

Article

A Class of Multi-Component Non-Isospectral TD Hierarchies and Their Bi-Hamiltonian Structures

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Abstract: By using the classical Lie algebra, the stationary zero curvature equation, and the Lenard recursion equations, we obtain the non-isospectral TD hierarchy. Two kinds of expanding higher-dimensional Lie algebras are presented by extending the classical Lie algebra. By solving the expanded non-isospectral zero curvature equations, the multi-component non-isospectral TD hierarchies are derived. The Hamiltonian structure for one of them is obtained by using the trace identity.

Keywords: multi-component non-isospectral TD hierarchies; trace identity; Bi-Hamiltonian structures; high-dimensional lie algebras

MSC: 35Q55; 37K30

1. Introduction

Nonlinear integrable evolution equations are important in many application fields, such as mathematics and theoretical physics. Deriving the nonlinear integrable evolution equations is a key subject. Magri [1] proposed that integrable systems can be obtained by applying Lax representation and zero curvature representation. Tu [2] found another effective method to generate integrable hierarchies of equations and the corresponding Hamiltonian structures. Ma [3,4] called this method the Tu scheme. By applying the Tu scheme, many important isospectral integrable systems were obtained [5–12]. It is obvious that non-isospectral equations play a significant role in mathematics and theoretical physics. Ma [13] offered a universally applicable method of constructing Lax representations for isospectral and non-isospectral hierarchies of evolution equations. In [14], he gave a systematic method to construct integrable systems from the zero curvature representation non-isospectral flows associated with a given isospectral hierarchy of integrable systems. By using the Lenard recursive scheme, Qiao [15] displayed the hierarchies of generalized nonlinear evolution equations associated with the Harry-Dym spectral problem and constructed the corresponding generalized Lax representations by solving a key operator equation. Based on the methods, a large number of non-isospectral integrable systems are considered [16–18].

As we all know, integrable couplings originated from an investigation on soliton equations and have received increasing attention. The authors in [19] constructed a suitable transformation of Lax pairs to obtain the integrable coupling of TD hierarchy, which is called the direct method (see [20] for reference). Ma once provided a method to construct integrable couplings associated with the non-semisimple Lie algebra, which has the triangular block matrix form

$$M_1(m_1, m_2) = \begin{pmatrix} m_1 & 0 \\ m_2 & m_1 \end{pmatrix},$$

where m_1 and m_2 are two arbitrary square matrices of the same order. By applying this non-semi-simple Lie algebra, many isospectral or non-isospectral integrable couplings



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have been deduced [21–25]. Wang [26] extended the above non-semi-simple Lie algebra to generalized non-semi-simple Lie algebra, which has the following form:

$$M_2(m_1, m_2) = \begin{pmatrix} m_1 & \varepsilon m_2 \\ m_2 & m_1 \end{pmatrix},$$

where ε is a constant. The generalized non-semi-simple Lie algebra, which depends on ε in a non-trivial way, is applied to construct the integrable couplings of isospectral and non-isospectral problems, the generalized integrable coupling of the MKdV isospectral problem and the Ablowitz-Kaup-Newell-Segur non-isospectral problem were constructed. Based on generalized non-semi-simple Lie algebra, a series of integrable coupling systems were obtained, such as two non-isospectral hierarchies of the Ablowitz-Kaup-Newell-Segur type and Kaup-Newell type, non-isospectral generalized nonlinear Schrödinger hierarchies [27]. Motivated by the generalized non-semi-simple Lie algebra, a new non-semi-simple Lie algebra that can extend the classical spectral problems to the multi-component spectral problems was determined. Wang [28] extended the integrable coupling MKdV isospectral and Ablowitz-Kaup-Newell-Segur non-isospectral hierarchies to the N-dimensional isospectral MKdV and non-isospectral Ablowitz-Kaup-Newell-Segur hierarchies. N-dimensional non-isospectral KdV and non-isospectral generalized nonlinear Schrödinger hierarchies were generated, respectively, in [27,29].

The rest of this paper is organized as follows. In Section 2, with the help of the simple Lie algebra and the non-isospectral problem, we derive a non-isospectral TD hierarchy from zero curvature equations, and the Bi-Hamiltonian structure is presented by making use of the trace identity. In Section 3, using generalized non-semi-simple Lie algebra, by considering the extended non-isospectral problem and solving the extended zero curvature equation, the integrable coupling of a non-isospectral TD hierarchy is obtained. By applying Z_2^ε trace identity, we obtained the corresponding Bi-Hamiltonian structures. In Section 4, we present a multi-component non-semi-simple Lie algebra that can be used to generate higher-dimensional isospectral and non-isospectral integrable hierarchies. We consider the special case of $\varepsilon = 0$ and derive the multi-component non-isospectral TD integrable coupling hierarchy by solving the enlarged zero curvature equations, for which the Bi-Hamiltonian structures are obtained. Section 5 provides the conclusions and discussions.

2. A Nonisospectral TD Integrable Hierarchy

Denote a simple Lie algebra, A , as follows:

$$A = \text{span}\{h_1, h_2, h_3, h_4\}.$$

The basis of Lie algebra can be constituted by the following matrices:

$$h_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad h_2 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \quad h_3 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad h_4 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad (1)$$

and this satisfies the commutator relations

$$[h_1, h_2] = 0, \quad [h_1, h_3] = -[h_2, h_3] = h_3, \quad [h_1, h_4] = -[h_2, h_4] = -h_4, \quad [h_3, h_4] = h_1 - h_2,$$

where $[a, b] = ab - ba, a, b \in A$.

We consider the 2×2 matrix spectral problem:

$$\begin{cases} \psi_x = M\psi, & M = \begin{pmatrix} -\lambda + r & q \\ q & \lambda \end{pmatrix} = -\lambda h_1 + \lambda h_2 + r h_1 + q(h_3 + h_4), \\ \psi_t = N\psi, & N = \begin{pmatrix} a & b \\ c & -a \end{pmatrix} = -a h_1 + a h_2 + b h_3 + c h_4, \\ \lambda_t = \sum_{m \geq 0} k_m(t) \lambda^{-m}, \end{cases} \quad (2)$$

where $\psi = (\psi_1, \psi_2)^T, u = (q, r)^T$.

To derive a nonlinear integrable hierarchy, the following stationary zero curvature equation needs to be solved

$$N_x = \frac{\partial M}{\partial \lambda} \lambda_t + [M, N]. \quad (3)$$

By substituting M and N in (2) into (3), we obtain

$$\begin{cases} a_x = q(c - b) - \lambda_t, \\ b_x = -2b\lambda + rb - 2qa, \\ c_x = 2c\lambda - rc + 2qa. \end{cases} \quad (4)$$

By assuming $a = \sum_{m \geq 0} a_m \lambda^{-m}, b = \sum_{m \geq 0} b_m \lambda^{-m}, c = \sum_{m \geq 0} c_m \lambda^{-m}$, inserting them into (4), and comparing the same powers of λ , we have

$$\begin{cases} a_{mx} = q(c_m - b_m) - k_m(t), \\ b_{mx} = -2b_{m+1} + rb_m - 2qa_m, \\ c_{mx} = 2c_{m+1} - rc_m + 2qa_m, \end{cases} \quad (5)$$

which leads to the following recurrence relations:

$$\begin{cases} a_m = \partial_x^{-1}(qc_m - qb_m) - k_m(t)x, \\ b_{m+1} = \frac{1}{2}(rb_m - b_{mx} - 2qa_m), \\ c_{m+1} = \frac{1}{2}(rc_m + c_{mx} - 2qa_m). \end{cases} \quad (6)$$

By taking the initial conditions

$$a_0 = -1, \quad b_0 = 0, \quad c_0 = 0, \quad k_0 = 0,$$

and imposing the integrable conditions

$$a_i|_{q=r=0} = 0, \quad b_i|_{q=r=0} = 0, \quad c_i|_{q=r=0} = 0, \quad i \geq 0,$$

$\{a_i, b_i, c_i | i > 0\}$ can be gained from the relations in (6). First, we list three sets as follows:

$$\begin{aligned} b_1 &= q, \quad c_1 = q, \quad a_1 = -k_1(t)x, \\ b_2 &= qk_1(t)x + \frac{1}{2}(rq - q_x), \quad c_2 = qk_1(t)x + \frac{1}{2}(rq + q_x), \quad a_2 = \frac{1}{2}q^2 - k_2(t)x, \\ b_3 &= qk_2(t)x + \frac{1}{2}(rqx - xq_x - q)k_1(t) + \frac{1}{4}(r^2q - rq_x - (rq)_x + q_{xx}) - \frac{1}{2}q^3, \\ c_3 &= qk_2(t)x + \frac{1}{2}(rqx + xq_x + q)k_1(t) + \frac{1}{4}(r^2q + rq_x + (rq)_x + q_{xx}) - \frac{1}{2}q^3, \\ a_3 &= \partial^{-1}q(xq_x + q)k_1(t) + \frac{1}{2}rq^2 - k_3(t)x, \\ &\dots \end{aligned}$$

Denoting

$$N_+^{(n)} = \sum_{m=0}^n (a_m(-h_1 + h_2)\lambda^{n-m} + b_m h_3 \lambda^{n-m} + c_m h_4 \lambda^{n-m}), \quad \lambda_{t,+}^{(n)} = \sum_{m=0}^n k_m(t)\lambda^{n-m},$$

Then, (3) can be decomposed as

$$\frac{\partial M}{\partial \lambda} \lambda_{t,+}^{(n)} - N_{x,+}^{(n)} + [M, N_+^{(n)}] = -\frac{\partial M}{\partial \lambda} \lambda_{t,-}^{(n)} + N_{x,-}^{(n)} - [M, N_-^{(n)}]. \quad (7)$$

The gradations of the left-hand side of (7) are obtained as follows:

$$\deg \frac{\partial M}{\partial \lambda} \lambda_{t,+}^{(n)} \geq 0, \quad \deg N_+^{(n)} \geq 0, \quad \deg ([M, N_+^{(n)}]) \geq 0.$$

One can find that the minimum gradation of the left-hand side of (7) is zero. The maximum gradation of the right-hand side of (7) has no choice but to be zero. Thus, we obtain the following equation by taking the gradations of these terms to be zero:

$$\frac{\partial M}{\partial \lambda} \lambda_{t,+}^{(n)} - N_{+,x}^{(n)} + [M, N_+^{(n)}] = 2b_{n+1}h_3 - 2c_{n+1}h_4.$$

Then, we take the modified term $\Delta_n = q^{-1}(b_{n+1} + c_{n+1})h_1$ so that for $N^{(n)} = N_+^{(n)} + \Delta_n$. By solving the non-isospectral zero curvature equation

$$\frac{\partial M}{\partial u} u_t + \frac{\partial M}{\partial \lambda} \lambda_t^{(n)} - N_x^{(n)} + [M, N^{(n)}] = 0,$$

we obtain the well-known non-isospectral TD hierarchy:

$$\begin{cases} q_{t_n} = c_{n+1} - b_{n+1} = q^{-1}(\partial a_{n+1} + k_{n+1}(t)), \\ r_{t_n} = \partial[q^{-1}(c_{n+1} + b_{n+1})]. \end{cases} \quad (8)$$

Here, we have used the first equation of (5).

