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A Generalization of the First Tits Construction

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Abstract: Let F be a field of characteristic, not 2 or 3. The first Tits construction is a well-known tripling process to construct separable cubic Jordan algebras, especially Albert algebras. We generalize the first Tits construction by choosing the scalar employed in the tripling process outside of the base field. This yields a new family of non-associative unital algebras which carry a cubic map, and maps that can be viewed as generalized adjoint and generalized trace maps. These maps display properties often similar to the ones in the classical setup. In particular, the cubic norm map permits some kind of weak Jordan composition law.

Keywords: non-associative algebras; first Tits construction; Jordan algebras; generalized cubic algebras

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1. Introduction

Let F be a field of characteristic, not 2 or 3. Separable cubic Jordan algebras over F play an important role in Jordan theory (where separable means that their trace defines a non-degenerate bilinear form). It is well known that every separable cubic Jordan algebra can be obtained by either a first or a second Tits construction [1] (IX, Section 39). In particular, exceptional simple Jordan algebras, also called Albert algebras, are separable cubic Jordan algebras. The role of Albert algebras in the structure theory of Jordan algebras is similar to the role of octonion algebras in the structure theory of alternative algebras. Moreover, their automorphism group is an exceptional algebraic group of type F_4, and their cubic norms have isometry groups of type E_6. For some recent developments, see [2–6].

In this paper, we canonically generalize the first Tits construction $J(A,\mu)$. The first Tits construction starts with a separable associative cubic algebra A and uses a scalar $\mu \in F^\times$ in its definition. Our construction also starts with A and employs the same algebra multiplication as that used for the classical first Tits construction, but now allows also $\mu \in A^\times$.

We obtain a new class of non-associative unital algebras which we again denote by $J(A,\mu)$. They carry a cubic map $N : J(A,\mu) \to A$ that generalizes the classical norm, a map $T : J(A,\mu) \to F$ that generalizes the classical trace, and a map $\sharp : J(A,\mu) \to J(A,\mu)$ that generalizes the classical adjoint of a Jordan algebra. Starting with a cubic étale algebra E, the algebras obtained this way can be viewed as generalizations of special nine-dimensional Jordan algebras. Starting with a central simple algebra A of degree three, the algebras obtained this way can be viewed as generalizations of Albert algebras.

Cubic Jordan algebras carry a cubic norm that satisfies some Jordan composition law involving the U-operator. Curiously, the cubic map $N : J(A,\mu) \to A$ of our generalized construction still allows some sort of generalized weak Jordan composition law, and some of the known identities of cubic Jordan algebras involving a generalized trace map and adjoint can be at least partially recovered.

We point out that there already exists a canonical non-associative generalization of associative central simple cyclic algebras of degree three, involving skew polynomials: the non-associative cyclic algebras $(K/F, \sigma, \mu)$, where K/F is a cubic separable field extension or a C_3-Galois algebra, and $\mu \in K \setminus F$, were first studied over finite fields [7], and then later over arbitrary base fields and rings [8–11] and applied in space-time block coding [12,13]. Their "norm maps" reflect some of the properties of the non-associative cyclic algebra $(K/F, \sigma, \mu)$ and are isometric to the "norm maps" $N : J(K, \mu) \to K$ of the generalized Tits construction $J(K, \mu)$. We show that these algebras are not related, however.

Some obvious questions like if and when the algebras obtained through a generalized first Tits construction are division algebras seem to be very difficult to answer. We will not address these here and only discuss some straightforward implications.

The contents of the paper are as follows: After introducing the terminology in Section 2 and reviewing the classical first Tits construction, we generalize the classical construction in Section 3 and obtain unital non-associative algebras $J(A, \mu)$, where $\mu \in A^\times$. The algebras $J(A, \mu)$ carry maps that satisfy some of the same identities we know from the classical setup. If $A \neq F$, then $\mathrm{Nuc}_l(J(A, \mu)) = \mathrm{Nuc}_r(J(A, \mu)) = F$ for all $\mu \in A^\times$. If A is a central simple associative division algebra of degree three, then $\mathrm{Nuc}_m(J(A, \mu)) = F$ (Theorems 3 and 4). Some necessary conditions on when $J(A, \mu)$ is a division algebra are listed in Theorem 6: If $J(A, \mu)$ is a division algebra, then $\mu \notin N_A(A^\times)$ and A must be a division algebra. If N is anisotropic, then A is a division algebra and $\mu \notin N_A(A^\times)$. If there exist elements $0 \neq x = (x_0, x_1, x_2) \in J(A, \mu)$ such that $x^\sharp = 0$, we show that either A must have zero divisors, or A is a division algebra and $\mu \in N_A(A^\times)$. Moreover, if A is a division algebra over F and 1, μ, and μ^2 are linearly independent over F, then N must be anisotropic.

We investigate in which special cases several classical identities carry over in Section 4.

In Section 5, we compare the algebras obtained from a generalized first Tits construction starting with a cyclic field extension with the algebras $(K/F, \sigma, \mu)^+$, where $(K/F, \sigma, \mu)$ is a non-associative cyclic algebra over F of degree three. If $\mu \in F^\times$, then it is well known that these algebras are isomorphic. For $\mu \in K \setminus F$, they are not isomorphic, but their norms are isometric.

This construction was briefly investigated for the first time in Andrew Steele's PhD thesis [11]. We improved and corrected most of their results, and added many new ones.

2. Preliminaries

2.1. Non-Associative Algebras

Throughout the paper, F is a field of characteristic, not 2 or 3. An *algebra* over F is an F-vector space A together with an F-bilinear map $A \times A \to A$, $(x, y) \mapsto x \cdot y$, denoted simply by juxtaposition of xy, the *multiplication* of A. An algebra A is *unital* if there exists an element in A, denoted by 1, such that $1x = x1 = x$ for all $x \in A$.

A non-associative algebra $A \neq 0$ is called a *division algebra* if for any $a \in A$, $a \neq 0$, the left multiplication with a, $L_a(x) = ax$, and the right multiplication with a, $R_a(x) = xa$, are bijective. We will only consider unital finite-dimensional algebras, which implies that A is a division algebra if and only if A has no zero divisors. Define $[x, y, z] = (xy)z - x(yz)$. The subalgebras $\mathrm{Nuc}_l(A) = \{x \in A \mid [x, A, A] = 0\}$, $\mathrm{Nuc}_m(A) = \{x \in A \mid [A, x, A] = 0\}$, and $\mathrm{Nuc}_r(A) = \{x \in A \mid [A, A, x] = 0\}$ of A are called the *left*, *middle*, and *right nuclei* of A, $\mathrm{Nuc}(A) = \{x \in A \mid [x, A, A] = [A, x, A] = [A, A, x] = 0\}$ is called the *nucleus* of A. The *center* of A is defined as $C(A) = \{x \in A \mid xy = yx \text{ for all } y \in A\} \cap \mathrm{Nuc}(A)$ [14]. All algebras we consider will be unital.

A non-associative unital algebra J is called a *cubic Jordan algebra* over F, if J is a Jordan algebra, i.e., $xy = yx$ and $(x^2y)x = x^2(yx)$ for all $x, y \in J$, and if its generic minimal polynomial has degree three. Given an associative algebra A over F, its multiplication written simply by juxtaposition, we can define a Jordan algebra over F denoted by A^+ on the F-vector space underlying the algebra A via $x \cdot y = \frac{1}{2}(xy + yx)$. A Jordan algebra J is called *special*, if it is a subalgebra of A^+ for some associative algebra A over F; otherwise, J is *exceptional*. An exceptional Jordan algebra is called an *Albert algebra*.

