, \ldots, s-1$ and M_s be the path from z to y'. Since $EFH[V(L_{i+4}) \cup V(R_{i+4})]$ is connected, there exists a tree $\overline{T}_i$ containing $\bar{x}_i$ and $\bar{z}_i$ in $EFH[V(L_{i+4}) \cup V(R_{i+4})]$ for $i = 1, 2, \ldots, s-1$. Take $T_i = \overline{T}_i \cup P_i \cup M_i \cup x_i\bar{x}_i \cup z_i\bar{z}_i$ for $i = 1, 2, \ldots, s-1$. Noting that $y = x_s$, then $\bar{y} \in V(R_{s+4})$. Since $EFH[V(L_2) \cup V(R_{s+4}) \cup V(R_4)]$ is connected, there exists a tree $\overline{T}_s$ containing $z', \bar{y}$ and $\bar{x}$ in $EFH[V(L_2) \cup V(R_{s+4}) \cup V(R_4)]$. Take $T_s = \overline{T}_s \cup zz' \cup y\bar{y} \cup x\bar{x}$ and $T_{s+1} = P_s \cup yy' \cup M_s$. Then, $T_1, T_2, \ldots, T_{s+1}$ are $s+1$ internally edge disjoint S-trees. Thus, $\kappa_3(EFH) \geq s+1$.

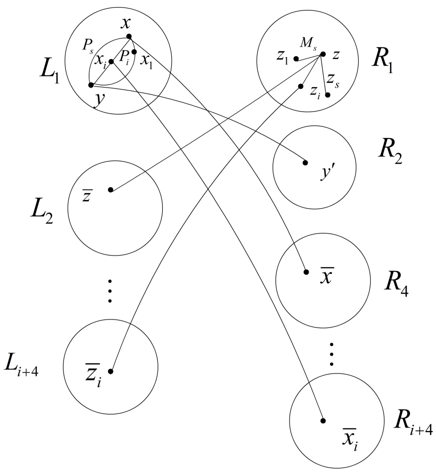

Figure 4. The illustration of Subcase 2.1.1 (I).

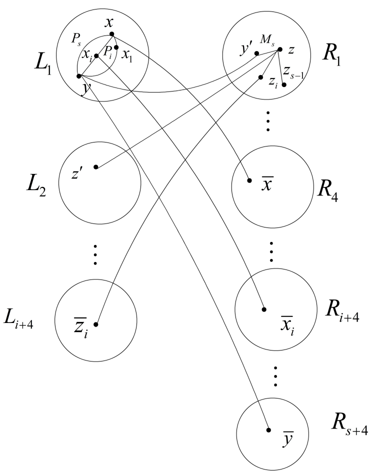

Figure 5. The illustration of Subcase 2.1.1 (II).

Subcase 2.1.2. $y \neq x_i$ for each $i = 1, 2, \ldots, s$.

By Lemma 7(4), we can show $\bar{y} \notin V(R_{i+4})$ for $i = 0, 1, \ldots, s$. Without loss of generality, let $\bar{y} \in V(R_1) \cup V(R_2)$.

First, we let $\bar{y} \in V(R_2)$. By Lemma 7(3), $z' \notin V(L_1)$ or $\bar{z} \notin V(L_1)$. Without loss of generality, let $\bar{z} \notin V(L_1)$. Suppose $\bar{z} \in V(L_2)$. Take s vertices $z_1, z_2, \ldots, z_s$ in $V(R_1)$, such that $\bar{z}_i \in V(L_{i+4})$ for $i = 1, 2, \ldots, s$. Let $Z = \{z_1, z_2, \ldots, z_s\}$. By Lemma 6, there exist s internally disjoint paths $M_1, M_2, \ldots, M_s$ from z to Z in R_1. Let M_i be the path from z to z_i for $i = 1, 2, \ldots, s$. Since $EFH[V(L_{i+4}) \cup V(R_{i+4})]$ is connected, there exists a tree $\overline{T}_i$ containing $\bar{x}_i$ and $\bar{z}_i$ in $EFH[V(L_{i+4}) \cup V(R_{i+4})]$ for $i = 1, 2, \ldots, s$. Take $T_i = \overline{T}_i \cup P_i \cup M_i \cup x_i\bar{x}_i \cup z_i\bar{z}_i$ for $i = 1, 2, \ldots, s$. Since $EFH[V(L_2) \cup V(R_2) \cup V(R_4)]$ is connected, there exists a tree $\overline{T}_{s+1}$ containing $\bar{z}, \bar{y}$ and $\bar{x}$ in $EFH[V(L_2) \cup V(R_2) \cup V(R_4)]$. Take $T_{s+1} = \overline{T}_{s+1} \cup x\bar{x} \cup z\bar{z} \cup y\bar{y}$. Then, $T_1, T_2, \ldots, T_{s+1}$ are $s+1$ internally edge disjoint S-trees. Thus, $\kappa_3(EFH) \geq s+1$.