The first nontrivial system in the hierarchy (8) is as follows:

$$\begin{cases} q_{t_2} = (xq_x + q)k_1(t) + \frac{1}{2}(rq_x + (rq)_x), \\ r_{t_2} = 2k_2(t) + (xr_x + r)k_1(t) + \frac{1}{2}(r^2 + q^{-1}q_{xx} - 2q^2)_x. \end{cases} \quad (9)$$

Set $k_1(t) = k_2(t) = 0, t = 2\tilde{t}$, and (9) reduces to

$$\begin{cases} q_{\tilde{t}_2} = rq_x + (rq)_x, \\ r_{\tilde{t}_2} = (r^2 + q^{-1}q_{xx} - 2q^2)_x, \end{cases} \quad (10)$$

which is the TD equation [2].

When $n = 3$, (8) becomes

$$\begin{cases} q_{t_3} = (\frac{1}{2}xqr_x + rq + xrq_x)k_1(t) + (xq_x + q)k_2(t) + \frac{1}{4}(q_{xxx} - 6q^2q_x + 3r(rq)_x), \\ r_{t_3} = 2k_3(t) + \partial(xr_x + r)k_2(t) + \partial(\frac{1}{2}xq^{-1}q_{xx} + \frac{1}{2}xr^2 + q^{-1}q_x - 2\partial^{-1}xqq_x - 2\partial^{-1}q^2)k_1(t) \\ + \frac{1}{4}(r^3 + 3q^{-1}r_xq_x + 3rq^{-1}q_{xx} + r_{xx} - 6q^2r)_x. \end{cases} \quad (11)$$

Let $k_1(t) = k_2(t) = k_3(t) = 0, r = 0, t = -4\tilde{t}, q = i\tilde{q}$, (11) reduces to

$$\tilde{q}_{\tilde{t}_3} + \tilde{q}_{xxx} + 6\tilde{q}^2\tilde{q}_x = 0,$$

which is the MKdV equation.

Bi-Hamiltonian Structures

In this subsection, we apply the trace identity discussed in [2] to consider the Bi-Hamiltonian structure of the nonlinear integrable hierarchy (8).

We calculate the following quantities:

$$\frac{\partial M}{\partial \lambda} = -h_1 + h_2, \quad \frac{\partial M}{\partial q} = h_3 + h_4, \quad \frac{\partial M}{\partial r} = h_1,$$

$$\operatorname{tr}\left(N\frac{\partial M}{\partial \lambda}\right) = -2a, \quad \operatorname{tr}\left(N\frac{\partial M}{\partial q}\right) = b + c, \quad \operatorname{tr}\left(N\frac{\partial M}{\partial r}\right) = a.$$

Bearing in mind that the trace identity

$$\frac{\delta}{\delta u} \int \operatorname{tr}\left(N\frac{\partial M}{\partial \lambda}\right) dx = \lambda^{-\gamma} \frac{\partial}{\partial \lambda} \lambda^{\gamma} \left(\operatorname{tr}\left(N\frac{\partial M}{\partial q}\right), \operatorname{tr}\left(N\frac{\partial M}{\partial r}\right) \right)^T,$$

we arrive at

$$\frac{\delta}{\delta u} \int -2adx = \lambda^{-\gamma} \frac{\partial}{\partial \lambda} \lambda^{\gamma} \begin{pmatrix} b+c \\ a \end{pmatrix}.$$

By comparing the coefficients of λ^{-m-1} in the above equation, we obtain

$$\frac{\delta}{\delta u} \int -2a_{m+1}dx = (\gamma - m) \begin{pmatrix} b_m + c_m \\ a_m \end{pmatrix}.$$

We figured out $\gamma = 0$ via inserting $m = 1$ into the above equation. Thus, we have

$$\frac{\delta}{\delta u} \int \frac{a_{m+2}}{m+1} dx = \frac{1}{2} \begin{pmatrix} b_{m+1} + c_{m+1} \\ a_{m+1} \end{pmatrix} = \frac{\delta H_{m+1}}{\delta u}, \quad (12)$$

where $H_{m+1} = \int \frac{a_{m+2}}{m+1} dx$. A recurrence relationship is generated by a direct calculation, as follows

$$\begin{pmatrix} b_{m+1} + c_{m+1} \\ a_{m+1} \end{pmatrix} = L \begin{pmatrix} b_m + c_m \\ a_m \end{pmatrix} + Rk_m(t) + R_0k_{m+1}(t), \quad m \geq 0, \quad (13)$$

where

$$L = \frac{1}{2} \begin{pmatrix} r & \partial q^{-1} \partial - 4q \\ \partial^{-1} q \partial & \partial^{-1} r \partial \end{pmatrix}, \quad R = \frac{1}{2} \begin{pmatrix} \partial q^{-1} \\ \partial^{-1} r \end{pmatrix}, \quad R_0 = \begin{pmatrix} 0 \\ x \end{pmatrix}.$$

According to (12) and (13), the non-isospectral TD hierarchy (8) can be represented as the following:

$$\begin{aligned} \begin{pmatrix} q \\ r \end{pmatrix}_{t_n} &= \begin{pmatrix} q^{-1}(\partial a_{n+1} + k_{n+1}(t)) \\ \partial[q^{-1}(b_{n+1} + c_{n+1})] \end{pmatrix} = J \begin{pmatrix} b_{n+1} + c_{n+1} \\ a_{n+1} \end{pmatrix} + R_1k_{n+1}(t) \\ &=: 2J \frac{\delta H_{n+1}}{\delta u} + R_1k_{n+1}(t) \\ &= J \left(2L \frac{\delta H_n}{\delta u} + Rk_n(t) + R_0k_{n+1}(t) \right) + R_1k_{n+1}(t) \\ &=: Q \frac{\delta H_n}{\delta u} + JRk_n(t) + (JR_0 + R_1)k_{n+1}(t) \\ &= JL^n \begin{pmatrix} b_1 + c_1 \\ a_1 \end{pmatrix} + \sum_{i=0}^n JL^{n-i} (Rk_i(t) + R_0k_{i+1}(t)) + R_1k_{n+1}(t), \end{aligned}$$

with R_1 and the Hamiltonian operators J and Q as follows:

$$R_1 = \begin{pmatrix} q^{-1} \\ 0 \end{pmatrix}, \quad J = \begin{pmatrix} 0 & q^{-1} \partial \\ \partial q^{-1} & 0 \end{pmatrix}, \quad Q = \begin{pmatrix} \partial & q^{-1} r \partial \\ \partial q^{-1} r & (\partial q^{-1})^2 \partial - 4 \partial \end{pmatrix}.$$

3. An Integrable Coupling of the TD Non-Isospectral Hierarchy

Similar to [26], in terms of the definition of Lie algebra, the Lie algebra A can be enlarged into a higher-dimensional Lie algebra.

The enlarged Lie algebra $A_{12} = \text{span}\{h_i\}_{i=1}^8$ has the following basis:

$$\begin{aligned}\tilde{h}_1 &= \begin{pmatrix} h_1 & 0 \\ 0 & h_1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad \tilde{h}_2 = \begin{pmatrix} h_2 & 0 \\ 0 & h_2 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \\ \tilde{h}_3 &= \begin{pmatrix} h_3 & 0 \\ 0 & h_3 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad \tilde{h}_4 = \begin{pmatrix} h_4 & 0 \\ 0 & h_4 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \\ \tilde{h}_5 &= \begin{pmatrix} 0 & \varepsilon h_1 \\ h_1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & \varepsilon & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad \tilde{h}_6 = \begin{pmatrix} 0 & \varepsilon h_2 \\ h_2 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \varepsilon \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}, \\ \tilde{h}_7 &= \begin{pmatrix} 0 & \varepsilon h_3 \\ h_3 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & \varepsilon \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad \tilde{h}_8 = \begin{pmatrix} 0 & \varepsilon h_4 \\ h_4 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & \varepsilon & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix},\end{aligned}$$

where the commutator is given by

$$\begin{aligned}[\tilde{h}_1, \tilde{h}_2] &= 0, \quad [\tilde{h}_1, \tilde{h}_3] = \tilde{h}_3, \quad [\tilde{h}_1, \tilde{h}_4] = -\tilde{h}_4, \quad [\tilde{h}_1, \tilde{h}_5] = 0, \quad [\tilde{h}_1, \tilde{h}_6] = 0, \\ [\tilde{h}_1, \tilde{h}_7] &= \tilde{h}_7, \quad [\tilde{h}_1, \tilde{h}_8] = -\tilde{h}_8, \quad [\tilde{h}_2, \tilde{h}_3] = -\tilde{h}_3, \quad [\tilde{h}_2, \tilde{h}_4] = \tilde{h}_4, \quad [\tilde{h}_2, \tilde{h}_5] = 0, \\ [\tilde{h}_2, \tilde{h}_6] &= 0, \quad [\tilde{h}_2, \tilde{h}_7] = -\tilde{h}_7, \quad [\tilde{h}_2, \tilde{h}_8] = \tilde{h}_8, \quad [\tilde{h}_3, \tilde{h}_4] = \tilde{h}_1 - \tilde{h}_2, \quad [\tilde{h}_3, \tilde{h}_5] = -\tilde{h}_7, \\ [\tilde{h}_3, \tilde{h}_6] &= \tilde{h}_7, \quad [\tilde{h}_3, \tilde{h}_7] = 0, \quad [\tilde{h}_3, \tilde{h}_8] = \tilde{h}_5 - \tilde{h}_6, \quad [\tilde{h}_4, \tilde{h}_5] = \tilde{h}_8, \quad [\tilde{h}_4, \tilde{h}_6] = -\tilde{h}_8, \\ [\tilde{h}_4, \tilde{h}_7] &= -\tilde{h}_5 + \tilde{h}_6, \quad [\tilde{h}_4, \tilde{h}_8] = 0, \quad [\tilde{h}_5, \tilde{h}_6] = 0, \quad [\tilde{h}_5, \tilde{h}_7] = \varepsilon \tilde{h}_3, \\ [\tilde{h}_5, \tilde{h}_8] &= -\varepsilon \tilde{h}_4, \quad [\tilde{h}_6, \tilde{h}_7] = -\varepsilon \tilde{h}_3, \quad [\tilde{h}_6, \tilde{h}_8] = \varepsilon \tilde{h}_4, \quad [\tilde{h}_7, \tilde{h}_8] = \varepsilon(\tilde{h}_1 - \tilde{h}_2).\end{aligned}$$