The following easy observation is included for the sake of the reader:

Lemma 1. *Let A be an associative algebra over F such that A^+ is a division algebra. Then, A is a divison algebra.*

Proof. Suppose that $xy = 0$ for some $x, y \in A$. Then, $(yx) \cdot (yx) = y(xy)x = 0$, and since A^+ is a division algebra, we obtain $yx = 0$. This implies that $x \cdot y = \frac{1}{2}(xy + yx) = 0$. Using again that A^+ is a division algebra, we deduce that $x = 0$ or $y = 0$. $\square$

A *non-associative cyclic algebra* $(K/F, \sigma, c)$ of degree m over F is an m-dimensional K-vector space $(K/F, \sigma, c) = K \oplus Kz \oplus Kz^2 \oplus \cdots \oplus Kz^{m-1}$, with multiplication given by the relations $z^m = c$, $zl = \sigma(l)z$ for all $l \in K$. The algebra $(K/F, \sigma, c)$ is a unital F-central algebra and associative if and only if $c \in F^\times$. The algebra $(K/F, \sigma, c)$ is a division algebra for all $c \in F^\times$, such that $c^s \notin N_{K/F}(K^\times)$ for all s which are prime divisors of m, $1 \leq s \leq m-1$. If $c \in K \setminus F$, then $(K/F, \sigma, c)$ is a division algebra for all $c \in K \setminus F$ such that $1, c, \ldots, c^{m-1}$ are linearly independent over F [10]. If m is prime, then $(K/F, \sigma, c)$ is a division algebra for all $c \in K \setminus F$.

2.2. Cubic Maps

Let V and W be two finite-dimensional vector spaces over F. A trilinear map $M : V \times V \times V \to W$ is called *symmetric* if $M(x, y, z)$ is invariant under all permutations of its variables. A map $M : V \to W$ is called a *cubic map* over F, if $M(ax) = a^3 M(x)$ for all $a \in F$, $x \in V$, and if the associated map $M : V \times V \times V \to W$ defined by

$$M(x,y,z) = \frac{1}{6}(M(x+y+z) - M(x+y) - M(x+z) - M(y+z) + M(x) + M(y) + M(z))$$

is a (symmetric) F-trilinear map. We canonically identify symmetric trilinear maps $M : V \times V \times V \to W$ with the corresponding cubic maps $M : V \to W$.

A cubic map $M : V \to F$ is called a *cubic form* and a trilinear map $M : V \times V \times V \to F$ a *trilinear form* over F. A cubic map is called *non-degenerate* if $v = 0$ is the only vector such that $M(v, v_2, , v_3) = 0$ for all $v_i \in V$. A cubic map M is called *anisotropic* if $M(x) = 0$ implies that $x = 0$; otherwise, it is *isotropic*. For a non-associative algebra A over F together with a non-degenerate cubic form $M : A \to F$, M is called *multiplicative*, if $M(vw) = M(v)M(w)$ for all $v, w \in A$.

2.3. Associative Cubic Algebras

(cf. for Instance [1,15] (Chapter C.4)) Let A be a unital separable associative algebra over F with cubic norm $N_A : A \to F$. Let $x, y \in A$ and let Z be an indeterminate. The linearization $N_A(x + Zy) = N_A(x) + ZN_A(x; y) + Z^2 N_A(y; x) + Z^3 N_A(y)$ of N_A, i.e., the coefficient of Z in the above expansion, is quadratic in x and linear in y, and is denoted by $N_A(x; y)$. Indeed, we have

$$N_A(x + Zx) = N_A((1 + Z)x) = (1 + Z)^3 N_A(x) = (1 + 3Z + 3Z^2 + Z^3)N_A(x),$$

so $N_A(1; 1) = 3N_A(1) = 3$. Linearize $N_A(x; y)$ to obtain a symmetric trilinear map $N_A : A \times A \times A \to F$, $N_A(x, y, z) = N_A(x + z; y) - N_A(x; y) - N_A(z; y)$. We define

$$\begin{aligned}
T_A(x) &= N_A(1; x), \\
T_A(x, y) &= T_A(x)T_A(y) - N_A(1, x, y), \\
S_A(x) &= N_A(x; 1), \\
x^\sharp &= x^2 - T_A(x)x + S_A(x)1,
\end{aligned}$$

for all $x, y \in A$. We call $x^\sharp$ the *adjoint* of x, and define the *sharp map* $\sharp : A \times A \to A$, $x \sharp y = (x+y)^\sharp - x^\sharp - y^\sharp$ as the linearization of the adjoint. We observe that $T_A(1) = S_A(1) = 3$. Since the trilinear map $N_A(x,y,z)$ is symmetric,

$$T_A(x,y) = T_A(y,x) \tag{1}$$

for all $x, y \in A$.

The algebra A is called an (associative) *cubic algebra* (respectively, called an algebra *of degree three* in [15] (p. 490)), if the following three axioms are satisfied for all $x, y \in A$:

$$x^3 - T_A(x)x^2 + S_A(x)x - N_A(x)1 = 0 \text{ (\textit{degree 3 identity})}, \tag{2}$$

$$T_A(x^\sharp, y) = N_A(x; y) \text{ (\textit{trace-sharp formula})}, \tag{3}$$

$$T_A(x,y) = T_A(xy) \text{ (\textit{trace-product formula})}. \tag{4}$$

For the rest of Section 2.3, we assume that A is a separable cubic algebra over F with cubic norm $N_A : A \to F$. Note that (2) is equivalent to the condition that

$$xx^\sharp = x^\sharp x = N_A(x)1, \tag{5}$$

and combining (1) with (4) gives

$$T_A(xy) = T_A(yx). \tag{6}$$

An element $x \in A$ is invertible if and only if $N_A(x) \neq 0$. The inverse of $x \in A$ is $x^{-1} = N_A(x)^{-1}x^\sharp$. It can be shown that

$$(xy)^\sharp = y^\sharp x^\sharp \tag{7}$$

for all $x, y \in A$. Notice that

$$T_A(x^\sharp) = T_A(x^\sharp, 1) = N_A(x; 1) = S_A(x), \tag{8}$$

using (3) and (4). We also have $S_A(x) = T_A(x^\sharp) = T_A(x^2) - T_A(x)^2 + 3S_A(x)$, so

$$2S_A(x) = T_A(x)^2 - T_A(x^2). \tag{9}$$

A straightforward calculation shows that

$$x \sharp y = 2(x \cdot y) - T_A(x)y - T_A(y)x + (T_A(x)T_A(y) - T_A(x \cdot y))1 \tag{10}$$

for all $x, y \in A$. In particular,

$$x \cdot y = \frac{1}{2}(xy + yx) = \frac{1}{2}(x \sharp y + T_A(x)y + T_A(y)x - (T_A(x)T_A(y) - T_A(x,y))1)$$

for all $x, y \in A$ and by employing (5) and the adjoint identity in A, we see that the norm N_A satisfies the relation

$$N_A(x^\sharp) = N_A(x)^2. \tag{11}$$

A^+ satisfies the *adjoint identity*

$$(x^\sharp)^\sharp = N_A(x)x \tag{12}$$

for all $x \in A$. By (11), we have $N_A(x^\sharp)1 = x^\sharp(x^\sharp)^\sharp = x^\sharp N_A(x)x = N_A(x)^2 1$. For $x, y \in A$, we define the operators $U_x : A \to A$, $U_x(y) = T_A(x,y)x - x^\sharp \sharp y$ and $U_{x,y} : A \to A$, $U_{x,y}(z) = U_{x+y}(z) - U_x(z) - U_y(z)$. Then, we have $x \cdot y = \frac{1}{2}U_{x,y}(1)$ for all $x, y \in A$ and

$$xyx = T_A(x,y)x - x^\sharp \sharp y, \tag{13}$$

Hence, $U_x(y) = xyx$ for all $x, y \in A^\times$.