Now, we let $\bar{y} \in V(R_1)$.

If $\bar{y} = z$, then $z' \notin V(L_1)$. We can let $z' \in V(L_2)$. Taking $T_1, T_2, \ldots, T_s$ to be the same as above, since $EFH[V(L_2) \cup V(R_4)]$ is connected, there exists a tree $\overline{T}_{s+1}$ containing $\bar{x}$ and z' in $EFH[V(L_2) \cup V(R_4)]$. Take $T_{s+1} = \overline{T}_{s+1} \cup x\bar{x} \cup yzz'$. Then, $T_1, T_2, \ldots, T_{s+1}$ are $s+1$ internally edge disjoint S-trees. Thus, $\kappa_3(EFH) \geq s+1$.

If $\bar{y} \neq z$. By Lemma 7(3), suppose $\bar{y}' \in V(L_2)$, where $\bar{y}'$ is the hypercube neighbor of $\bar{y}$. By Lemma 7(4), $\bar{z} \notin V(L_1)$. Without loss of generality, let $\bar{z} \in V(L_2) \cup V(L_3)$. Take $s-1$ vertices $z_1, z_2, \ldots, z_{s-1}$ in $V(R_1)$, such that $\bar{z}_i \in V(L_{i+4})$ for $i = 1, 2, \ldots, s-1$. Let $Z = \{z_1, z_2, \ldots, z_{s-1}, \bar{y}\}$. By Lemma 6, there exist s internally disjoint paths $M_1, M_2, \ldots, M_s$ from z to Z in R_1. Let M_i be the path from z to z_i for $i = 1, 2, \ldots, s-1$ and M_s be the path from z to $\bar{y}$. Since $EFH[V(L_{i+4}) \cup V(R_{i+4})]$ is connected, there exists a tree $\overline{T}_i$ containing $\bar{x}_i$ and $\bar{z}_i$ in $EFH[V(L_{i+4}) \cup V(R_{i+4})]$ for $i = 1, 2, \ldots, s-1$. Take $T_i = \overline{T}_i \cup P_i \cup M_i \cup x_i\bar{x}_i \cup z_i\bar{z}_i$ for $i = 1, 2, \ldots, s-1$. If $\bar{z} \in V(L_3)$ (Figure 6), noting that $\bar{x}_s \in V(R_{s+4})$, since $EFH[V(L_3) \cup V(R_{s+4})]$ is connected, there exists a tree $\overline{T}_s$ containing $\bar{z}$ and $\bar{x}_s$ in $EFH[V(L_3) \cup V(R_{s+4})]$. Take $T_s = \overline{T}_s \cup P_s \cup x_s\bar{x}_s \cup z\bar{z}$. Since $EFH[V(L_2) \cup V(R_4)]$ is connected, there exists a tree $\overline{T}_{s+1}$ containing $\bar{x}$ and $\bar{y}'$ in $EFH[V(L_2) \cup V(R_4)]$. Take $T_{s+1} = \overline{T}_{s+1} \cup x\bar{x} \cup y\bar{y}\bar{y}' \cup M_s$. Then, $T_1, T_2, \ldots, T_{s+1}$ are $s+1$ internally edge disjoint S-trees. If $\bar{z} \in V(L_2)$ (Figure 7), since $\bar{y} \neq z$, then $\bar{y}' \neq \bar{z}$ by Lemma 7(3). Since $L_2 \cong Q_s$, we can partition L_2 into L_{21} and L_{22}, such that $L_{21} \cong Q_{s-1}, L_{22} \cong Q_{s-1}$ and $\bar{y}' \in V(L_{21}), \bar{z} \in V(L_{22})$. In L_{21}, there exists a spanning tree T_{21} containing $\bar{y}'$. Since $|V(T_{21})| = |V(L_{21})| = 2^{s-1} \geq s+1$ for $s \geq 3$, there exists a vertex $u \in V(L_{21})$, such that $u' \notin V(R_1) \cup V(R_{i+4})$ for $i = 1, \ldots, s$ by Lemma 7(4). Let $u' \in V(R_2) \cup V(R_4)$. Similarly, there exists a spanning tree T_{22} containing $\bar{z}$ in L_{22}. Since $|V(T_{22})| = |V(L_{22})| = 2^{s-1} \geq s+1$ for $s \geq 3$, there exists a vertex $v \in V(L_{22})$, such that $v' \notin V(R_1) \cup V(R_2) \cup V(R_{i+4})$ for $i = 0, 1, \ldots, s-1$ by Lemma 7(4). Let $v' \in V(R_3) \cup V(R_{s+4})$. Since $EFH[V(R_2) \cup V(R_4) \cup V(L_3)]$ is connected, there exists a tree $\overline{T}_s$ containing u' and $\bar{x}$. Take $T_s = \overline{T}_s \cup x\bar{x} \cup T_{21} \cup uu' \cup y\bar{y}\bar{y}' \cup M_s$. Since $EFH[V(R_3) \cup V(R_{s+4}) \cup V(L_4)]$ is connected, there exists a tree $\overline{T}_{s+1}$ containing v' and $\bar{x}_s$. Take $T_{s+1} = \overline{T}_{s+1} \cup vv' \cup T_{22} \cup z\bar{z} \cup P_s \cup x_s\bar{x}_s$. Then, $T_1, T_2, \ldots, T_{s+1}$ are $s+1$ internally edge disjoint S-trees. Thus, $\kappa_3(EFH) \geq s+1$.