Assume $G_1 = \text{span}\{\tilde{h}_1, \tilde{h}_2, \tilde{h}_3, \tilde{h}_4\}$, $G_2 = \text{span}\{\tilde{h}_5, \tilde{h}_6, \tilde{h}_7, \tilde{h}_8\}$, then $A_{12} = G_1 \oplus G_2$. It is obvious that

$$[G_1, G_1] \subseteq G_1, \quad [G_1, G_2] \subseteq G_2, \quad [G_2, G_2] \subseteq G_1.$$

We introduce an enlarged non-isospectral problem:

$$\begin{cases} \psi_x = U\psi, & U = -\lambda\tilde{h}_1 + \lambda\tilde{h}_2 + r_1\tilde{h}_1 + q_1\tilde{h}_3 + q_1\tilde{h}_4 - \lambda\tilde{h}_5 + \lambda\tilde{h}_6 + r_2\tilde{h}_5 + q_2\tilde{h}_7 + q_2\tilde{h}_8, \\ \psi_t = V\psi, & V = a\tilde{h}_1 - a\tilde{h}_2 + b\tilde{h}_3 + c\tilde{h}_4 + d\tilde{h}_5 - d\tilde{h}_6 + e\tilde{h}_7 + f\tilde{h}_8, \\ \lambda_t = \sum_{m \geq 0} k_m(t)\lambda^{-m}, \end{cases} \quad (14)$$

where

$$\begin{aligned}U &= \begin{pmatrix} U_1 & \varepsilon U_2 \\ U_2 & U_1 \end{pmatrix}, \quad V = \begin{pmatrix} V_1 & \varepsilon V_2 \\ V_2 & V_1 \end{pmatrix}, \\ U_1 &= -\lambda h_1 + \lambda h_2 + r_1 h_1 + q_1 h_3 + q_1 h_4, \quad U_2 = -\lambda h_1 + \lambda h_2 + r_2 h_1 + q_2 h_3 + q_2 h_4, \\ V_1 &= a h_1 - a h_2 + b h_3 + c h_4, \quad V_2 = \hat{a} h_1 - \hat{a} h_2 + \hat{b} h_3 + \hat{c} h_4.\end{aligned} \quad (15)$$

and $\tilde{u} = (q_1, r_1, q_2, r_2)^T$. Choosing $a = \sum_{m \geq 0} a_m \lambda^{-m}$, $b = \sum_{m \geq 0} b_m \lambda^{-m}$, $c = \sum_{m \geq 0} c_m \lambda^{-m}$, $\hat{a} = \sum_{m \geq 0} \hat{a}_m \lambda^{-m}$, $\hat{b} = \sum_{m \geq 0} \hat{b}_m \lambda^{-m}$, $\hat{c} = \sum_{m \geq 0} \hat{c}_m \lambda^{-m}$, and solving the stationary zero curvature equation

$$V_x = \frac{\partial U}{\partial \lambda} \lambda_t + [U, V], \quad (16)$$

we have

$$\begin{cases} a_{mx} = q_1(c_m - b_m) + \varepsilon q_2(\widehat{c}_m - \widehat{b}_m) - k_m(t), \\ b_{mx} = -2b_{m+1} - 2\varepsilon\widehat{b}_{m+1} + r_1b_m + \varepsilon r_2\widehat{b}_m - 2q_1a_m - 2\varepsilon q_2\widehat{a}_m, \\ c_{mx} = 2c_{m+1} + 2\varepsilon\widehat{c}_{m+1} - r_1c_m - \varepsilon r_2\widehat{c}_m + 2q_1a_m + 2\varepsilon q_2\widehat{a}_m, \\ \widehat{a}_{mx} = q_1(\widehat{c}_m - \widehat{b}_m) + q_2(c_m - b_m) - k_m(t), \\ \widehat{b}_{mx} = -2b_{m+1} - 2\widehat{b}_{m+1} + r_1\widehat{b}_m + r_2b_m - 2q_1\widehat{a}_m - 2q_2a_m, \\ \widehat{c}_{mx} = 2c_{m+1} + 2\widehat{c}_{m+1} - r_1\widehat{c}_m - r_2c_m + 2q_1\widehat{a}_m + 2q_2a_m. \end{cases} \quad (17)$$

A straightforward computation yields the following recursion equations:

$$\begin{cases} a_m = \partial^{-1}[q_1(c_m - b_m) + \varepsilon q_2(\widehat{c}_m - \widehat{b}_m)] - k_m(t)x, \\ b_{m+1} = \frac{1}{2(\varepsilon-1)}[b_{mx} - \varepsilon\widehat{b}_{mx} + 2q_1(a_m - \varepsilon\widehat{a}_m) - r_1(b_m - \varepsilon\widehat{b}_m) + \varepsilon(2q_2(\widehat{a}_m - a_m) - r_2(\widehat{b}_m - b_m))], \\ c_{m+1} = \frac{1}{2(\varepsilon-1)}[\varepsilon\widehat{c}_{mx} - c_{mx} + 2q_1(a_m - \varepsilon\widehat{a}_m) - r_1(c_m - \varepsilon\widehat{c}_m) + \varepsilon(2q_2(\widehat{a}_m - a_m) - r_2(\widehat{c}_m - c_m))], \\ \widehat{a}_m = \partial^{-1}[q_1(\widehat{c}_m - \widehat{b}_m) + q_2(c_m - b_m)] - k_m(t)x, \\ \widehat{b}_{m+1} = \frac{1}{2(\varepsilon-1)}[\widehat{b}_{mx} - b_{mx} + 2q_2(a_m - \varepsilon\widehat{a}_m) - r_2(b_m - \varepsilon\widehat{b}_m) + 2q_1(\widehat{a}_m - a_m) - r_1(\widehat{b}_m - b_m)], \\ \widehat{c}_{m+1} = \frac{1}{2(\varepsilon-1)}[c_{mx} - \widehat{c}_{mx} + 2q_2(a_m - \varepsilon\widehat{a}_m) - r_2(c_m - \varepsilon\widehat{c}_m) + 2q_1(\widehat{a}_m - a_m) - r_1(\widehat{c}_m - c_m)]. \end{cases} \quad (18)$$

Assuming

$$a_0 = -1, \quad b_0 = 0, \quad c_0 = 0, \quad k_0 = 0, \quad \widehat{a}_0 = -1, \quad \widehat{b}_0 = 0, \quad \widehat{c}_0 = 0,$$

then (18) gives rise to

$$\begin{aligned} b_1 &= q_1, \quad c_1 = q_1, \quad \widehat{b}_1 = q_2, \quad \widehat{c}_1 = q_2, \quad a_1 = -k_1(t)x, \quad \widehat{a}_1 = -k_1(t)x, \\ b_2 &= q_1k_1(t)x + \frac{1}{2(\varepsilon-1)}[q_{1x} - \varepsilon q_{2x} - r_1(q_1 - \varepsilon q_2) - \varepsilon r_2(q_2 - q_1)], \\ c_2 &= q_1k_1(t)x + \frac{1}{2(\varepsilon-1)}[\varepsilon q_{2x} - q_{1x} - r_1(q_1 - \varepsilon q_2) - \varepsilon r_2(q_2 - q_1)], \\ \widehat{b}_2 &= q_2k_1(t)x + \frac{1}{2(\varepsilon-1)}[q_{2x} - q_{1x} - r_2(q_1 - \varepsilon q_2) - r_1(q_2 - q_1)], \\ \widehat{c}_2 &= q_2k_1(t)x + \frac{1}{2(\varepsilon-1)}[q_{1x} - q_{2x} - r_2(q_1 - \varepsilon q_2) - r_1(q_2 - q_1)], \\ a_2 &= \frac{1}{2(\varepsilon-1)}[-q_1^2 + \varepsilon q_2^2 + 2\varepsilon q_2(q_1 - q_2)] - k_2(t)x, \\ \widehat{a}_2 &= \frac{1}{2(\varepsilon-1)}[q_1^2 - \varepsilon q_2^2 + 2q_2(\varepsilon q_2 - q_1)] - k_2(t)x, \\ &\dots \end{aligned}$$

Note that

$$\begin{aligned} V_+^{(n)} &= a_m\widetilde{h}_1\lambda^{n-m} - a_m\widetilde{h}_2\lambda^{n-m} + b_m\widetilde{h}_3\lambda^{n-m} + c_m\widetilde{h}_4\lambda^{n-m} \\ &\quad + \widehat{a}_m\widetilde{h}_5\lambda^{n-m} - \widehat{a}_m\widetilde{h}_6\lambda^{n-m} + \widehat{b}_m\widetilde{h}_7\lambda^{n-m} + \widehat{c}_m\widetilde{h}_8\lambda^{n-m}, \end{aligned}$$

then, we have

$$\begin{aligned} -V_{+,x}^{(n)} + \frac{\partial U}{\partial \lambda}\lambda_{t,+}^{(n)} + [U, V_+^{(n)}] &= V_{-,x}^{(n)} - \frac{\partial U}{\partial \lambda}\lambda_{t,-}^{(n)} - [U, V_-^{(n)}] \\ &= 2(b_{n+1} + \varepsilon\widehat{b}_{n+1})\widetilde{h}_3 - 2(c_{n+1} + \varepsilon\widehat{c}_{n+1})\widetilde{h}_4 \\ &\quad + 2(b_{n+1} + \widehat{b}_{n+1})\widetilde{h}_7 - 2(c_{n+1} + \widehat{c}_{n+1})\widetilde{h}_8. \end{aligned}$$