Define
$$x \times y = \frac{1}{2}(x \sharp y),$$
and
$$\bar{x} = \frac{1}{2}(T_A(x)1 - x)$$
for $x, y \in A$. (Note that some literature does not include the factor $\frac{1}{2}$ in the definition of $\times$, e.g., [16]). By (10), we then have
$$x \times y = x \cdot y - \frac{1}{2}T_A(x)y - \frac{1}{2}T_A(y)x + \frac{1}{2}(T_A(x)T_A(y) - T_A(x \cdot y))1$$
for all $x, y \in A$; hence,
$$x \times x = x^2 - T_A(x)x + \frac{1}{2}(T_A(x)^2 - T_A(x^2)) = x^\sharp, \tag{14}$$
using (9).

2.4. The First Tits Construction

Let A be a separable cubic associative algebra over F with norm N_A, trace T_A and adjoint map $\sharp$. Let $\mu \in F^\times$ and define the F-vector space $J = J(A, \mu) = A_0 \oplus A_1 \oplus A_2$, where $A_i = A$ for $i = 0, 1, 2$. Then, $J(A, \mu)$ together with the multiplication
$$\begin{aligned}&(x_0, x_1, x_2)(y_0, y_1, y_2)\\ &= (x_0 \cdot y_0 + \overline{x_1 y_2} + \overline{y_1 x_2}, \overline{x_0} y_1 + \overline{y_0} x_1 + \mu^{-1}(x_2 \times y_2), x_2 \overline{y_0} + y_2 \overline{x_0} + \mu(x_1 \times y_1))\end{aligned}$$
becomes a separable cubic Jordan algebra over F. $J(A, \mu)$ is called a *first Tits construction*. A^+ is a subalgebra of $J(A, \mu)$ by canonically identifying it with A_0. If A is a cubic etale algebra, then $J(A, \mu) \cong D^+$ for with D an associative cyclic algebra D of degree three. If A is a central simple algebra of degree three then $J(A, \mu)$ is an Albert algebra.

We define the *cubic norm form* $N : J(A, \mu) \to F$, the *trace* $T : J(A, \mu) \to F$, and the quadratic map $\sharp : J(A, \mu) \to J(A, \mu)$ (the *adjoint*) by
$$\begin{aligned}N((x_0, x_1, x_2)) &= N_A(x_0) + \mu N_A(x_1) + \mu^{-1} N_A(x_2) - T_A(x_0 x_1 x_2)\\ T((x_0, x_1, x_2)) &= T_A(x_0),\\ (x_0, x_1, x_2)^\sharp &= (x_0^\sharp - x_1 x_2, \mu^{-1} x_2^\sharp - x_0 x_1, \mu x_1^\sharp - x_2 x_0).\end{aligned}$$

The *intermediate quadratic form* $S : J(A, \mu) \to F$, $S(x_0) = N(x; 1)$, linearizes to a map $S : J(A, \mu) \times J(A, \mu) \to F$. The *sharp map* $\sharp : J(A, \mu) \times J(A, \mu) \to J(A, \mu)$ is the linearization $x \sharp y = (x + y)^\sharp - x^\sharp - y^\sharp$ of the adjoint. For every $x = (x_0, x_1, x_2) \in J(A, \mu)$, we have $x \sharp 1 = T(x)1 - x$ and
$$x \sharp y = (x_0 \sharp y_0 - x_1 y_2 - y_1 x_2, \mu^{-1}(x_2 \sharp y_2) - x_0 y_1 - y_0 x_1, \mu(x_1 \sharp y_1) - x_2 y_0 - y_2 x_0)$$
for all $x = (x_0, x_1, x_2), y = (y_0, y_1, y_2) \in J(A, \mu)$. We define the *trace symmetric bilinear form* $T : J(A, \mu) \times J(A, \mu) \to F$, $T(x, y) = T_A(x_0 y_0) + T_A(x_1 y_2) + T_A(x_2 y_1)$. Then, for all $x, y \in J(A, \mu)$, we have
$$T(x, y) = T(xy). \tag{15}$$

Remark 1. $(N, \sharp, 1)$ *is a cubic form with adjoint and base point* $(1, 0, 0)$ *on* $J(A, \mu)$ *which makes* $J(A, \mu)$ *into a cubic Jordan algebra* $J(N, \sharp, 1)$.

3. The Generalized First Tits Construction $J(A,\mu)$

Let A be a separable associative cubic algebra over F with norm N_A, trace T_A and adjoint map $\sharp$.

We now generalize the first Tits construction by choosing the scalar $\mu \in A^\times$. Then, the F-vector space $J(A,\mu) = A_0 \oplus A_1 \oplus A_2$, where again $A_i = A$ for $i = 0,1,2$, becomes a unital non-associative algebra over F together with the multiplication given by

$$\begin{aligned}&(x_0,x_1,x_2)(y_0,y_1,y_2)\\&= (x_0 \cdot y_0 + \overline{x_1 y_2} + \overline{y_1 x_2}, \overline{x_0} y_1 + \overline{y_0} x_1 + \mu^{-1}(x_2 \times y_2), x_2\overline{y_0} + y_2\overline{x_0} + \mu(x_1 \times y_1)).\end{aligned}$$

The algebra $J(A,\mu)$ is called a *generalized first Tits construction*. The special Jordan algebra A^+ is a subalgebra of $J(A,\mu)$ by canonically identifying it with A_0. If $\mu \in F^\times$, then $J(A,\mu)$ is the first Tits construction from Section 2.4.

We define a *(generalized) cubic norm map* $N : J(A,\mu) \to A$, a *(generalized) trace* $T : J(A,\mu) \to F$, and a quadratic map $\sharp : J(A,\mu) \to J(A,\mu)$ via

$$N((x_0,x_1,x_2)) = N_A(x_0) + \mu N_A(x_1) + \mu^{-1} N_A(x_2) - T_A(x_0 x_1 x_2) \tag{16}$$

$$T((x_0,x_1,x_2)) = T_A(x_0), \tag{17}$$

$$(x_0,x_1,x_2)^\sharp = (x_0^\sharp - x_1 x_2, \mu^{-1} x_2^\sharp - x_0 x_1, \mu x_1^\sharp - x_2 x_0). \tag{18}$$

Put $\sharp : J(A,\mu) \times J(A,\mu) \to J(A,\mu)$, $x \sharp y = (x+y)^\sharp - x^\sharp - y^\sharp$; then, it can be verified by a direct computation that

$$x \sharp y = (x_0 \sharp y_0 - x_1 y_2 - y_1 x_2, \mu^{-1}(x_2 \sharp y_2) - x_0 y_1 - y_0 x_1, \mu(x_1 \sharp y_1) - x_2 y_0 - y_2 x_0)$$

for all $x = (x_0,x_1,x_2), y = (y_0,y_1,y_2) \in J(A,\mu)$. We also define a symmetric F-bilinear form $T : J(A,\mu) \times J(A,\mu) \to F$ via $T(x,y) = T_A(x_0 y_0) + T_A(x_1 y_2) + T_A(x_2 y_1)$.

The quadratic form $S_A : A \to F$, $S_A(x_0) = N_A(x;1)$, linearizes to $S_A : A \times A \to F$, and we have $S_A(x_0) = T_A(x_0^\sharp)$ for all $x_0 \in A$. We extend S_A to $J(A,\mu)$ by defining the quadratic map $S : J(A,a) \to A$, $S(x) = N(x;1)$. As in the classical case, we obtain:

Theorem 1.

(i) [11] *(Proposition 5.2.2) For all $x \in J(A,\mu)$, we have $S(x) = T(x^\sharp)$ and the linearization $S : J(A,\mu) \times J(A,\mu) \to A$ satisfies*

$$S(x,y) = T(x)T(y) - T(x,y)$$

for all $y \in J(A,\mu)$.

(ii) [11] *(Lemma 5.2.3) For all $x,y \in J(A,\mu)$, we have $T(x,Y) = T(xy)$.*

(iii) [11] *(Lemma 5.2.3) For all $x \in J(A,\mu)$, we have $x \sharp 1 = T(x)1 - x$.*

Proof.