By symmetry and $t \geq s$, if $x, y \in V(R_i), z \in V(L_j)$ for some $i \in \{1, 2, \ldots, 2^s\}$ and some $j \in \{1, 2, \ldots, 2^t\}$, we can also obtain $\kappa_3(EFH) \geq s+1$.

Subcase 2.2. $z \in V(L_j)$ for some $j \in \{2, \ldots, 2^t\}$.

Without loss of generality, we let $z \in V(L_2)$ (Figure 8), and suppose $\bar{x} \in V(R_3), \bar{x}_i \in V(R_{i+3}), x' \in V(R_{s+4}), x'_i \in V(R_{s+i+4})$ for $i = 1, 2, \ldots, s$. Then, $z' \notin V(R_{i+3})$ or $\bar{z} \notin V(R_{i+3})$ or $z' \notin V(R_{s+i+4})$ or $\bar{z} \notin V(R_{s+i+4})$ for $i = 0, 1, \ldots, s$. Without loss of generality, let $\bar{z} \notin V(R_{i+3})$ for $i = 0, 1, \ldots, s$. Suppose $\bar{z} \in V(R_2)$. If $y = x_i$ for some $i \in \{1, 2, \ldots, s\}$, then $\bar{y} = \bar{x}_i$ for some $i \in \{1, 2, \ldots, s\}$. Then, $y' \notin V(R_{i+3})$ for $i = 0, 1, \ldots, s$. If $y \neq x_i$ for each $i = 1, 2, \ldots, s$, then $\bar{y} \notin V(R_{i+3})$ for $i = 0, 1, \ldots, s$ by Lemma 7(4). Without loss of generality, let $\bar{y} \notin V(R_{i+3})$ for $i = 0, 1, \ldots, s$. Suppose $\bar{y} \in V(R_1) \cup V(R_2)$.