Letting

$$V^{(n)} = V_+^{(n)} + \widetilde{\Delta}_n$$

and the modified term

$$\begin{aligned}\tilde{\Delta}_n &= \frac{[q_1(b_{n+1} + \varepsilon \hat{b}_{n+1} + c_{n+1} + \varepsilon \hat{c}_{n+1}) - \varepsilon q_2(b_{n+1} + \hat{b}_{n+1} + c_{n+1} + \hat{c}_{n+1})] \tilde{h}_1}{q_1^2 - \varepsilon q_2^2} \\ &+ \frac{[q_1(b_{n+1} + \hat{b}_{n+1} + c_{n+1} + \hat{c}_{n+1}) - q_2(b_{n+1} + \varepsilon \hat{b}_{n+1} + c_{n+1} + \varepsilon \hat{c}_{n+1})] \tilde{h}_5}{q_1^2 - \varepsilon q_2^2},\end{aligned}$$

the non-isospectral zero curvature equation

$$U_{t_n} + \frac{\partial U}{\partial \lambda} \lambda_t^{(n)} - V_x^{(n)} + [U, V^{(n)}] = 0,$$

gives the integrable coupling of non-isospectral TD hierarchy

$$\left\{ \begin{aligned} q_{1t_n} &= c_{n+1} + \varepsilon \hat{c}_{n+1} - b_{n+1} - \varepsilon \hat{b}_{n+1} \\ &= \frac{q_1(a_{n,x} + \varepsilon \hat{a}_{n,x}) - \varepsilon q_2(a_{n,x} + \hat{a}_{n,x}) + (q_1 - \varepsilon q_2)k_n(t) + \varepsilon(q_1 - q_2)k_n(t)}{q_1^2 - \varepsilon q_2^2}, \\ r_{1t_n} &= \partial \left(\frac{q_1(b_{n+1} + \varepsilon \hat{b}_{n+1} + c_{n+1} + \varepsilon \hat{c}_{n+1}) - \varepsilon q_2(b_{n+1} + \hat{b}_{n+1} + c_{n+1} + \hat{c}_{n+1})}{q_1^2 - \varepsilon q_2^2} \right), \\ q_{2t_n} &= c_{n+1} + \hat{c}_{n+1} - b_{n+1} - \hat{b}_{n+1} \\ &= \frac{q_1(a_{n,x} + \hat{a}_{n,x}) - q_2(a_{n,x} + \varepsilon \hat{a}_{n,x}) + (q_1 - q_2)k_n(t) + (q_1 - \varepsilon q_2)k_n(t)}{q_1^2 - \varepsilon q_2^2}, \\ r_{2t_n} &= \partial \left(\frac{q_1(b_{n+1} + \hat{b}_{n+1} + c_{n+1} + \hat{c}_{n+1}) - q_2(b_{n+1} + \varepsilon \hat{b}_{n+1} + c_{n+1} + \varepsilon \hat{c}_{n+1})}{q_1^2 - \varepsilon q_2^2} \right). \end{aligned} \right. \quad (19)$$

When $n = 2$, (19) leads to

$$\left\{ \begin{aligned} q_{1,t_2} &= (xq_{1,x} + q_1)k_1(t) \\ &+ \frac{1}{2(\varepsilon - 1)} \left[2r_1(\varepsilon q_{2,x} - q_{1,x}) + 2\varepsilon r_2(q_{1,x} - q_{2,x}) + q_1(\varepsilon r_{2,x} - r_{1,x}) + \varepsilon q_2(r_{1,x} - r_{2,x}) \right], \\ r_{1,t_2} &= 2k_2(t) + (r_1 + xr_{1,x})k_1(t) \\ &+ \frac{1}{2(\varepsilon - 1)} \partial \left[2\varepsilon r_1 r_2 - r_1^2 - \varepsilon r_2^2 - 4\varepsilon q_1 q_2 - \frac{q_1(q_{1,xx} - \varepsilon q_{2,xx}) + \varepsilon q_2(q_{1,xx} - q_{2,xx}) + 2\varepsilon^2 q_2^4 - 2q_1^4}{q_1^2 - \varepsilon q_2^2} \right], \\ q_{2,t_2} &= (xq_{2,x} + q_2)k_1(t) \\ &+ \frac{1}{2(\varepsilon - 1)} \left[2r_1(q_{1,x} - q_{2,x}) + 2r_2(\varepsilon q_{2,x} - q_{1,x}) + q_1(r_{1,x} - r_{2,x}) + q_2(\varepsilon r_{2,x} - r_{1,x}) \right], \\ r_{2,t_2} &= 2k_2(t) + (r_2 + xr_{2,x})k_1(t) \\ &+ \frac{1}{2(\varepsilon - 1)} \partial \left[r_1^2 + \varepsilon r_2^2 - 2r_1 r_2 + 4q_1 q_2 + \frac{q_1(q_{1,xx} - q_{2,xx}) + q_2(q_{1,xx} - \varepsilon q_{2,xx}) + 2\varepsilon^2 q_2^4 - 2q_1^4}{q_1^2 - \varepsilon q_2^2} \right]. \end{aligned} \right. \quad (20)$$

Let $\varepsilon = 0$; the first and second equations in (20) coincide with (9). Therefore, the system (20) is an integrable coupling of the TD non-isospectral equations. Thus, (19) is an integrable coupling of the TD non-isospectral hierarchy. Let $n = 3$, and choose a suitable transformation; then, (19) becomes a coupled mKdV equation.

Bi-Hamiltonian Structures

With the aid of the following Z_2^ε -trace identities, which were discussed in [26], we shall construct the Bi-Hamiltonian structures of the integrable coupling of the non-isospectral hierarchy (19):

$$\begin{aligned} \frac{\delta}{\delta \tilde{u}} \int \left(\text{tr}(V_1 \frac{\partial U_2}{\partial \lambda}) + \text{tr}(V_2 \frac{\partial U_1}{\partial \lambda}) \right) dx &= \lambda^{-\gamma} \frac{\partial}{\partial \lambda} \lambda^\gamma \left(\text{tr}(V_1 \frac{\partial U_2}{\partial \tilde{u}}) + \text{tr}(V_2 \frac{\partial U_1}{\partial \tilde{u}}) \right), \\ \frac{\delta}{\delta \tilde{u}} \int \left(\text{tr}(V_1 \frac{\partial U_1}{\partial \lambda}) + \varepsilon \text{tr}(V_2 \frac{\partial U_2}{\partial \lambda}) \right) dx &= \lambda^{-\gamma} \frac{\partial}{\partial \lambda} \lambda^\gamma \left(\text{tr}(V_1 \frac{\partial U_1}{\partial \tilde{u}}) + \varepsilon \text{tr}(V_2 \frac{\partial U_2}{\partial \tilde{u}}) \right). \end{aligned} \quad (21)$$

From (15), we obtain

$$\begin{aligned} \text{tr}(V_1 \frac{\partial U_2}{\partial q_1}) + \text{tr}(V_2 \frac{\partial U_1}{\partial q_1}) &= \hat{b} + \hat{c}, \quad \text{tr}(V_1 \frac{\partial U_2}{\partial q_2}) + \text{tr}(V_2 \frac{\partial U_1}{\partial q_2}) = b + c, \\ \text{tr}(V_1 \frac{\partial U_2}{\partial r_1}) + \text{tr}(V_2 \frac{\partial U_1}{\partial r_1}) &= \hat{a}, \quad \text{tr}(V_1 \frac{\partial U_2}{\partial r_2}) + \text{tr}(V_2 \frac{\partial U_1}{\partial r_2}) = a, \\ \text{tr}(V_1 \frac{\partial U_1}{\partial q_1}) + \varepsilon \text{tr}(V_2 \frac{\partial U_2}{\partial q_1}) &= b + c, \quad \text{tr}(V_1 \frac{\partial U_1}{\partial q_2}) + \varepsilon \text{tr}(V_2 \frac{\partial U_2}{\partial q_2}) = \varepsilon(\hat{b} + \hat{c}), \\ \text{tr}(V_1 \frac{\partial U_1}{\partial r_1}) + \varepsilon \text{tr}(V_2 \frac{\partial U_2}{\partial r_1}) &= a, \quad \text{tr}(V_1 \frac{\partial U_1}{\partial r_2}) + \varepsilon \text{tr}(V_2 \frac{\partial U_2}{\partial r_2}) = \varepsilon \hat{a}, \\ \text{tr}(V_1 \frac{\partial U_2}{\partial \lambda}) + \text{tr}(V_2 \frac{\partial U_1}{\partial \lambda}) &= -2(a + \hat{a}), \quad \text{tr}(V_1 \frac{\partial U_1}{\partial \lambda}) + \varepsilon \text{tr}(V_2 \frac{\partial U_2}{\partial \lambda}) = -2(a + \varepsilon \hat{a}). \end{aligned}$$

Substituting the above equations into (21) and collecting the terms with the same powers of λ produces

$$\begin{aligned} \frac{\delta}{\delta \tilde{u}} \int -2(a_{m+1} + \hat{a}_{m+1}) dx &= (\gamma - m) \begin{pmatrix} \hat{b}_m + \hat{c}_m \\ \hat{a}_m \\ b_m + c_m \\ a_m \end{pmatrix}, \\ \frac{\delta}{\delta \tilde{u}} \int -2(a_{m+1} + \varepsilon \hat{a}_{m+1}) dx &= (\gamma - m) \begin{pmatrix} b_m + c_m \\ a_m \\ \varepsilon(\hat{b}_m + \hat{c}_m) \\ \varepsilon \hat{a}_m \end{pmatrix}. \end{aligned}$$

Set $m = 1$; we have $\gamma = 0$. Therefore, we conclude that

$$\begin{aligned} \frac{\delta}{\delta \tilde{u}} \int \frac{a_{m+2} + \hat{a}_{m+2}}{m+1} dx &= \frac{1}{2} \begin{pmatrix} \hat{b}_{m+1} + \hat{c}_{m+1} \\ \hat{a}_{m+1} \\ b_{m+1} + c_{m+1} \\ a_{m+1} \end{pmatrix} = \frac{\delta H_{m+1}}{\delta \tilde{u}}, \\ \frac{\delta}{\delta \tilde{u}} \int \frac{a_{m+2} + \varepsilon \hat{a}_{m+2}}{m+1} dx &= \frac{1}{2} \begin{pmatrix} b_{m+1} + c_{m+1} \\ a_{m+1} \\ \varepsilon(\hat{b}_{m+1} + \hat{c}_{m+1}) \\ \varepsilon \hat{a}_{m+1} \end{pmatrix} = \frac{\delta \hat{H}_{m+1}}{\delta \tilde{u}}, \end{aligned} \quad (22)$$

where

$$H_{m+1} = \int \frac{a_{m+2} + \hat{a}_{m+2}}{m+1} dx, \quad \hat{H}_{m+1} = \int \frac{a_{m+2} + \varepsilon \hat{a}_{m+2}}{m+1} dx.$$