(i) Let $x = (x_0,x_1,x_2), y = (y_0,y_1,y_2) \in J(A,a)$, then

$$\begin{aligned}N(x;y) =& N_A(x_0;y_0) + \mu N_A(x_1;y_1) + \mu^{-1} N_A(x_2;y_2)\\&- T_A(x_0 x_1 y_2) - T_A(x_0 y_1 x_2) - T_A(y_0 x_1 x_2),\end{aligned}$$

and since $S(x) = N(x;1)$ we obtain $S(x) = N_A(x_0;1) - T_A(x_1 x_2) = S_A(x_0) - T_A(x_1 x_2)$. On the other hand,

$$T(x^\sharp) = T_A(x_0^\sharp - x_1 x_2) = T_A(x_0^\sharp) - T_A(x_1 x_2) = S_A(x_0) - T_A(x_1 x_2) = S(x).$$

We have $S_A(x_0,y_0) = T_A(x_0)T_A(y_0) - T_A(x_0,y_0)$ for all $x_0,y_0 \in A$. Linearizing S gives $S(x,y) = S_A(x_0,y_0) - T_A(x_1 y_2) - T_A(y_1 x_2) = T_A(x_0)T_A(y_0) - T_A(x_0,y_0) -$

$T_A(x_1y_2) - T_A(y_1x_2) = T(x)T(y) - T(x,y)$ using the definitions of $T_A(x_i)$ and $T_A(x_i, y_i)$ and the fact that $T_A(x_0, y_0) = T_A(x_0y_0)$.

(ii) Let $x = (x_0, x_1, x_2), y = (y_0, y_1, y_2) \in J(A, \mu)$. Since T_A is linear, we obtain

$$\begin{aligned} T(xy) &= T_A(x_0 \cdot y_0) + T_A(\overline{x_1y_2}) + T_A(\overline{y_1x_2}) \\ &= \frac{1}{2}(T_A(x_0y_0) + T_A(y_0x_0)) + \frac{1}{2}(T_A(x_1y_2)T_A(1) - T_A(x_1y_2)) \\ &\quad + \frac{1}{2}(T_A(y_1x_2)T_A(1) - T_A(y_1x_2)). \end{aligned}$$

By (6) we obtain $T_A(x_0y_0) = T_A(y_0x_0)$ and $T_A(y_1x_2) = T_A(x_2y_1)$. Since we have $T_A(1) = 3$ we obtain $T(xy) = T_A(x_0y_0) + T_A(x_1y_2) + T_A(x_2y_1) = T(x, y)$.

(iii) Let $x = (x_0, x_1, x_2) \in J(A, \mu)$. By (10), we have $x_0 \sharp 1 = T_A(x_0)1 - x_0$. Thus, $x \sharp 1 = (x_0 \sharp 1, -x_1, -x_2) = T(x)1 - x$.

□

Theorem 2. *Let $\mu \in A^\times$, and let $x, y \in J(A, \mu)$. Then,*

(i) $x^\sharp = x^2 - T(x)x + S(x)1$,

(ii) $S(x) = T(x^\sharp)$,

(iii) $T(x \times y) = \frac{1}{2}(T(x)T(y) - T(xy))$.

Note that these are relations that also hold for a cubic form with adjoint and base point $(N, \sharp, 1)$ [15,17].

Proof. Let $x = (x_0, x_1, x_2), y = (y_0, y_1, y_2) \in J(A, \mu)$.

(i) We have that $x^2 - T(x)x + S(x)1 = (x_0, x_1, x_2)^2 - T_A(x_0)(x_0, x_1, x_2) + (S_A(x_0)1 - T_A(x_1x_2)1, 0, 0) = (x_0^2 - T_A(x_0)x_0 + S_A(x_0)1 + 2\overline{x_1x_2} - T_A(x_1x_2)1, \mu^{-1}x_2^\sharp + 2\overline{x_0}x_1 - T_A(x_0)x_1, \mu x_1^\sharp + 2x_2\overline{x_0} - T_A(x_0)x_2) = (x_0^\sharp - x_1x_2, \mu^{-1}x_2^\sharp - x_0x_1, \mu x_1^\sharp - x_2\overline{x_0}) = x^\sharp$.

(ii) As for the classical construction,

$$T(x^\sharp) = T_A(x_0^\sharp - x_1x_2) = T_A(x_0^\sharp) - T_A(x_1x_2) = S_A(x_0) - T_A(x_1x_2) = S(x).$$

(iii) Since $x \times y = \frac{1}{2}(x \sharp y) = \frac{1}{2}(x_0 \sharp y_0 - x_1y_2 - y_1x_2, \mu^{-1}(x_2 \sharp y_2) - x_0y_1 - y_0x_1, \mu(x_1 \sharp y_1) - x_2y_0 - y_2x_0)$, we obtain $T(x \times y) = T_A(x_0 \times y_0) - \frac{1}{2}T_A(x_1y_2) - \frac{1}{2}T_A(y_1x_2) = \frac{1}{2}(T_A(x_0)T_A(y_0) - T_A(x_0y_0) - T_A(x_1y_2) - T_A(y_1x_2)) = \frac{1}{2}(T(x)T(y) - T(xy))$.

□

Define operators $U_x, U_{x,y} : J(A, \mu) \to J(A, \mu)$ via

$$U_x(y) = T(x, y)x - x^\sharp \sharp y, \quad U_{x,y}(z) = U_{x+y}(z) - U_x(z) - U_y(z)$$

for all $z \in J(A, \mu)$.

Proposition 1. *(cf. [11] (Proposition 5.2.4) without factor $\frac{1}{2}$ because of slightly different terminology) For all $x, y \in J(A, \mu)$, we have $xy = \frac{1}{2}U_{x,y}(1)$.*

This generalizes the classical setup. Our proof is different to the one of [11] (Proposition 5.2.4), which also proves this result without the factor $\frac{1}{2}$ because of the slightly different definition of the multiplication.

Proof. We find that $U_x(1) = T(x,1)x - x^\sharp \sharp 1 = T(x)x - T(x^\sharp)1 + x^\sharp$; in the second equality, we have used Theorem 1 and the fact that $T(x,1) = T(x)$ by Theorem 1. So

$$\begin{aligned} U_{x,y}(1) &= U_{x+y}(1) - U_x(1) - U_y(1) \\ &= T(x+y)(x+y) - T((x+y)^\sharp)1 + (x+y)^\sharp - T(x)x + T(x^\sharp)1 - x^\sharp \\ &\quad - T(y)y + T(y^\sharp)1 - y^\sharp \\ &= T(x)y + T(y)x + x\sharp y - T(x\sharp y)1. \end{aligned}$$

We look at the first component of xy and $U_{x,y}(1)$: let $x = (x_0, x_1, x_2)$ and $y = (y_0, y_1, y_2)$. Then, the first component of $U_{x,y}(1) = T(x)y + T(y)x + x\sharp y - T(x\sharp y)1$ is

$$T_A(x_0)y_0 + T_A(y_0)x_0 + x_0\sharp y_0 - x_1y_2 - y_1x_2 - T_A(x_0\sharp y_0 - x_1y_2 - y_1x_2)1. \tag{19}$$

Using (10), the linearity of T_A and (6), we obtain—after some simplification—that (19) is equal to

$$2(x_0 \cdot y_0) + T_A(x_1y_2) - x_1y_2 + T_A(y_1x_2) - y_1x_2 = 2(x_0 \cdot y_0) + 2\,\overline{x_1y_2} + 2\,\overline{y_1x_2}.$$

This is equal to 2 times the first component of xy. Now, we look at the second component of xy and $U_{x,y}(1)$: the second component of $U_{x,y}(1) = T(x)y + T(y)x + x\sharp y - T(x\sharp y)1$ is

$$T_A(x_0)y_1 + T_A(y_0)x_1 + \mu^{-1}(x_2\sharp y_2) - x_0y_1 - y_0x_1 = 2\overline{x_0}y_1 + 2\overline{y_0}x_1 + 2\mu^{-1}(x_2 \times y_2).$$

This is precisely equal to 2 times the second component of xy. Finally, the third component of $2xy$ and $U_{x,y}(1)$ are equal, too. The third component of $U_{x,y}(1) = T(x)y + T(y)x + x\sharp y - T(x\sharp y)1$ is

$$T_A(x_0)y_2 + T_A(y_0)x_2 + \mu(x_1\sharp y_1) - x_2y_0 - y_2x_0 = 2x_2\overline{y_0} + 2y_2\overline{x_0} + 2\mu(x_1 \times y_1).$$

This is precisely equal to 2 times the third component of xy. $\square$

Theorem 3. *If $\mu \in A^\times$ and $A \neq F$, then* $\mathrm{Nuc}_l(J(A,\mu)) = \mathrm{Nuc}_r(J(A,\mu)) = F$.