Choose s vertices $z_1, z_2, \ldots, z_s$ in $V(L_2)$, such that $\bar{z}_i \in V(R_{i+3})$ for $i = 1, 2, \ldots, s$. Denote $Z = \{z_1, z_2, \ldots, z_s\}$. By Lemma 6, there exist s internally disjoint paths $M_1, M_2, \ldots, M_s$ from z to Z in L_2. Let M_i be the path from z to z_i for $i = 1, 2, \ldots, s$. Since R_{i+3} is connected, there exists a tree $\overline{T}_i$ containing $\bar{x}_i$ and $\bar{z}_i$ in R_{i+3} for $i = 1, 2, \ldots, s$. Take $T_i = \overline{T}_i \cup P_i \cup M_i \cup x_i\bar{x}_i \cup z_i\bar{z}_i$ for $i = 1, 2, \ldots, s$. Since $EFH[V(R_1) \cup V(R_2) \cup V(R_3) \cup V(L_3)]$ is connected, there exists a tree $\overline{T}_{s+1}$ containing $\bar{x}, \bar{y}$ and $\bar{z}$ in $EFH[V(R_1) \cup V(R_2) \cup V(R_3) \cup V(L_3)]$. Take $T_{s+1} = \overline{T}_{s+1} \cup x\bar{x} \cup y\bar{y} \cup z\bar{z}$. Then, $T_1, T_2, \ldots, T_{s+1}$ are $s+1$ internally edge disjoint S-trees. Thus, $\kappa_3(EFH) \geq s+1$.

By symmetry and $t \geq s$, if $x, y \in V(R_i), z \in V(R_j)$ for some $i, j \in \{1, 2, \ldots, 2^s\}$ and $i \neq j$, we can also obtain $\kappa_3(EFH) \geq s+1$.

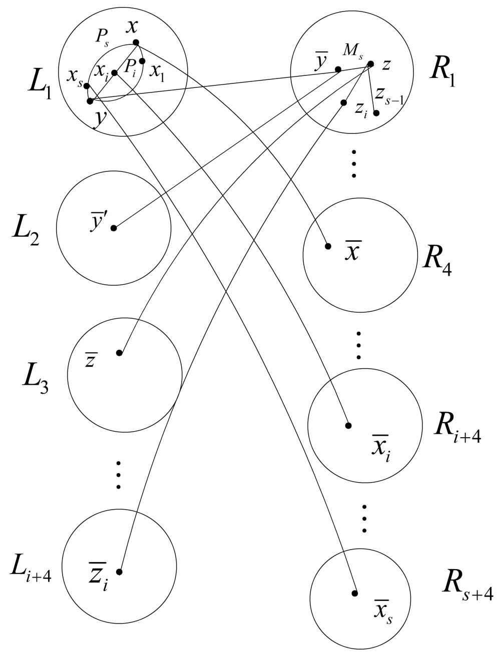

Figure 6. The illustration of Subcase 2.1.2 (I).

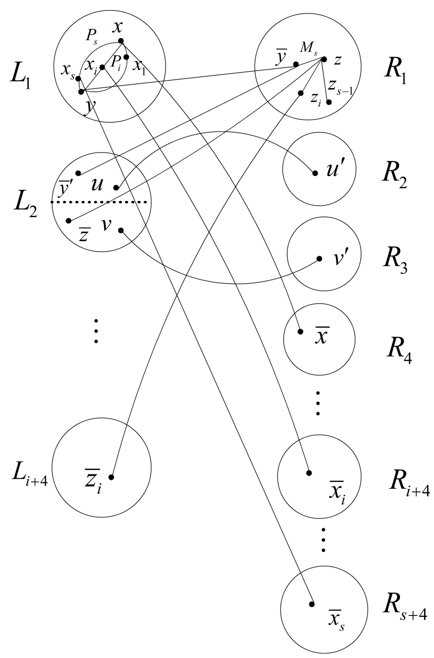

Figure 7. The illustration of Subcase 2.1.2 (II).

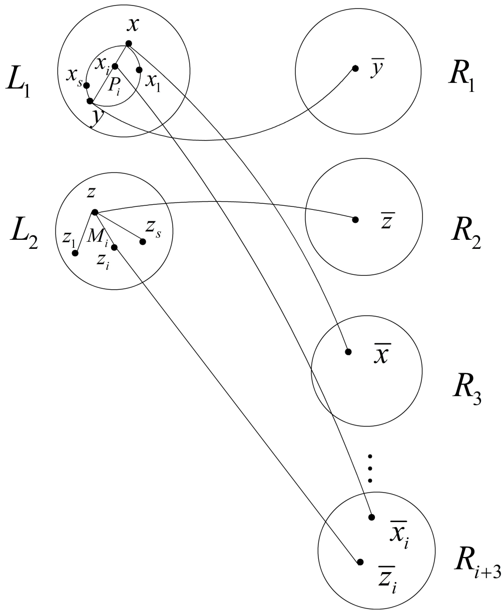

Figure 8. The illustration of Subcase 2.2.