From (18), we obtain two kind of recursive relationships:

$$\begin{pmatrix} \hat{b}_{m+1} + \hat{c}_{m+1} \\ \hat{a}_{m+1} \\ b_{m+1} + c_{m+1} \\ a_{m+1} \end{pmatrix} = \tilde{L}_1 \begin{pmatrix} \hat{b}_m + \hat{c}_m \\ \hat{a}_m \\ b_m + c_m \\ a_m \end{pmatrix} + \tilde{R}_1 k_m(t) + \tilde{R}_0 k_{m+1}(t), \quad m \geq 0, \quad (23)$$

$$\begin{pmatrix} b_{m+1} + c_{m+1} \\ a_{m+1} \\ \varepsilon(\hat{b}_{m+1} + \hat{c}_{m+1}) \\ \varepsilon \hat{a}_{m+1} \end{pmatrix} = \tilde{L}_1 \begin{pmatrix} b_m + c_m \\ a_m \\ \varepsilon(\hat{b}_m + \hat{c}_m) \\ \varepsilon \hat{a}_m \end{pmatrix} + \hat{R}_1 k_m(t) + \hat{R}_0 k_{m+1}(t), \quad m \geq 0, \quad (24)$$

where the recursion operator $\tilde{L}_1$ becomes

$$\tilde{L}_1 = \frac{1}{2(\varepsilon - 1)} \begin{pmatrix} \tilde{L}_{11} & \tilde{L}_{12} & \tilde{L}_{13} & \tilde{L}_{14} \\ \tilde{L}_{21} & \tilde{L}_{22} & \tilde{L}_{23} & \tilde{L}_{24} \\ \varepsilon \tilde{L}_{13} & \varepsilon \tilde{L}_{14} & \tilde{L}_{11} & \tilde{L}_{12} \\ \varepsilon \tilde{L}_{23} & \varepsilon \tilde{L}_{24} & \tilde{L}_{21} & \tilde{L}_{22} \end{pmatrix},$$

with

$$\tilde{L}_{11} = -r_1 + \varepsilon r_2, \quad \tilde{L}_{12} = -\partial \left(\frac{q_1 + \varepsilon q_2}{q_1^2 - \varepsilon q_2^2} \right) \partial + 4q_1 - 4\varepsilon q_2, \quad \tilde{L}_{13} = r_1 - r_2,$$

$$\tilde{L}_{14} = \partial \left(\frac{q_1 + q_2}{q_1^2 - \varepsilon q_2^2} \right) \partial - 4q_1 + 4q_2, \quad \tilde{L}_{21} = \partial^{-1}(\varepsilon q_2 - q_1) \partial,$$

$$\tilde{L}_{22} = \partial^{-1}(\varepsilon r_2 - r_1) \partial, \quad \tilde{L}_{23} = \partial^{-1}(q_1 - q_2) \partial, \quad \tilde{L}_{24} = \partial^{-1}(r_1 - r_2) \partial,$$

and

$$\tilde{R}_1 = \frac{1}{2} \begin{pmatrix} -\partial \left(\frac{q_2}{q_1^2 - \varepsilon q_2^2} \right) \\ \partial^{-1} r_2 \\ \partial \left(\frac{q_1}{q_1^2 - \varepsilon q_2^2} \right) \\ \partial^{-1} r_1 \end{pmatrix}, \quad \hat{R}_1 = \frac{1}{2} \begin{pmatrix} \partial \left(\frac{q_1}{q_1^2 - \varepsilon q_2^2} \right) \\ \partial_x^{-1} r_1 \\ -\varepsilon \partial \left(\frac{q_2}{q_1^2 - \varepsilon q_2^2} \right) \\ \varepsilon \partial^{-1} r_2 \end{pmatrix}, \quad \tilde{R}_0 = \begin{pmatrix} 0 \\ x \\ 0 \\ x \end{pmatrix}, \quad \hat{R}_0 = \begin{pmatrix} 0 \\ x \\ 0 \\ \varepsilon x \end{pmatrix}.$$

Then, the hierarchy in (19) determines the following two kinds of Bi-Hamiltonian structures. The first one is

$$\begin{aligned} \begin{pmatrix} q_1 \\ r_1 \\ q_2 \\ r_2 \end{pmatrix}_{t_n} &= \begin{pmatrix} \frac{q_1(a_{n+1,x} + \varepsilon \hat{a}_{n+1,x}) - \varepsilon q_2(a_{n+1,x} + \hat{a}_{n+1,x}) + (q_1 - \varepsilon q_2)k_{n+1} + \varepsilon(q_1 - q_2)k_{n+1}}{q_1^2 - \varepsilon q_2^2} \\ \partial \left(\frac{q_1(b_{n+1} + \varepsilon \hat{b}_{n+1} + c_{n+1} + \varepsilon \hat{c}_{n+1}) - \varepsilon q_2(b_{n+1} + \hat{b}_{n+1} + c_{n+1} + \hat{c}_{n+1})}{q_1^2 - \varepsilon q_2^2} \right) \\ \frac{q_1(a_{n+1,x} + \hat{a}_{n+1,x}) - q_2(a_{n+1,x} + \varepsilon \hat{a}_{n+1,x}) + (q_1 - q_2)k_{n+1} + (q_1 - \varepsilon q_2)k_{n+1}}{q_1^2 - \varepsilon q_2^2} \\ \partial \left(\frac{q_1(b_{n+1} + \hat{b}_{n+1} + c_{n+1} + \hat{c}_{n+1}) - q_2(b_{n+1} + \varepsilon \hat{b}_{n+1} + c_{n+1} + \varepsilon \hat{c}_{n+1})}{q_1^2 - \varepsilon q_2^2} \right) \end{pmatrix} \\ &= \tilde{J}_1 \begin{pmatrix} \hat{b}_{n+1} + \hat{c}_{m+1} \\ \hat{a}_{n+1} \\ b_{n+1} + c_{n+1} \\ a_{n+1} \end{pmatrix} + \tilde{R} k_{n+1} =: 2\tilde{J}_1 \frac{\delta H_{n+1}}{\delta \tilde{u}} + \tilde{R} k_{n+1} \\ &= \tilde{J}_1 (2\tilde{L}_1 \frac{\delta H_n}{\delta \tilde{u}} + \tilde{R}_1 k_n(t) + \tilde{R}_0 k_{n+1}(t)) + \tilde{R} k_{n+1}(t) =: \tilde{Q}_1 \frac{\delta H_n}{\delta \tilde{u}} + \tilde{J}_1 \tilde{R}_1 k_n(t) + (\tilde{J}_1 \tilde{R}_0 + \tilde{R}) k_{n+1} \\ &= \tilde{J}_1 \tilde{L}_1^n \begin{pmatrix} \hat{b}_1 + \hat{c}_1 \\ \hat{a}_1 \\ b_1 + c_1 \\ a_1 \end{pmatrix} + \sum_{i=0}^n \tilde{J}_1 \tilde{L}_1^{n-i} (\tilde{R}_1 k_i(t) + \tilde{R}_0 k_{i+1}(t)) + \tilde{R} k_{n+1}, \end{aligned}$$

with the Hamiltonian operators $\tilde{J}_1$, $\tilde{Q}_1$, and $\tilde{R}$, as follows:

$$\tilde{J}_1 = \begin{pmatrix} 0 & \frac{\varepsilon(q_1 - q_2)}{q_1^2 - \varepsilon q_2^2} \partial & 0 & \frac{q_1 - \varepsilon q_2}{q_1^2 - \varepsilon q_2^2} \partial \\ \varepsilon \partial \left(\frac{q_1 - q_2}{q_1^2 - \varepsilon q_2^2} \right) & 0 & \partial \left(\frac{q_1 - \varepsilon q_2}{q_1^2 - \varepsilon q_2^2} \right) & 0 \\ 0 & \frac{q_1 - \varepsilon q_2}{q_1^2 - \varepsilon q_2^2} \partial & 0 & \frac{q_1 - q_2}{q_1^2 - \varepsilon q_2^2} \partial \\ \partial \left(\frac{q_1 - \varepsilon q_2}{q_1^2 - \varepsilon q_2^2} \right) & 0 & \partial \left(\frac{q_1 - q_2}{q_1^2 - \varepsilon q_2^2} \right) & 0 \end{pmatrix},$$

$$\tilde{Q}_1 = \frac{1}{\varepsilon - 1} \begin{pmatrix} \varepsilon \tilde{Q}_{11} & \varepsilon \tilde{Q}_{12} & \tilde{Q}_{13} & \tilde{Q}_{14} \\ \varepsilon \tilde{Q}_{21} & \varepsilon \tilde{Q}_{22} & \tilde{Q}_{23} & \tilde{Q}_{24} \\ \tilde{Q}_{13} & \tilde{Q}_{14} & \tilde{Q}_{11} & \tilde{Q}_{12} \\ \tilde{Q}_{23} & \tilde{Q}_{24} & \tilde{Q}_{21} & \tilde{Q}_{22} \end{pmatrix}, \quad \tilde{R} = \frac{1}{q_1^2 - \varepsilon q_2^2} \begin{pmatrix} (1 + \varepsilon)q_1 - 2\varepsilon q_2 \\ 0 \\ 2q_1 - (1 + \varepsilon)q_2 \\ 0 \end{pmatrix},$$

$$\begin{aligned} \tilde{Q}_{11} &= \frac{q_1 - q_2}{q_1^2 - \varepsilon q_2^2} \partial \tilde{L}_{21} + \frac{q_1 - \varepsilon q_2}{q_1^2 - \varepsilon q_2^2} \partial \tilde{L}_{23}, & \tilde{Q}_{12} &= \frac{q_1 - q_2}{q_1^2 - \varepsilon q_2^2} \partial \tilde{L}_{22} + \frac{q_1 - \varepsilon q_2}{q_1^2 - \varepsilon q_2^2} \partial \tilde{L}_{24}, \\ \tilde{Q}_{13} &= \frac{\varepsilon(q_1 - q_2)}{q_1^2 - \varepsilon q_2^2} \partial \tilde{L}_{23} + \frac{q_1 - \varepsilon q_2}{q_1^2 - \varepsilon q_2^2} \partial \tilde{L}_{21}, & \tilde{Q}_{14} &= \frac{\varepsilon(q_1 - q_2)}{q_1^2 - \varepsilon q_2^2} \partial \tilde{L}_{24} + \frac{q_1 - \varepsilon q_2}{q_1^2 - \varepsilon q_2^2} \partial \tilde{L}_{22}, \\ \tilde{Q}_{21} &= \partial \left(\frac{q_1 - q_2}{q_1^2 - \varepsilon q_2^2} \right) \tilde{L}_{11} + \partial \left(\frac{q_1 - \varepsilon q_2}{q_1^2 - \varepsilon q_2^2} \right) \tilde{L}_{13}, & \tilde{Q}_{22} &= \partial \left(\frac{q_1 - q_2}{q_1^2 - \varepsilon q_2^2} \right) \tilde{L}_{12} + \partial \left(\frac{q_1 - \varepsilon q_2}{q_1^2 - \varepsilon q_2^2} \right) \tilde{L}_{14}, \\ \tilde{Q}_{23} &= \varepsilon \partial \left(\frac{q_1 - q_2}{q_1^2 - \varepsilon q_2^2} \right) \tilde{L}_{13} + \partial \left(\frac{q_1 - \varepsilon q_2}{q_1^2 - \varepsilon q_2^2} \right) \tilde{L}_{11}, & \tilde{Q}_{24} &= \varepsilon \partial \left(\frac{q_1 - q_2}{q_1^2 - \varepsilon q_2^2} \right) \tilde{L}_{14} + \partial \left(\frac{q_1 - \varepsilon q_2}{q_1^2 - \varepsilon q_2^2} \right) \tilde{L}_{12}. \end{aligned}$$