Proof. Let $(x_0, x_1, x_2) \in \mathrm{Nuc}_l(J(A,\mu))$, then

$$(x_0, x_1, x_2)[(0,1,0)(0,0,1)] = [(x_0, x_1, x_2)(0,1,0)](0,0,1)$$

implies that

$$(x_0, x_1, x_2) = (\overline{\overline{x_0}}, \mu^{-1}(\overline{\mu\overline{x_1}}), \overline{\overline{x_2}}),$$

that means $x_0 = \overline{\overline{x_0}}$ and $x_2 = \overline{\overline{x_2}}$. Using the definition of $\overline{\overline{x_0}}$, we obtain $\overline{\overline{x_0}} = \frac{1}{4}(\mathrm{T}_A(x_0) + x_0)$, so $x_0 = \frac{1}{4}(\mathrm{T}_A(x_0) + x_0)$. Thus, $x_0 = \frac{1}{3}\mathrm{T}_A(x_0) \in F$. Furthermore, since $x_2 = \overline{\overline{x_2}}$, we find in a similar way that $x_2 = \frac{1}{3}\mathrm{T}_A(x_2) \in F$. Next, since $x = (x_0, x_1, x_2) \in \mathrm{Nuc}_l(J(A,\mu))$, we have that

$$(x_0, x_1, x_2)[(0,0,1)(0,1,0)] = [(x_0, x_1, x_2)(0,0,1)](0,1,0).$$

This implies that

$$(x_0, x_1, x_2) = (\overline{\overline{x_0}}, \overline{\overline{x_1}}, \mu(\overline{\mu^{-1}\overline{x_2}})),$$

and so $x_1 = \overline{\overline{x_1}}$. We now find in a similar way that $x_1 = \frac{1}{3}\mathrm{T}_A(x_1) \in F$, thus $\mathrm{Nuc}_l(J(A,\mu)) \subseteq \{(x_0, x_1, x_2) \in J \mid x_0, x_1, x_2 \in F\}$. Let $x = (x_0, x_1, x_2) \in \mathrm{Nuc}_l(J(A,\mu))$, and let $a \in A \setminus F$. Then

$$(x_0, x_1, x_2)[(0,0,1)(0,a,0)] = [(x_0, x_1, x_2)(0,0,1)](0,a,0)$$

which implies that

$$(x_0 \cdot \overline{a}, \overline{\overline{a}}x_1, x_2\overline{\overline{a}}) = (\overline{a\overline{x_0}}, \overline{\overline{x_1}}a, \mu(\mu^{-1}\overline{x_2} \times a)),$$

and so $\overline{\overline{a}}x_1 = \overline{\overline{x_1}}a$. Assume towards a contradiction that $x_1 \neq 0$. Since $x_1 \in F$, this implies that x_1 is invertible and $\overline{\overline{x_1}} = x_1$. Thus, the condition $\overline{\overline{a}}x_1 = \overline{\overline{x_1}}a$ yields $a = \overline{\overline{a}}$, and so $a = \frac{1}{3}\,\mathrm{T}_A(a) \in F$ which is a contradiction. Next, since $(x_0, x_1, x_2) \in \mathrm{Nuc}_l(J(A,\mu))$, we know that

$$(x_0, x_1, x_2)[(0,1,0)(0,0,a)] = [(x_0, x_1, x_2)(0,1,0)](0,0,a)$$

which implies that

$$(x_0 \cdot \overline{a}, \overline{\overline{a}}x_1, x_2\overline{\overline{a}}) = (\overline{\overline{x_0}a}, \mu^{-1}(\mu\overline{x_1} \times a), a\overline{\overline{x_2}}),$$

and so $x_2\overline{\overline{a}} = a\overline{\overline{x_2}}$. Assume towards a contradiction that $x_2 \neq 0$. Then, since $x_2 \in F$, x_2 is invertible and $\overline{\overline{x_2}} = x_2$. Thus, the condition $x_2\overline{\overline{a}} = a\overline{\overline{x_2}}$ yields $a = \overline{\overline{a}}$, and so $a = \frac{1}{3}\,\mathrm{T}_A(a) \in F$ which is a contradiction. Therefore, $x = (x_0, 0, 0)$, $x_0 1 \in F$ which shows that $\mathrm{Nuc}_l(J(A,\mu)) = F$.

Let $(x_0, x_1, x_2) \in \mathrm{Nuc}_r(J(A,\mu))$. Then,

$$(0,0,1)[(0,1,0)(x_0, x_1, x_2)] = [(0,0,1)(0,1,0)](x_0, x_1, x_2)$$

implies that

$$(\overline{\overline{x_0}}, \mu^{-1}(\overline{\mu x_1}), \overline{\overline{x_2}}) = (x_0, x_1, x_2).$$

Hence, $x_0 = \overline{\overline{x_0}}$ and $x_2 = \overline{\overline{x_2}}$. Using the definition of $\overline{\overline{x_0}}$, we find that $\overline{\overline{x_0}} = \frac{1}{4}(\mathrm{T}_A(x_0) + x_0)$, so the condition $x_0 = \overline{\overline{x_0}}$ gives that $x_0 = \frac{1}{4}(\mathrm{T}_A(x_0) + x_0)$. Thus, $x_0 = \frac{1}{3}\,\mathrm{T}_A(x_0) \in F$. Furthermore, since $x_2 = \overline{\overline{x_2}}$, we find in a similar way that $x_2 = \frac{1}{3}\,\mathrm{T}_A(x_2) \in F$. Next, since $x = (x_0, x_1, x_2) \in \mathrm{Nuc}_r(J(A,\mu))$, we have that

$$(0,1,0)[(0,0,1)(x_0, x_1, x_2)] = [(0,1,0)(0,0,1)](x_0, x_1, x_2).$$

This implies that

$$(\overline{\overline{x_0}}, \overline{\overline{x_1}}, \mu(\overline{\mu^{-1}\overline{x_2}})) = (x_0, x_1, x_2),$$

and thus $x_1 = \overline{\overline{x_1}}$. We find in a similar way that $x_1 = \frac{1}{3}\,\mathrm{T}_A(x_1) \in F$, i.e. $\mathrm{Nuc}_r(J(A,\mu)) \subseteq \{(x_0, x_1, x_2) \in J \mid x_0, x_1, x_2 \in F\}$.