Case 3. $x \in V(L_i), y \in V(L_j)$, and $z \in V(R_k)$ for some $i, j \in \{1, 2, \ldots, 2^t\}$ with $i \neq j$ and some $k \in \{1, 2, \ldots, 2^s\}$.

Without loss of generality, let $x \in V(L_1), y \in V(L_2), z \in V(R_1)$.

Subcase 3.1. $z', \bar{z} \in V(L_1) \cup V(L_2)$.

By Lemma 7(3), without loss of generality, let $\bar{z} \in V(L_1)$, $z' \in V(L_2)$.

We first consider $\bar{z} = x$ or $z' = y$. Without loss of generality, let $\bar{z} = x$. By Lemma 7(3), we can let $x' \in V(R_2)$ and $y' \notin V(R_1)$ or $\bar{y} \notin V(R_1)$. Suppose $\bar{y} \notin V(R_1)$. Then, put $\bar{y} \in V(R_2) \cup V(R_3)$. Choose $x_1, x_2, \ldots, x_s$ in $V(L_1) \setminus \{x\}$, such that $\bar{x}_i \in V(R_{i+3})$ for $i = 1, 2, \ldots, s$. Denote $X = \{x_1, x_2, \ldots, x_s\}$. Choose $y_1, y_2, \ldots, y_s$ in $V(L_2) \setminus \{y\}$, such that $\bar{y}_i \in V(R_{i+3})$ for $i = 1, 2, \ldots, s$. Denote $Y = \{y_1, y_2, \ldots, y_s\}$. Choose $z_1, z_2, \ldots, z_s$ in $V(R_1) \setminus \{z\}$, such that $\bar{z}_i \in V(L_{i+3})$ for $i = 1, 2, \ldots, s$. Denote $Z = \{z_1, z_2, \ldots, z_s\}$. By Lemma 6, there exist s paths $P_1, P_2, \ldots, P_s$ from x to X in L_1, s paths $N_1, N_2, \ldots, N_s$ from y to Y in L_2, s paths $M_1, M_2, \ldots, M_s$ from z to Z in R_1. Let P_i, N_i, M_i be the paths from x to x_i, from y to y_i, and from z to z_i, respectively, for $i = 1, 2, \ldots, s$. Since $EFH[V(L_{i+3}) \cup V(R_{i+3})]$ is connected, there exists a tree $\overline{T}_i$ containing $\bar{x}_i, \bar{y}_i$, and $\bar{z}_i$ in $EFH[V(L_{i+3}) \cup V(R_{i+3})]$ for $i = 1, 2, \ldots, s$. Take $T_i = \overline{T}_i \cup P_i \cup N_i \cup M_i \cup x_i\bar{x}_i \cup y_i\bar{y}_i \cup z_i\bar{z}_i$ for $i = 1, 2, \ldots, s$. Since $EFH[V(R_2) \cup V(R_3) \cup V(L_3)]$ is connected, there exists a tree $\overline{T}_{s+1}$ containing $x', \bar{y}$ in $EFH[V(R_2) \cup V(R_3) \cup V(L_3)]$. Take $T_{s+1} = \overline{T}_{s+1} \cup y\bar{y} \cup xx' \cup xz$. Then, $T_1, T_2, \ldots, T_{s+1}$ are $s+1$ internally edge disjoint S-trees. Thus, $\kappa_3(EFH) \geq s+1$.