Similarly, for the second component, we have

$$\begin{pmatrix} q_1 \\ r_1 \\ q_2 \\ r_2 \end{pmatrix}_{t_n} = \begin{pmatrix} \frac{q_1(a_{n+1,x} + \varepsilon \hat{a}_{n+1,x}) - \varepsilon q_2(a_{n+1,x} + \hat{a}_{n+1,x}) + (q_1 - \varepsilon q_2)k_{n+1} + \varepsilon(q_1 - q_2)k_{n+1}}{q_1^2 - \varepsilon q_2^2} \\ \partial \left(\frac{q_1(b_{n+1} + \varepsilon \hat{b}_{n+1} + c_{n+1} + \varepsilon \hat{c}_{n+1}) - \varepsilon q_2(b_{n+1} + \hat{b}_{n+1} + c_{n+1} + \hat{c}_{n+1})}{q_1^2 - \varepsilon q_2^2} \right) \\ \frac{q_1(a_{n+1,x} + \hat{a}_{n+1,x}) - q_2(a_{n+1,x} + \varepsilon \hat{a}_{n+1,x}) + (q_1 - q_2)k_{n+1} + (q_1 - \varepsilon q_2)k_{n+1}}{q_1^2 - \varepsilon q_2^2} \\ \partial \left(\frac{q_1(b_{n+1} + \hat{b}_{n+1} + c_{n+1} + \hat{c}_{n+1}) - q_2(b_{n+1} + \varepsilon \hat{b}_{n+1} + c_{n+1} + \varepsilon \hat{c}_{n+1})}{q_1^2 - \varepsilon q_2^2} \right) \end{pmatrix}$$

$$= \hat{J}_1 \begin{pmatrix} b_{n+1} + c_{n+1} \\ a_{n+1} \\ \varepsilon(\hat{b}_{n+1} + \hat{c}_{n+1}) \\ \varepsilon \hat{a}_{n+1} \end{pmatrix} + \tilde{R}k_{n+1} =: 2\hat{J}_1 \frac{\delta \hat{H}_{n+1}}{\delta \tilde{u}} + \tilde{R}k_{n+1}$$

$$= \hat{J}_1 (2\tilde{L}_1 \frac{\delta \hat{H}_n}{\delta \tilde{u}} + \hat{R}_1 k_n(t) + \hat{R}_0 k_{n+1}(t)) + \tilde{R}k_{n+1} =: \hat{Q}_1 \frac{\delta \hat{H}_n}{\delta \tilde{u}} + \hat{J}_1 (\hat{R}_1 k_n(t) + \hat{R}_0 k_{n+1}(t)) + \tilde{R}k_{n+1}$$

$$= \hat{J}_1 \tilde{L}_1^n \begin{pmatrix} b_1 + c_1 \\ a_1 \\ \varepsilon(\hat{b}_1 + \hat{c}_1) \\ \varepsilon \hat{a}_1 \end{pmatrix} + \sum_{i=0}^n \hat{J}_1 \tilde{L}_1^{n-i} (\hat{R}_1 k_i(t) + \hat{R}_0 k_{i+1}(t)) + \tilde{R}k_{n+1},$$

with the Hamiltonian operators $\hat{J}_1$ and $\hat{Q}_1$, as follows:

$$\hat{J}_1 = \begin{pmatrix} 0 & \frac{q_1 - \varepsilon q_2}{q_1^2 - \varepsilon q_2^2} \partial & 0 & \frac{q_1 - q_2}{q_1^2 - \varepsilon q_2^2} \partial \\ \partial \left(\frac{q_1 - \varepsilon q_2}{q_1^2 - \varepsilon q_2^2} \right) & 0 & \partial \left(\frac{q_1 - q_2}{q_1^2 - \varepsilon q_2^2} \right) & 0 \\ 0 & \frac{q_1 - q_2}{q_1^2 - \varepsilon q_2^2} \partial & 0 & \frac{q_1 - \varepsilon q_2}{\varepsilon(q_1^2 - \varepsilon q_2^2)} \partial \\ \partial \left(\frac{q_1 - q_2}{q_1^2 - \varepsilon q_2^2} \right) & 0 & \partial \left(\frac{q_1 - \varepsilon q_2}{\varepsilon(q_1^2 - \varepsilon q_2^2)} \right) & 0 \end{pmatrix},$$

$$\hat{Q}_1 = \frac{1}{\varepsilon - 1} \begin{pmatrix} \varepsilon \hat{Q}_{11} & \varepsilon \hat{Q}_{12} & \hat{Q}_{13} & \hat{Q}_{14} \\ \varepsilon \hat{Q}_{21} & \varepsilon \hat{Q}_{22} & \hat{Q}_{23} & \hat{Q}_{24} \\ \hat{Q}_{13} & \hat{Q}_{14} & \hat{Q}_{11} & \hat{Q}_{12} \\ \hat{Q}_{23} & \hat{Q}_{24} & \hat{Q}_{21} & \hat{Q}_{22} \end{pmatrix},$$

$$\begin{aligned} \hat{Q}_{11} &= \frac{q_1 - q_2}{q_1^2 - \varepsilon q_2^2} \partial \tilde{L}_{23} + \frac{q_1 - \varepsilon q_2}{\varepsilon(q_1^2 - \varepsilon q_2^2)} \partial \tilde{L}_{21}, & \hat{Q}_{12} &= \frac{q_1 - q_2}{q_1^2 - \varepsilon q_2^2} \partial \tilde{L}_{24} + \frac{q_1 - \varepsilon q_2}{\varepsilon(q_1^2 - \varepsilon q_2^2)} \partial \tilde{L}_{22}, \\ \hat{Q}_{13} &= \frac{q_1 - q_2}{q_1^2 - \varepsilon q_2^2} \partial \tilde{L}_{21} + \frac{q_1 - \varepsilon q_2}{q_1^2 - \varepsilon q_2^2} \partial \tilde{L}_{23}, & \hat{Q}_{14} &= \frac{q_1 - q_2}{q_1^2 - \varepsilon q_2^2} \partial \tilde{L}_{22} + \frac{q_1 - \varepsilon q_2}{q_1^2 - \varepsilon q_2^2} \partial \tilde{L}_{24}, \\ \hat{Q}_{21} &= \partial \left(\frac{q_1 - q_2}{q_1^2 - \varepsilon q_2^2} \right) \tilde{L}_{13} + \partial \left(\frac{q_1 - \varepsilon q_2}{\varepsilon(q_1^2 - \varepsilon q_2^2)} \right) \tilde{L}_{11}, & \hat{Q}_{22} &= \partial \left(\frac{q_1 - q_2}{q_1^2 - \varepsilon q_2^2} \right) \tilde{L}_{14} + \partial \left(\frac{q_1 - \varepsilon q_2}{\varepsilon(q_1^2 - \varepsilon q_2^2)} \right) \tilde{L}_{12}, \\ \hat{Q}_{23} &= \partial \left(\frac{q_1 - q_2}{q_1^2 - \varepsilon q_2^2} \right) \tilde{L}_{11} + \partial \left(\frac{q_1 - \varepsilon q_2}{q_1^2 - \varepsilon q_2^2} \right) \tilde{L}_{13}, & \hat{Q}_{24} &= \partial \left(\frac{q_1 - q_2}{q_1^2 - \varepsilon q_2^2} \right) \tilde{L}_{12} + \partial \left(\frac{q_1 - \varepsilon q_2}{q_1^2 - \varepsilon q_2^2} \right) \tilde{L}_{14}. \end{aligned}$$

4. A Multi-Components Integrable TD Nonisospectral Hierarchy

In order to obtain the multi-components non-isospectral problem, we introduce the Lie algebra A_{1N} associated with the N -component non-isospectral problem.

First of all, a brief introduction to the Lie algebra A_{1N} related to the Lie algebra A will be given. Let us define a $N \times N$ square matrix:

$$M(m_1, m_2, \dots, m_N) = \begin{pmatrix} m_1 & \varepsilon m_N & \varepsilon m_{N-1} & \cdots & \varepsilon m_4 & \varepsilon m_3 & \varepsilon m_2 \\ m_2 & m_1 & \varepsilon m_N & \cdots & \varepsilon m_5 & \varepsilon m_4 & \varepsilon m_3 \\ m_3 & m_2 & m_1 & \cdots & \varepsilon m_6 & \varepsilon m_5 & \varepsilon m_4 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ m_{N-2} & m_{N-3} & m_{N-4} & \cdots & m_1 & \varepsilon m_N & \varepsilon m_{N-1} \\ m_{N-1} & m_{N-2} & m_{N-3} & \cdots & m_2 & m_1 & \varepsilon m_N \\ m_N & m_{N-1} & m_{N-2} & \cdots & m_3 & m_2 & m_1 \end{pmatrix} \quad (25)$$

$$=: [m_1, m_2, \dots, m_N]^T,$$

where $m_n (1 \leq n \leq N)$ represents n arbitrary square matrices of the same order.

For the arbitrary matrices $P = [P_1, P_2, \dots, P_N]^T, Q = [Q_1, Q_2, \dots, Q_N]^T$, one has the Lie product:

$$[P, Q] = \begin{pmatrix} R_1 & \varepsilon R_N & \varepsilon R_{N-1} & \cdots & \varepsilon R_4 & \varepsilon R_3 & \varepsilon R_2 \\ R_2 & R_1 & \varepsilon R_N & \cdots & \varepsilon R_5 & \varepsilon R_4 & \varepsilon R_3 \\ R_3 & R_2 & R_1 & \cdots & \varepsilon R_6 & \varepsilon R_5 & \varepsilon R_4 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ R_{N-2} & R_{N-3} & R_{N-4} & \cdots & R_1 & \varepsilon R_N & \varepsilon R_{N-1} \\ R_{N-1} & R_{N-2} & R_{N-3} & \cdots & R_2 & R_1 & \varepsilon R_N \\ R_N & R_{N-1} & R_{N-2} & \cdots & R_3 & R_2 & R_1 \end{pmatrix} =: [R_1, R_2, \dots, R_N]^T,$$

where

$$R_l = \sum_{\substack{k+j=l+1 \\ 1 \leq k, j \leq l}} [P_k, Q_j] + \sigma \varepsilon \sum_{\substack{s+t=l+1+N \\ l+1 \leq s, t \leq N}} [P_s, Q_t],$$

is subject to

$$\sigma = \begin{cases} 1, & 1 \leq l \leq N-1, \\ 0, & l = N. \end{cases}$$

Similarly, the matrix product PQ can be easily written as

$$PQ = [P_1 Q_1 + \sigma \varepsilon \sum_{\substack{s+t=2+N \\ 2 \leq s, t \leq N}} P_s Q_t, \sum_{\substack{k+j=3 \\ 1 \leq k, j \leq 2}} P_k Q_j + \sigma \varepsilon \sum_{\substack{s+t=3+N \\ 3 \leq s, t \leq N}} P_s Q_t, \\ \dots, \sum_{\substack{k+j=l \\ 1 \leq k, j \leq l}} P_k Q_j + \sigma \varepsilon \sum_{\substack{s+t=l+1+N \\ l+1 \leq s, t \leq N}} P_s Q_t, \dots, \sum_{\substack{k+j=l+1 \\ 1 \leq k, j \leq l}} P_k Q_j]^T.$$