Let $x = (x_0, x_1, x_2) \in \mathrm{Nuc}_r(J(A,\mu))$, and let $a \in A \setminus F$. Then, $(0,a,0)[(0,0,1)(x_0, x_1, x_2)] = [(0,a,0)(0,0,1)](x_0, x_1, x_2)$ which implies that

$$(\overline{a\overline{x_0}}, \overline{\overline{x_1}}a, \mu(a \times \mu^{-1}\overline{x_2})) = (\overline{a} \cdot x_0, \overline{\overline{a}}x_1, x_2\overline{\overline{a}});$$

therefore, $\overline{\overline{a}}x_1 = \overline{\overline{x_1}}a$. Assume towards a contradiction that $x_1 \neq 0$. Then, since $x_1 \in F$, x_1 is invertible and $\overline{\overline{x_1}} = x_1$. Thus, the condition $\overline{\overline{a}}x_1 = \overline{\overline{x_1}}a$ yields $a = \overline{\overline{a}}$, and so $a = \frac{1}{3}\,\mathrm{T}_A(a) \in F$ which is a contradiction. Next, since $(x_0, x_1, x_2) \in \mathrm{Nuc}_r(J(A,\mu))$, we know that

$$(0,0,a)[(0,1,0)(x_0, x_1, x_2)] = [(0,0,a)(0,1,0)](x_0, x_1, x_2)$$

which implies that

$$(\overline{\overline{x_0}a}, \mu^{-1}(a \times \mu\overline{x_1}), a\overline{\overline{x_2}}) = (\overline{a} \cdot x_0, \overline{\overline{a}}x_1, x_2\overline{\overline{a}}),$$

and so $x_2\overline{\overline{a}} = a\overline{\overline{x_2}}$. Assume towards a contradiction that $x_2 \neq 0$. Then, since $x_2 \in F$, x_2 is invertible and $\overline{\overline{x_2}} = x_2$. Thus, the condition $x_2\overline{\overline{a}} = a\overline{\overline{x_2}}$ yields $a = \overline{\overline{a}}$, and so $a = \frac{1}{3}\,\mathrm{T}_A(a) \in F$ which is a contradiction. Therefore, $x = (x_0, 0, 0) = x_0 1 \in F$ which shows the assertion. □

Theorem 4. *Let $A \neq F$ be a central simple division algebra of degree three and $\mu \in A^\times$. Then, $\mathrm{Nuc}_m(J(A,\mu)) = F$.*

Proof. Let $x = (x_0, x_1, x_2) \in \mathrm{Nuc}_m(J(A,\mu))$, and let $y_0 \notin C(A)$. Then, there exists $z_0 \in A$ such that $y_0 z_0 \neq z_0 y_0$. Since $(x_0, x_1, x_2) \in \mathrm{Nuc}_m(J(A,\mu))$, we know that

$$(y_0, 0, 0)[(x_0, x_1, x_2)(z_0, 0, 0)] = [(y_0, 0, 0)(x_0, x_1, x_2)](z_0, 0, 0)$$

which implies that

$$(y_0 \cdot (x_0 \cdot z_0), \overline{y_0}(\overline{z_0}x_1), (x_2\overline{z_0})\overline{y_0}) = ((y_0 \cdot x_0) \cdot z_0, \overline{z_0}(\overline{y_0}x_1), (x_2\overline{y_0})\overline{z_0}).$$

Comparing the second and third components yields

$$\overline{y_0}(\overline{z_0}x_1) = \overline{z_0}(\overline{y_0}x_1), \tag{20}$$

$$(x_2\overline{z_0})\overline{y_0} = (x_2\overline{y_0})\overline{z_0}. \tag{21}$$

Now, assume towards a contradiction that $x_1 \neq 0$. Since A is a division algebra, x_1 is invertible. Since A is associative, (20) implies that $\overline{y_0}\ \overline{z_0} = \overline{z_0}\ \overline{y_0}$. By definition, this yields

$$(T_A(y_0)1 - y_0)(T_A(z_0)1 - z_0) = (T_A(z_0)1 - z_0)(T_A(y_0)1 - y_0).$$

Hence, $y_0z_0 = z_0y_0$ which is a contradiction. Now, in a similar way we assume towards a contradiction that $x_2 \neq 0$. Then, since A is a division algebra, x_2 is invertible. Since A is associative, (21) implies again that $\overline{y_0}\ \overline{z_0} = \overline{z_0}\ \overline{y_0}$. Hence, $y_0z_0 = z_0y_0$ which is a contradiction. Next, since $x = (x_0, 0, 0) \in \mathrm{Nuc}_m(J(A,\mu))$, we also have that

$$(0,1,0)[(x_0,0,0)(0,0,y_2)] = [(0,1,0)(x_0,0,0)](0,0,y_2)$$

for each $y_2 \in A$. This implies $(\overline{y_2\overline{x_0}}, 0, 0) = (\overline{\overline{x_0}y_2}, 0, 0)$, and so $\overline{y_2\overline{x_0}} = \overline{\overline{x_0}y_2}$. By definition, this means that

$$\frac{1}{2}T_A(\overline{y_2}x_0)1 - \frac{1}{2}\overline{y_2}x_0 = \frac{1}{2}T_A(x_0\overline{y_2})1 - \frac{1}{2}x_0\overline{y_2}. \tag{22}$$

We know that $T_A(\overline{y_2}x_0) = T_A(x_0\overline{y_2})$ (see (6)), and so (22) gives that $\overline{y_2}x_0 = x_0\overline{y_2}$. By using the definition of $\overline{y_2}$, this implies that $y_2x_0 = x_0y_2$. Hence, $x_0 \in C(A)$. Therefore, $x = (x_0, 0, 0) = x_01 \in C(A)$. Since $F \subseteq \mathrm{Nuc}_m(J(A,\mu))$ this implies the assertion if A is a central simple division algebra. □

Theorem 5. *([18] (Chapter IX, Section 12), [15] (Chapter C.5)) For $\mu \in F^\times$, $J(A,\mu)$ is a division algebra if and only if $\mu \notin N_A(A^\times)$ and A is a division algebra, if and only if N is anisotropic.*

The general situation is much harder to figure out and we were only able to obtain some obvious necessary conditions:

Theorem 6. *Let $\mu \in A^\times$.*

(i) If $J(A,\mu)$ is a division algebra, then $\mu \notin N_A(A^\times)$ and A is a division algebra.
(ii) Let A be a division algebra over F. If 1, μ, μ^2 are linearly independent over F then N is anisotropic.
(iii) If N is anisotropic then A is a division algebra and $\mu \notin N_A(A^\times)$.
(iv) Let $0 \neq x = (x_0, x_1, x_2) \in J(A,\mu)$. Then, $x^\sharp = 0$ implies that A has zero divisors, or A is a division algebra and $\mu \in N_A(A^\times)$.

Proof.

(i) Suppose that $J(A,\mu)$ is a division algebra, then so is A^+ and thus A (Lemma 1). Assume towards a contradiction that $\mu = N_A(x_0)1$ for some $x_0 \in A^\times$. Then, $\mu \in F^\times$ and $J(A,\mu)$ is not a division algebra by Theorem 5. Hence, $\mu \notin N_A(A)1$.
(ii) Since A is a division algebra, N_A is anisotropic. So, let $N((x_0, x_1, x_2)) = 0$; then, the assumption means that $N(x_0) = 0$, which implies that $x_0 = 0$. This immediately means that $x_1 = x_2 = 0$, too.
(iii) If N is anisotropic, then so is N_A; so, A is clearly a division algebra. Moreover, $\mu \notin N_A(A^\times)$ by Theorem 5.

(iv) Let $0 \neq x = (x_0, x_1, x_2) \in J(A, \mu)$. Then, $x^\sharp = 0$ implies that

$$x_0^\sharp = x_1 x_2 \tag{23}$$
$$\mu^{-1} x_2^\sharp = x_0 x_1 \tag{24}$$
$$\mu x_1^\sharp = x_2 x_0. \tag{25}$$

We can now multiply (23) (resp. (24), (25)) by x_0 (resp. x_2, x_1) on the right and left to obtain two new equations. Additionally, using the fact that $N_A(x_i) = x_i x_i^\sharp = x_i^\sharp x_i$ for all $i = 0, 1, 2$, we obtain the following six equations:

$$\begin{aligned} N_A(x_0) &= x_1 x_2 x_0 & \qquad N_A(x_0) &= x_0 x_1 x_2 \\ \mu^{-1} N_A(x_2) &= x_0 x_1 x_2 & \qquad \mu^{-1} N_A(x_2) &= x_2 x_0 x_1 \\ \mu N_A(x_1) &= x_2 x_0 x_1 & \qquad \mu N_A(x_1) &= x_1 x_2 x_0. \end{aligned} \tag{26}$$

These imply that $N_A(x_0) = \mu^{-1} N_A(x_2) = \mu N_A(x_1)$. This means that either $N_A(x_0) = \mu^{-1} N_A(x_2) = \mu N_A(x_1) = 0$ and so N_A is isotropic, or $N_A(x_0) = \mu^{-1} N_A(x_2) = \mu N_A(x_1) \neq 0$ and N_A is anisotropic. In the later case, x_0, x_1, x_2 are all invertible in A, $N_A(x_i) \neq 0$ for all $i = 0, 1, 2$ and it follows that $\mu \in N_A(A^\times)$. This proves the assertion.