Now, we consider $\bar{z} \neq x$ and $z' \neq y$ (Figure 9). Since $L_1 \cong Q_s$ and $L_2 \cong Q_s$, we can partition L_1 into L_{11} and L_{12}, such that $L_{11} \cong Q_{s-1}$, $L_{12} \cong Q_{s-1}$ and $\bar{z} \in V(L_{11})$, $x \in V(L_{12})$. Similarly, we partition L_2 into L_{21} and L_{22}, such that $L_{21} \cong Q_{s-1}$, $L_{22} \cong Q_{s-1}$ and $z' \in V(L_{21}), y \in V(L_{22})$. By Lemma 7(4), we can let $\bar{x} \in V(R_2)$ and $y' \in V(R_2) \cup V(R_3)$. Choose $x_1, x_2, \ldots, x_{s-1}$ in $V(L_{12}) \setminus \{x\}$ such that $\bar{x}_i \notin V(R_1) \cup V(R_2) \cup V(R_3)$ for $i = 1, 2, \ldots, s-1$. This can be performed since $2^{s-1} - 1 \geq 3$ with $s \geq 3$. Let $\bar{x}_i \in V(R_{i+3})$ for $i = 1, 2, \ldots, s-1$. Denote $X = \{x_1, x_2, \ldots, x_{s-1}\}$. Choose $y_1, y_2, \ldots, y_{s-1}$ in $V(L_{22}) \setminus \{y\}$, such that $y'_i \notin V(R_1) \cup V(R_2) \cup V(R_3)$ for $i = 1, 2, \ldots, s-1$. Without loss of generality, for simplicity of description, we can let $y'_1 \in V(R_4)$ and $y'_i \in V(R_{s+i+1})$ for $i = 2, \ldots, s-1$. Note that $\bar{x}_1 \in V(R_4)$ and $\bar{x}_i \in V(R_{i+3})$ for $i = 2, \ldots, s-1$. Denote $Y = \{y_1, y_2, \ldots, y_{s-1}\}$. Choose $z_1, z_2, \ldots, z_s \in V(R_1) \setminus \{z\}$ such that $\bar{z}_i \in V(L_{i+3})$ for $i = 1, 2, \ldots, s$. Denote $Z = \{z_1, z_2, \ldots, z_s\}$. By Lemma 6 and $\kappa(L_{12}) = \kappa(L_{22}) = s-1, \kappa(R_1) = s$, there exist

$s-1$ paths $P_1, P_2, \ldots, P_{s-1}$ from x to X in L_{12}, $s-1$ paths $N_1, N_2, \ldots, N_{s-1}$ from y to Y in L_{22}, s paths $M_1, M_2, \ldots, M_s$ from z to Z in R_1. Let P_i, N_i, M_i be the paths from x to x_i, from y to y_i, and from z to z_i, respectively, for $i = 1, 2, \ldots, s-1$ and M_s be the path from z to z_s. Since $EFH[V(R_4) \cup V(L_4)]$ is connected, there exists a tree $\overline{T}_1$ containing $\bar{x}_1, y_1'$ and $\bar{z}_1$ in $EFH[V(R_4) \cup V(L_4)]$. Take $T_1 = \overline{T}_1 \cup P_1 \cup N_1 \cup M_1 \cup x_1\bar{x}_1 \cup y_1y_1' \cup z_1\bar{z}_1$. Since $EFH[V(R_{i+3}) \cup V(R_{s+i+1}) \cup V(L_{i+3})]$ is connected for $i = 2, 3, \ldots, s-1$, there exists a tree $\overline{T}_i$ containing $\bar{x}_i, y_i'$ and $\bar{z}_i$ in $EFH[V(R_{i+3}) \cup V(R_{s+i+1}) \cup V(L_{i+3})]$ for $i = 2, 3, \ldots, s-1$. Take $T_i = \overline{T}_i \cup P_i \cup N_i \cup M_i \cup x_i\bar{x}_i \cup y_iy_i' \cup z_i\bar{z}_i$ for $i = 2, 3, \ldots, s-1$. Since $EFH[V(R_2) \cup V(R_3) \cup V(L_{s+3})]$ is connected, there exists a tree $\overline{T}_s$ containing $\bar{x}, y'$ and $\bar{z}_s$ in $EFH[V(R_2) \cup V(R_3) \cup V(L_{s+3})]$. Take $T_s = \overline{T}_s \cup M_s \cup x\bar{x} \cup yy' \cup z_s\bar{z}_s$. Let u be the neighbor of x in $V(L_{11})$ and v be the neighbor of y in $V(L_{21})$. Suppose that T_{11} is a spanning tree of L_{11} and T_{21} is a spanning tree of L_{21}. Take $T_s = T_{11} \cup T_{21} \cup ux \cup vy \cup z\bar{z} \cup zz'$. Then, $T_1, T_2, \ldots, T_{s+1}$ are $s+1$ internally edge disjoint S-trees. Thus, $\kappa_3(EFH) \geq s+1$.