When $\varepsilon = 0$, $[R_1, R_2, \dots, R_N]^T$ is a lower triangular matrix, $R_l = \sum_{\substack{k+j=l+1 \\ 1 \leq k, j \leq l}} [P_k, Q_j]$, and

$$PQ = [P_1 Q_1, \sum_{\substack{k+j=3 \\ 1 \leq k, j \leq 2}} P_k Q_j, \dots, \sum_{\substack{k+j=l \\ 1 \leq k, j \leq l}} P_k Q_j, \sum_{\substack{k+j=l+1 \\ 1 \leq k, j \leq N}} P_k Q_j]^T.$$

Secondly, we denote $A_{1N} = \text{span}\{\hat{h}_i\}_{i=1}^{4N}$, with commutators

$$\begin{aligned} \hat{h}_1 &= M(h_1, 0, \dots, 0), & \hat{h}_2 &= M(h_2, 0, \dots, 0), & \hat{h}_3 &= M(h_3, 0, \dots, 0), & \hat{h}_4 &= M(h_4, 0, \dots, 0), \\ \hat{h}_5 &= M(0, h_1, \dots, 0), & \hat{h}_6 &= M(0, h_2, \dots, 0), & \hat{h}_7 &= M(0, h_3, \dots, 0), & \hat{h}_8 &= M(0, h_4, \dots, 0), \\ & \dots \end{aligned}$$

$$\hat{h}_{4N-3} = M(0, 0, \dots, h_1), \quad \hat{h}_{4N-2} = M(0, 0, \dots, h_2),$$

$$\hat{h}_{4N-1} = M(0, 0, \dots, h_3), \quad \hat{h}_{4N} = M(0, 0, \dots, h_4),$$

$$[\hat{h}_{4l-3}, \hat{h}_{4k-2}] = [\hat{h}_{4l-3}, \hat{h}_{4k-3}] = [\hat{h}_{4l-2}, \hat{h}_{4k-2}] = [\hat{h}_{4l-1}, \hat{h}_{4k-1}] = [\hat{h}_{4l}, \hat{h}_{4k}] = 0, \quad l, k = 1, 2, \dots, N,$$

$$[\hat{h}_{4l-3}, \hat{h}_{4k-1}] = -[\hat{h}_{4l-2}, \hat{h}_{4k-1}] = \begin{cases} \hat{h}_{4(k+l-1)-1}, & 1 \leq k \leq N-l+1, \\ \varepsilon \hat{h}_{4(k+l-1-N)-1}, & N-l+2 \leq k \leq N, \end{cases}$$

$$[\hat{h}_{4l-3}, \hat{h}_{4k}] = -[\hat{h}_{4l-2}, \hat{h}_{4k}] = \begin{cases} -\hat{h}_{4(k+l-1)}, & 1 \leq k \leq N-l+1, \\ -\varepsilon \hat{h}_{4(k+l-1-N)}, & N-l+2 \leq k \leq N, \end{cases}$$

$$[\hat{h}_{4l-1}, \hat{h}_{4k}] = \begin{cases} \hat{h}_{4(k+l-1)-3} - \hat{h}_{4(k+l-1)-2}, & 1 \leq k \leq N-l+1, \\ \varepsilon (\hat{h}_{4(k+l-1-N)-3} - \hat{h}_{4(k+l-1-N)-2}), & N-l+2 \leq k \leq N. \end{cases}$$

Let $\hat{G}_k = \text{span}\{\hat{h}_{4k-3}, \hat{h}_{4k-2}, \hat{h}_{4k-1}, \hat{h}_{4k}\}$, $k = 1, 2, \dots, N$, then $A_{1N} = \hat{G}_1 \oplus \hat{G}_2 \oplus \dots \oplus \hat{G}_N$. Thus, we obtain

$$[\hat{G}_s, \hat{G}_t] \subseteq \hat{G}_{s+t-1-\delta N}, \quad \delta = \begin{cases} 0, & 2 \leq s+t \leq N+1, \\ 1, & N+2 \leq s+t \leq 2N, \end{cases} \quad s, t = 1, 2, \dots, N.$$

In terms of A_{1N} under the case of $\varepsilon = 0$, we introduce the following N-component non-isospectral problem:

$$\begin{cases} \psi_x = \hat{U}\psi, & \hat{U} = \sum_{l=1}^N [-\lambda \hat{h}_{4l-3} + \lambda \hat{h}_{4l-2} + r_l \hat{h}_{4l-3} + q_l (\hat{h}_{4l-1} + \hat{h}_{4l})], \\ \psi_t = \hat{V}\psi, & \hat{V} = \sum_{l=1}^N [a_l \hat{h}_{4l-3} - a_l \hat{h}_{4l-2} + b_l \hat{h}_{4l-1} + c_l \hat{h}_{4l}], \\ \lambda_t = \sum_{m \geq 0} k_m(t) \lambda^{-m}, \end{cases} \quad (26)$$

where $\hat{u} = (q_1, r_1, q_2, r_2, \dots, q_N, r_N)^T$, $\hat{U} = [\hat{U}_1, \hat{U}_2, \dots, \hat{U}_N]^T$, $\hat{V} = [\hat{V}_1, \hat{V}_2, \dots, \hat{V}_N]^T$ and

$$\hat{U}_l = -\lambda h_1 + \lambda h_2 + r_l h_1 + q_l (h_3 + h_4), \quad \hat{V}_l = a_l h_1 - a_l h_2 + b_l h_3 + c_l h_4.$$

By choosing

$$a_l = \sum_{m \geq 0} a_{lm} \lambda^{-m}, b_l = \sum_{m \geq 0} b_{lm} \lambda^{-m}, c_l = \sum_{m \geq 0} c_{lm} \lambda^{-m}, \quad (27)$$

and solving the stationary zero curvature equation

$$\hat{V}_x = \frac{\partial \hat{U}}{\partial \lambda} \lambda_t + [\hat{U}, \hat{V}], \quad (28)$$

we arrive at the following recursion equations:

$$\begin{cases} a_{lm,x} = \sum_{\substack{i+j=l+1 \\ 1 \leq i,j \leq l}} [q_i (c_{jm} - b_{jm})] - k_m(t), \\ b_{lm,x} = -2 \sum_{1 \leq p \leq l} b_{p,m+1}, \\ c_{lm,x} = 2 \sum_{1 \leq p \leq l} c_{p,m+1}, \end{cases} \quad (29)$$

which are equal to

$$\begin{cases} a_{lm} = \sum_{\substack{i+j=l+1 \\ 1 \leq i,j \leq l}} \partial^{-1} [q_i (c_{jm} - b_{jm})] - k_m(t)x, \\ b_{1,m+1} = \frac{1}{2} (r_1 b_{1m} - b_{1m,x} - 2q_1 a_{1m}), \\ b_{l,m+1} = \frac{1}{2} [b_{(l-1)m,x} - b_{lm,x} - 2q_l a_{1m} + r_l b_{1m} - \sum_{\substack{k+j=l+1 \\ 1 \leq k \leq l \\ 1 < j \leq l}} (2q_k (a_{jm} - a_{j-1,m}) - r_k (b_{jm} - b_{j-1,m}))], \\ c_{1,m+1} = \frac{1}{2} (c_{1m,x} - 2q_1 a_{1m} + r_1 c_{1m}), \\ c_{l,m+1} = \frac{1}{2} [c_{lm,x} - c_{(l-1)m,x} - 2q_l a_{1m} + r_l c_{1m} - \sum_{\substack{k+j=l+1 \\ 1 \leq k \leq l \\ 1 < j \leq l}} (2q_k (a_{jm} - a_{j-1,m}) - r_k (c_{jm} - c_{j-1,m}))]. \end{cases} \quad (30)$$

By choosing the initial data

$$a_{l0} = -1, \quad b_{l0} = 0, \quad c_{l0} = 0, \quad k_0 = 0, \quad l = 1, 2, \dots, N,$$

letting

$$a_{lk}|_{q_k=r_k=0} = 0, \quad b_{lk}|_{q_k=r_k=0} = 0, \quad c_{lk}|_{q_k=r_k=0} = 0, \quad l = 1, 2, \dots, N,$$

and using the recursion relations (30), we can obtain

$$\begin{aligned} b_{l1} &= q_l, \quad c_{l1} = q_l, \quad a_{l1} = -k_1(t)x, \\ b_{1,2} &= q_1 k_1(t)x + \frac{1}{2}(r_1 q_1 - q_{1,x}), \\ b_{l,2} &= q_l k_1(t)x - \frac{1}{2}[q_{l,x} - q_{l-1,x} - r_l q_1 - \sum_{\substack{k+j=l+1 \\ 1 \leq k < l \\ 1 \leq j \leq l}} r_k(q_j - q_{j-1})], \\ c_{1,2} &= q_1 k_1(t)x + \frac{1}{2}(r_1 q_1 + q_{1,x}), \\ c_{l,2} &= q_l k_1(t)x - \frac{1}{2}[q_{l-1,x} - q_{l,x} - r_l q_1 - \sum_{\substack{k+j=l+1 \\ 1 \leq k < l \\ 1 \leq j \leq l}} r_k(q_j - q_{j-1})], \\ a_{2l-1,2} &= \frac{1}{2}[q_l^2 - 2 \sum_{\substack{i+j=2l-1 \\ 1 \leq i \leq [\frac{2l-1}{2}]} q_i(q_j - q_{j+1})] - k_2(t)x, \\ a_{2l,2} &= \frac{1}{2}[-q_l^2 + q_{2l} q_1 - 2 \sum_{\substack{i+j=2l \\ l < i < 2l}} q_i(q_j - q_{j+1})] - k_2(t)x, \\ &\dots \end{aligned}$$

By denoting

$$\widehat{V}_+^{(n)} = \sum_{k=1}^N \sum_{m=0}^n (a_{km} \widehat{h}_{4k-1} \lambda^{n-m} + a_{km} \widehat{h}_{4k-2} \lambda^{n-m} + b_{km} \widehat{h}_{4k-3} \lambda^{n-m} + c_{km} \widehat{h}_{4k} \lambda^{n-m}),$$

one can find that

$$-\widehat{V}_{+,x}^{(n)} + \frac{\partial \widehat{U}}{\partial \lambda} \lambda_{+,t}^{(n)} + [\widehat{U}, \widehat{V}_+^{(n)}] = 2 \sum_{k=1}^N \left(\sum_{p=1}^k b_{p,n+1} \widehat{h}_{4k-1} - \sum_{p=1}^k c_{p,n+1} \widehat{h}_{4k} \right).$$