□

In other words: If A is a division algebra and $\mu \notin N_A(A^\times)$, $0 \neq x = (x_0, x_1, x_2) \in J(A, \mu)$, then $x^\sharp \neq 0$. Note that (iv) was a substantial part of the classical result that if $\mu \in F^\times$, $\mu \notin N_A(A^\times)$ and A is a division algebra, then N is anisotropic. What is missing in order to generalize this result to the generalized first Tits construction is the adjoint identity $(x^\sharp)^\sharp = N(x)x$. This identity only holds in very special cases—see Lemma 4 below. It would be of course desirable to have conditions on when (or if at all) $J(A, \mu)$ is a division algebra.

4. Some More Identities

Lemma 2. *Let $x = (x_0, x_1, x_2)$, $y = (y_0, y_1, y_2)$, $z = (z_0, z_1, z_2) \in J(A, \mu)$ be such that one of x_1, y_1, z_1 is equal to zero and one of x_2, y_2, z_2 is equal to zero. Then, $T(x \times y, z) = T(x, y \times z)$.*

Proof. We find that

$$\begin{aligned} T(x \times y, z) =& \frac{1}{2} T_A((x_0 \sharp y_0) z_0 - x_1 y_2 z_0 - y_1 x_2 z_0) \\ &+ \frac{1}{2} T_A(\mu^{-1}(x_2 \sharp y_2) z_2 - x_0 y_1 z_2 - y_0 x_1 z_2) \\ &+ \frac{1}{2} T_A(\mu(x_1 \sharp y_1) z_1 - x_2 y_0 z_1 - y_2 x_0 z_1) \end{aligned} \tag{27}$$

and

$$\begin{aligned} T(x, y \times z) =& \frac{1}{2} T_A(x_0(y_0 \sharp z_0) - x_0 y_1 z_2 - x_0 z_1 y_2) \\ &+ \frac{1}{2} T_A(x_1 \mu(y_1 \sharp z_1) - x_1 y_2 z_0 - x_1 z_2 y_0) \\ &+ \frac{1}{2} T_A(x_2 \mu^{-1}(y_2 \sharp z_2) - x_2 y_0 z_1 - x_2 z_0 y_1). \end{aligned} \tag{28}$$

Using the definitions, we can show that $T_A((x_0 \sharp y_0) z_0) = T_A(x_0(y_0 \sharp z_0))$. Furthermore, since one of x_1, y_1, z_1 is equal to zero, we have that $T_A(\mu(x_1 \sharp y_1) z_1) = 0 = T_A(x_1 \mu(y_1 \sharp z_1))$. Finally, since one of x_2, y_2, z_2 is equal to zero, $T_A(\mu^{-1}(x_2 \sharp y_2) z_2) = 0 = T_A(x_2 \mu^{-1}(y_2 \sharp z_2))$. Therefore, applying these equalities and using (6), we deduce that (27) and (28) are equal, so $T(x \times y, z) = T(x, y \times z)$. □

We know that $xx^\sharp = x^\sharp x = N(x)1$ holds for all $x \in J(A,\mu)$ if $\mu \in F^\times$. We now show for which $x \in J(A,\mu)$ we still obtain $xx^\sharp = x^\sharp x = N(x)1$:

Lemma 3. *Let $\mu \in A^\times$ and suppose that $x \in J(A,\mu)$, such that one of the following holds:*

(i) $x = (x_0, 0, x_2) \in J(A,\mu)$, $x_0\mu = \mu x_0$ *and* $N_A(x_2) = 0$.

(ii) $x = (x_0, x_1, 0) \in J(A,\mu)$, $x_1\mu = \mu x_1$ *and* $N_A(x_1) = 0$.

Then, we have

$$xx^\sharp = x^\sharp x = N(x)1. \tag{29}$$

Moreover, assume that one of the following holds:

(iii) $x = (x_0, 0, x_2) \in J(A,\mu)$, $x_0\mu = \mu x_0$ *and* $N_A(x_2) \neq 0$.

(iv) $x = (x_0, x_1, 0) \in J(A,\mu)$, $x_1\mu = \mu x_1$ *and* $N_A(x_1) \neq 0$.

Then, $xx^\sharp = x^\sharp x = N(x)1$ *if and only if* $\mu \in F^\times$.

Proof. Let $x = (x_0, x_1, x_2) \in J(A,\mu)$, and let $xx^\sharp = (a_0, a_1, a_2)$; then, $x^\sharp = (x_0^\sharp - x_1x_2, \mu^{-1}x_2^\sharp - x_0x_1, \mu x_1^\sharp - x_2x_0)$. Thus, we have

$$\begin{aligned} a_0 = {} & \frac{1}{2}(x_0x_0^\sharp - x_0x_1x_2 + x_0^\sharp x_0 - x_1x_2x_0) \\ & + \frac{1}{2}(T_A(x_1\mu x_1^\sharp - x_1x_2x_0)1 - x_1\mu x_1^\sharp + x_1x_2x_0) \\ & + \frac{1}{2}(T_A(\mu^{-1}x_2^\sharp x_2 - x_0x_1x_2)1 - \mu^{-1}x_2^\sharp x_2 + x_0x_1x_2), \end{aligned} \tag{30}$$

$$\begin{aligned} a_1 = {} & \frac{1}{2}(T_A(x_0) - x_0)(\mu^{-1}x_2^\sharp - x_0x_1) + \frac{1}{2}(T_A(x_0^\sharp - x_1x_2) - x_0^\sharp + x_1x_2)x_1 \\ & + \mu^{-1}(x_2 \times (\mu x_1^\sharp - x_2x_0)), \end{aligned} \tag{31}$$

$$\begin{aligned} a_2 = {} & \frac{1}{2}x_2(T_A(x_0^\sharp - x_1x_2) - x_0^\sharp + x_1x_2) + \frac{1}{2}(\mu x_1^\sharp - x_2x_0)(T_A(x_0) - x_0) \\ & + \mu(x_1 \times (\mu^{-1}x_2^\sharp - x_0x_1)) \end{aligned} \tag{32}$$

by the definition of the multiplication on $J(A,\mu)$.