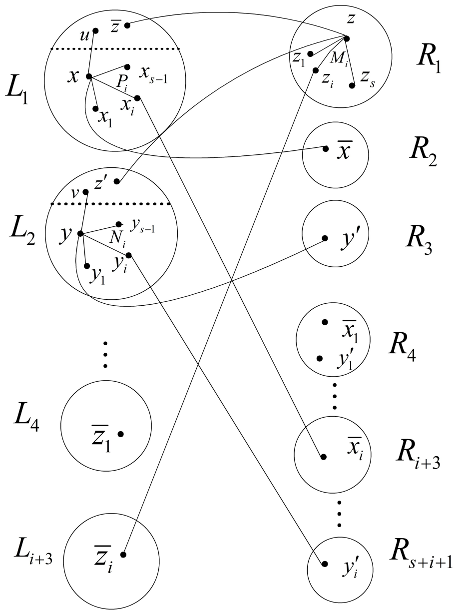

Figure 9. The illustration of Subcase 3.1.

Subcase 3.2. $z' \notin V(L_1) \cup V(L_2)$ or $\bar{z} \notin V(L_1) \cup V(L_2)$.

Without loss of generality, let $z' \notin V(L_1) \cup V(L_2)$. Suppose $z' \in V(L_3)$. By Lemma 7(3), $x' \notin V(R_1)$ or $\bar{x} \notin V(R_1)$, $y' \notin V(R_1)$ or $\bar{y} \notin V(R_1)$. Without loss of generality, we can let $\bar{x} \in V(R_2)$, $y' \in V(R_2) \cup V(R_3)$. Choose $x_1, x_2, \ldots, x_s \in V(L_1) \setminus \{x\}$, such that $\bar{x}_i \notin V(R_1) \cup V(R_2) \cup V(R_3)$ for $i = 1, 2, \ldots, s$. Suppose $\bar{x}_i \in V(R_{i+3})$ for $i = 1, 2, \ldots, s$. Denote $X = \{x_1, x_2, \ldots, x_s\}$. Choose $y_1, y_2, \ldots, y_s \in V(L_2)$, such that $\bar{y}_i \in V(R_{i+3})$ for $i = 1, 2, \ldots, s$. Denote $Y = \{y_1, y_2, \ldots, y_s\}$. Choose $z_1, z_2, \ldots, z_s \in V(R_1) \setminus \{z\}$, such that $z_i' \in V(L_{i+3})$ for $i = 1, 2, \ldots, s$. Denote $Z = \{z_1, z_2, \ldots, z_s\}$. By Lemma 6, there exist s paths $P_1, P_2, \ldots, P_s$ from x to X in L_1, s paths $N_1, N_2, \ldots, N_s$ from y to Y in L_2, s paths $M_1, M_2, \ldots, M_s$ from z to Z in R_1. Let P_i, N_i, M_i be the paths from x to x_i, from y to y_i, and from z to z_i, respectively, for $i = 1, 2, \ldots, s$. Note that if $y = y_i$ for some $i \in \{1, 2, \ldots, s\}$, we regard N_i as the vertex y. Since $EFH[V(L_{i+3}) \cup V(R_{i+3})]$ is connected, there exists a tree $\overline{T}_i$ containing $\bar{x}_i, \bar{y}_i$ and z_i' in $EFH[V(L_{i+3}) \cup V(R_{i+3})]$ for $i = 1, 2, \ldots, s$. Take $T_i = \overline{T}_i \cup P_i \cup N_i \cup M_i \cup x_i\bar{x}_i \cup y_i\bar{y}_i \cup z_iz_i'$ for $i = 1, 2, \ldots, s$. Since $EFH[V(R_2) \cup V(R_3) \cup V(L_3)]$ is connected, there exists a tree $\overline{T}_{s+1}$ containing $z', \bar{x}, y'$ in $EFH[V(R_2) \cup V(R_3) \cup V(L_3)]$. Take $T_{s+1} = \overline{T}_{s+1} \cup x\bar{x} \cup yy' \cup zz'$. Then, $T_1, T_2, \ldots, T_{s+1}$ are $s+1$ internally edge disjoint S-trees. Thus, $\kappa_3(EFH) \geq s+1$.

By symmetry and $t \geq s$, if $x \in V(R_i), y \in V(R_j), z \in V(L_k)$ for some $i, j \in \{1, 2, \ldots, 2^s\}$ with $i \neq j$ and some $k \in \{1, 2, \ldots, 2^t\}$, we can also obtain $\kappa_3(EFH) \geq s+1$.