We choose the modification term as

$$\begin{aligned} \widehat{\Delta}_n &= q_1^{-1} \sum_{i=1}^N \left[\sum_{p=1}^i (b_{p,n+1} + c_{p,n+1}) \right] \widehat{h}_{4i-3} \\ &+ \sum_{k=2}^N (-1)^{k-1} q_1^{-k} \sum_{i=2}^N \left[\sum_{\substack{s_1+\dots+s_k=i+k-1 \\ s_l \geq 2, l=1, \dots, k-1 \\ s_k > 0}} [\sum_{p=1}^{s_k} (b_{p,n+1} + c_{p,n+1}) q_{s_1} q_{s_2} \dots q_{s_{k-1}}] \right] \widehat{h}_{4i-3}. \end{aligned}$$

By taking $\widehat{V}^{(n)} = \widehat{V}_+^{(n)} + \widehat{\Delta}_n$, the non-isospectral zero curvature equation

$$\widehat{U}_{t_n} + \frac{\partial \widehat{U}}{\partial \lambda} \lambda_t^{(n)} - \widehat{N}_x + [\widehat{U}, \widehat{V}^{(n)}] = 0$$

leads to the following N-component non-isospectral TD hierarchy:

$$\begin{aligned}
q_{j,t_n} &= \sum_{p=1}^j (c_{p,n+1} - b_{p,n+1}) = q_1^{-1} \left(\sum_{i=1}^j a_{in,x} + jk_n(t) \right) \\
&+ \sum_{i=2}^j (-1)^{i-1} q_1^{-i} \sum_{\substack{s_1+\dots+s_i=j+i-1 \\ s_l \geq 2, l=1, \dots, i-1 \\ s_i > 0}} \left[\sum_{p=1}^{s_i} a_{pn,x} q_{s_1} q_{s_2} \cdots q_{s_{i-1}} + s_i q_{s_1} q_{s_2} \cdots q_{s_{i-1}} k_n(t) \right], \\
r_{j,t_n} &= \sum_{i=2}^j (-1)^{i-1} \partial q_1^{-i} \sum_{\substack{s_1+\dots+s_i=j+i-1 \\ s_l \geq 2, l=1, \dots, i-1 \\ s_i > 0}} \left[\sum_{p=1}^{s_i} (b_{p,n+1} + c_{p,n+1}) q_{s_1} q_{s_2} \cdots q_{s_{i-1}} \right] \\
&+ \partial q_1^{-1} \sum_{i=1}^j (c_{i,n+1} + b_{i,n+1}), \quad j = 1, \dots, N.
\end{aligned} \tag{31}$$

Letting $N = n = 2$, system (31) is reduced to system (19), in which $\varepsilon = 0$. Under the case of $n = 2$, we can infer that the system is a new N-component non-isospectral TD system. Therefore, (31) is a N-component non-isospectral TD integrable hierarchy.

Bi-Hamiltonian Structures

In order to furnish the Hamiltonian structures of the N-component non-isospectral TD hierarchy (31), we will use the Z_N^ε -trace identity

$$\frac{\delta}{\partial \hat{u}} \int \sum_{\substack{k+j=l+1 \\ 1 \leq k, j \leq l}} \text{tr} \left(\hat{V}_k \frac{\partial \hat{U}_j}{\partial \lambda} \right) dx = \lambda^{-\gamma} \frac{\partial}{\partial \lambda} \lambda^\gamma \sum_{\substack{k+j=l+1 \\ 1 \leq k, j \leq l}} \text{tr} \left(\hat{V}_k \frac{\partial \hat{U}_j}{\partial \hat{u}} \right). \tag{32}$$

From (26), we have

$$\sum_{\substack{k+j=l+1 \\ 1 \leq k, j \leq l}} \text{tr} \left(\hat{V}_k \frac{\partial \hat{U}_j}{\partial \lambda} \right) = -2 \sum_{1 \leq p \leq l} a_p, \tag{33}$$

and

$$\sum_{\substack{k+j=l+1 \\ 1 \leq k, j \leq l}} \text{tr} \left(\hat{V}_k \frac{\partial \hat{U}_j}{\partial \hat{u}} \right) = \left(\begin{array}{c} \sum_{1 \leq p \leq l} (b_p + c_p) \\ \sum_{1 \leq p \leq l} a_p \end{array} \right). \tag{34}$$

By substituting (33) and (34) into (32), we have

$$\frac{\delta}{\partial \hat{u}} \int -2 \sum_{1 \leq p \leq l} a_p dx = \lambda^{-\gamma} \frac{\partial}{\partial \lambda} \lambda^\gamma \left(\begin{array}{c} \sum_{1 \leq p \leq l} (b_p + c_p) \\ \sum_{1 \leq p \leq l} a_p \end{array} \right). \tag{35}$$

By inserting (27) into (35) and comparing the coefficients of λ , we obtain

$$\frac{\delta}{\partial \hat{u}} \int -2 \sum_{1 \leq p \leq l} a_{p,m+1} dx = (\gamma - m) \left(\begin{array}{c} \sum_{1 \leq p \leq l} (b_{p,m} + c_{p,m}) \\ \sum_{1 \leq p \leq l} a_{p,m} \end{array} \right). \tag{36}$$

By fixing the constant $m = 1$ in the above equation, we conclude that $\gamma = 0$. Then, we obtain

$$\frac{\delta}{\partial \hat{u}} \int \frac{\sum_{1 \leq p \leq l} a_{p,m+2}}{m+1} dx = \frac{1}{2} \left(\frac{\sum_{1 \leq p \leq l} (b_{p,m+1} + c_{p,m+1})}{\sum_{1 \leq p \leq l} a_{p,m+1}} \right) = \frac{\delta \hat{H}_{l,m+1}}{\partial \hat{u}}. \quad (37)$$

Therefore,

$$\hat{H}_{l,m+1} = \int \frac{\sum_{1 \leq p \leq l} a_{p,m+2}}{m+1} dx, \quad l = 1, 2, \dots, N.$$

With the aid of (30) and (37), the Bi-Hamiltonian structures of hierarchy (31) are obtained as follows:

$$\begin{aligned} \hat{u}_{t_n} &= \begin{pmatrix} q_1 \\ r_1 \\ q_2 \\ r_2 \\ \vdots \\ q_N \\ r_N \end{pmatrix}_{t_n} = \begin{pmatrix} q_1^{-1}(a_{1n+1,x} + k_{n+1}(t)) \\ \partial[q_1^{-1}(c_{1,n+1} + b_{1,n+1})] \\ q_1^{-1}(a_{1n+1,x} + a_{2n+1,x} + 2k_{n+1}(t)) - \frac{q_2(a_{1n+1,x} + k_{n+1}(t))}{q_1^2} \\ \partial \left[q_1^{-1} \sum_{i=1}^2 (c_{i,n+1} + b_{i,n+1}) - \frac{\sum_{\substack{s_1+s_2=3 \\ s_1 \geq 2, s_2 > 0}} \sum_{p=1}^{s_2} (c_{p,n+1} + b_{p,n+1}) q_{s_1}}{q_1^2} \right] \\ \vdots \\ A \\ B \end{pmatrix} \\ &= \bar{\Phi}_1^n \bar{J}_1 \frac{\delta}{\partial \hat{u}} \begin{pmatrix} \bar{H}_{1,1} \\ \bar{H}_{2,1} \\ \bar{H}_{3,1} \\ \bar{H}_{4,1} \\ \vdots \\ \bar{H}_{N-1,1} \\ \bar{H}_{N,1} \end{pmatrix} + \sum_{m=0}^n \bar{\Phi}_1^{n-m} \bar{J}_1 (\bar{R}_1 k_n(t) + \bar{R}_0 k_{n+1}(t)) + \bar{R} k_{n+1}(t), \end{aligned} \quad (38)$$

where $\bar{J}_1$ is the Hamiltonian operator:

$$\bar{J}_1 = \begin{pmatrix} J_{11} & 0 & \cdots & 0 & 0 \\ J_{12} & J_{11} & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ J_{1,N-1} & J_{1,N-2} & \cdots & J_{13} & J_{12} \\ J_{1N} & J_{1,N-1} & \cdots & J_{12} & J_{11} \end{pmatrix},$$

$$J_{11} = \begin{pmatrix} 0 & q_1^{-1} \partial \\ \partial q_1^{-1} & 0 \end{pmatrix}, J_{12} = \begin{pmatrix} 0 & -\frac{q_2 \partial}{q_1^2} \\ -\frac{\partial q_2}{q_1^2} & 0 \end{pmatrix}, \quad J_{1l} = \begin{pmatrix} 0 & C \partial \\ \partial C & 0 \end{pmatrix}, \quad l = 3, \dots, N,$$

$$C = -\frac{q_l}{q_1^2} + \sum_{i=3}^l (-1)^{i-1} q_1^{-i} \sum_{\substack{s_1+\dots+s_{i-1}=l+i-2 \\ s_j \geq 2, j=1, \dots, i-1}} q_{s_1} q_{s_2} \cdots q_{s_{i-1}},$$

$$\bar{R} = \begin{pmatrix} q_1^{-1} & & & & \\ 0 & & & & \\ 2q_1^{-1} - \frac{q_2}{q_1^2} & & & & \\ 0 & & & & \\ \vdots & & & & \\ Nq_1^{-1} + \sum_{i=2}^N (-1)^{i-1} q_1^{-i} & \sum_{\substack{s_1+\dots+s_i=N+i-1 \\ s_l \geq 2, l=1, \dots, i-1 \\ s_i > 0}} s_i q_{s_1} q_{s_2} \cdots q_{s_{i-1}} & & & \\ 0 & & & & \end{pmatrix},$$

$$\bar{Q}_1 = \bar{J}_1 \bar{L}_1 \text{ and } \bar{\Phi}_1 = \bar{J}_1 \bar{L}_1 \bar{J}_1^{-1}.$$

5. Conclusions and Discussion

In this paper, we consider three kinds of Lie algebras: classical Lie algebras, A , the enlarged Lie algebras, A_{12} , and Lie algebras, A_{1N} , which are associated with the N -component non-isospectral problem. Based on these Lie algebras, we derive three classes of non-isospectral TD hierarchies, and the corresponding Bi-Hamiltonian structures are established by using trace identity. It should be noted that the first nontrivial system in the first kind of TD hierarchy can be reduced to the TD equation, and the second nontrivial member can be reduced to the mKdV equation. We find that the second kind of integrable coupling of the TD hierarchy can be reduced to the integrable coupling of TD equations and the coupled mKdV equations. The method we used in this paper can be applied to generate other multi-component non-isospectral integrable hierarchies in the future.

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