Case 4. $x \in V(L_i), y \in V(L_j)$, and $z \in V(L_k)$ for some $i, j, k \in \{1, 2, \ldots, 2^t\}$ with $i \neq j \neq k$.

Let $x \in V(L_1), y \in V(L_2)$, and $z \in V(L_3)$ (Figure 10). Without loss of generality, suppose $\bar{x}, \bar{y}, \bar{z} \in V(R_1) \cup V(R_2) \cup V(R_3)$. Choose $x_i \in V(L_1) \setminus \{x\}, y_i \in V(L_2) \setminus \{y\}, z_i \in V(L_3) \setminus \{z\}$, such that $\bar{x}_i, \bar{y}_i, \bar{z}_i \in V(R_{i+3})$ for $i = 1, 2, \ldots, s$. Let $X = \{x_1, x_2, \ldots, x_s\}$, $Y = \{y_1, y_2, \ldots, y_s\}$ and $Z = \{z_1, z_2, \ldots, z_s\}$. By Lemma 6, there exist s paths $P_1, P_2, \ldots, P_s$ from x to X in L_1, s paths $N_1, N_2, \ldots, N_s$ from y to Y in L_2, s paths $M_1, M_2, \ldots, M_s$ from z to Z in L_3. Let P_i, N_i, M_i be the paths from x to x_i, from y to y_i, and from z to z_i, respectively, for $i = 1, 2, \ldots, s$. Since $EFH[V(R_{i+3})]$ is connected, there exists a tree $\overline{T}_i$ containing $\bar{x}_i, \bar{y}_i$ and $\bar{z}_i$ in $EFH[V(R_{i+3})]$ for $i = 1, 2, \ldots, s$. Take $T_i = \overline{T}_i \cup P_i \cup N_i \cup M_i \cup x_i\bar{x}_i \cup y_i\bar{y}_i \cup z_i\bar{z}_i$ for $i = 1, 2, \ldots, s$. Since $EFH[V(R_1) \cup V(R_2) \cup V(R_3) \cup V(L_4)]$ is connected, there exists a tree $\overline{T}_{s+1}$ containing $\bar{x}, \bar{y}$ and $\bar{z}$ in $EFH[V(R_1) \cup V(R_2) \cup V(R_3) \cup V(L_4)]$. Take $T_{s+1} = \overline{T}_{s+1} \cup x\bar{x} \cup y\bar{y} \cup z\bar{z}$. Then, $T_1, T_2, \ldots, T_{s+1}$ are $s+1$ internally edge disjoint S-trees. Thus, $\kappa_3(EFH) \geq s+1$.

By symmetry and $t \geq s$, if $x \in V(R_i), y \in V(R_j), z \in V(R_k)$ for some $i, j, k \in \{1, 2, \ldots, 2^s\}$ with $i \neq j \neq k$, we can also obtain $\kappa_3(EFH) \geq s+1$.

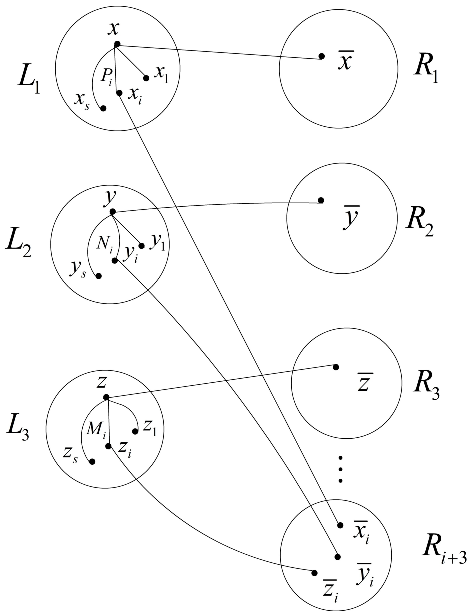

Figure 10. The illustration of Case 4.

We have completed the proof. $\square$

4. Conclusions

The exchanged folded hypercube is a variant of the hypercube and denoted by $EFH(s,t)$. It has many attractive properties to design interconnection networks. The generalized k-connectivity is an extension of the traditional connectivity. In this paper, we computed the generalized 3-connectivity of the exchanged folded hypercube. The study of the generalized k-connectivity of the exchanged folded hypercube for $k \geq 4$ is a meaningful and challenging problem.

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