(i) If $x_1 = 0$ and $N_A(x_2) = 0$, using the fact that $x_ix_i^\sharp = x_i^\sharp x_i = N_A(x_i)1$ for all $i = 0,1,2$ (see (5)), (30) simplifies to $a_0 = N_A(x_0)1 = N(x)$. Since we have $x_0\mu^{-1} = \mu^{-1}x_0$, (31) gives that $a_1 = \frac{1}{2}\mu^{-1}(\mathrm{T}_A(x_0) - x_0)x_2^\sharp - \mu^{-1}(x_2 \times (x_2x_0))$. By (10),

$$x_2^\sharp \sharp x_0 = x_2^\sharp x_0 + x_0x_2^\sharp - T_A(x_2^\sharp)x_0 - T_A(x_0)x_2^\sharp + (T_A(x_2^\sharp)T_A(x_0) - T_A(x_2^\sharp x_0))1. \tag{33}$$

Using the fact that $T_A(x_2^\sharp) = S_A(x_2)$ (by (8)) on the right-hand side of (33), we further obtain after some simplification that

$$x_2^\sharp \sharp x_0 = x_2^2x_0 - T_A(x_2)x_2x_0 + x_0x_2^\sharp - T_A(x_0)x_2^\sharp - T_A(x_2^2x_0)1 + T_A(x_2x_0)1. \tag{34}$$

Now, combining (13) with (34) yields $T_A(x_0)x_2^\sharp - x_0x_2^\sharp = x_2^2x_0 + x_2x_0x_2 - T_A(x_2)x_2x_0 - T_A(x_2x_0)x_2 + (T_A(x_2)T_A(x_2x_0) - T_A(x_2^2x_0))1 = 2(x_2 \times (x_2x_0))$, so $x_2 \times (x_2x_0) = \frac{1}{2}(T_A(x_0)x_2^\sharp - x_0x_2^\sharp)$. Hence, (4) implies $a_1 = 0$. For a_2, (32) yields $a_2 = \frac{1}{2}x_2(T_A(x_0^\sharp) - x_0^\sharp) - \frac{1}{2}x_2(T_A(x_0)x_0 - x_0^2)$. Then, using the definition of $x_0^\sharp$ and the fact that $2S_A(x_0) = \mathrm{T}_A(x_0)^2 - \mathrm{T}_A(x_0^2)$, we find that $\mathrm{T}_A(x_0^\sharp) - x_0^\sharp = \mathrm{T}_A(x_0)x_0 - x_0^2$. Therefore, $a_2 = 0$.

(ii) In this case, we have $x_2 = 0$, $x_1\mu = \mu x_1$ and $N_A(x_1) = 0$. So, (30) simplifies to $a_0 = N_A(x_0)1 = N(x)$. For a_1, (31) simplifies to $a_1 = -\frac{1}{2}(\mathrm{T}_A(x_0)x_0 - x_0^2)x_1 + \frac{1}{2}(\mathrm{T}_A(x_0^\sharp) - x_0^\sharp)x_1$. Then, in a similar way to how we found a_2 in (i), we find here that $a_1 = 0$.

For a_2, (32) simplifies to $a_2 = \frac{1}{2}\mu x_1^\sharp(T_A(x_0) - x_0) - \mu(x_1 \times (x_0 x_1))$. We now find in a similar way to how we found a_1 in (i) that $a_2 = 0$.

To prove that the claimed equivalence holds assuming (iii) or (iv), we only need to show the forward direction since we know from the classical first Tits construction that the reverse direction holds:

(iii) Here, (30) yields $a_0 = N_A(x_0)1 + \frac{1}{2}(\mathrm{T}_A(\mu^{-1})N_A(x_2) - \mu^{-1}N_A(x_2))$; thus, $xx^\sharp = N(x)1 = (N_A(x_0)1 + \mu^{-1}N_A(x_2))1$ gives that $N_A(x_0)1 + \frac{1}{2}(\mathrm{T}_A(\mu^{-1}) N_A(x_2) - \mu^{-1}N_A(x_2)) = a_0 = N_A(x_0)1 + \mu^{-1}N_A(x_2)$. Therefore, we have $\mu^{-1} = \frac{1}{3}\mathrm{T}_A(\mu^{-1}) \in F^\times$, so $\mu \in F^\times$.

(iv) In this case, (30) yields $a_0 = N_A(x_0)1 + \frac{1}{2}(\mathrm{T}_A(\mu)N_A(x_1) - \mu N_A(x_1))$; thus, $xx^\sharp = N(x)1 = (N_A(x_0)1 + \mu N_A(x_1))1$ yields $N_A(x_0)1 + \frac{1}{2}(\mathrm{T}_A(\mu)N_A(x_1) - \mu N_A(x_1)) = a_0 = N_A(x_0)1 + \mu N_A(x_1)$. Therefore, we obtain $\mu = \frac{1}{3}\mathrm{T}_A(\mu) \in F^\times$. The proof that $x^\sharp x = N(x)1$ is performed similarly.

□

Corollary 1. *Let $\mu \in A^\times$. Suppose that $x \in J(A,\mu)$ satisfies $N(x) \neq 0$, and assume that one of the following holds:*

(i) $x = (x_0, x_1, 0) \in J(A,\mu)$, $x_1\mu = \mu x_1$ *and* $N_A(x_1) = 0$.
(ii) $x = (x_0, 0, x_2) \in J(A,\mu)$, $x_0\mu = \mu x_0$ *and* $N_A(x_2) = 0$.

Then, x is invertible in $J(A,\mu)$ with $x^{-1} = N(x)^{-1}x^\sharp$.

Proof. Let $\mu \in A^\times$ and suppose that $x \in J(A,\mu)$ satisfies (i) or (ii); then, $xx^\sharp = x^\sharp x = N(x)1$. Since $F = C(J(A,\mu))$ this yields the assertion. □

In particular, if N is anisotropic, then every $0 \neq xx = (x_0, x_1, 0) \in J(A,\mu)$ in (i) or (ii) is of the type $x = (x_0, 0, 0) \in J(A,\mu)$, i.e., lies in A; so, this result then becomes trivial.

Corollary 2. *Let $\mu \in A^\times$ and suppose that $x \in J(A,\mu)$, such that one of the following holds:*

(i) $x = (x_0, 0, x_2) \in J(A,\mu)$, $x_0\mu = \mu x_0$ *and* $N_A(x_2) = 0$.
(ii) $x = (x_0, x_1, 0) \in J(A,\mu)$, $x_1\mu = \mu x_1$ *and* $N_A(x_1) = 0$.

Then, we have

$$x^3 - T(x)x^2 + S(x)x - N(x)1 = 0.$$

Proof. Using the fact that $x^\sharp = x^2 - T(x)x + S(x)1$ from Theorem 2 (i), we have that $x^3 - T(x)x^2 + S(x)x - N(x)1 = 0$ if and only if $xx^\sharp = x^\sharp x = N(x)1$. Thus, the result now follows as a consequence of Lemma 3. □

Theorem 7. *The identity $xx^\sharp = x^\sharp x = N(x)1$ holds for all $x \in J(A,\mu)$ if and only if $\mu \in F^\times$.*

Proof. If $\mu \in F^\times$, then $xx^\sharp = x^\sharp x = (N(x), 0, 0)$ for all $x \in J(A,\mu)$. Conversely, suppose that $xx^\sharp = x^\sharp x = (N(x), 0, 0)$ holds for all $x \in J(A,\mu)$. Take $x = (0, 1, 0)$. Then, $x^\sharp = (0, 0, \mu)$, and so

$$xx^\sharp = (\bar{\mu}, 0, 0) = (\frac{1}{2}(T_A(\mu)1 - \mu), 0, 0).$$

We also know that by definition, $N(x) = \mu N_A(1) = \mu$, so the condition $xx^\sharp = (N(x), 0, 0)$ gives that $\mu = \frac{1}{2}(T_A(\mu)1 - \mu)$. Hence $\mu = \frac{1}{3}T_A(\mu)1 \in F^\times$. □

We know that the adjoint identity $(x^\sharp)^\sharp = N(x)x$ holds for all $x \in J(A,\mu)$, if $\mu \in F^\times$ [15] (Chapter C.4). In the general construction, it holds only in very special cases:

Lemma 4. *Let $\mu \in A^\times$ and suppose that $x \in J(A,\mu)$, such that one of the following holds:*

(i) $x = (0, x_1, 0) \in J(A,\mu)$ *and* $N_A(x_1) = 0$.
(ii) $x = (x_0, x_1, 0) \in J(A,\mu)$ *and* $N_A(x_1) = 0$ *and* $x_1\mu = \mu x_1$.

(iii) $x = (x_0, 0, x_2) \in J(A, \mu)$ *and* $N_A(x_2) = 0$ *and* $x_0\mu = \mu x_0$.
Then, we have $(x^\sharp)^\sharp = N(x)x$.
Moreover, if one of the following holds:
(iv) $x = (x_0, x_1, 0) \in J(A, \mu)$, $N_A(x_1) \neq 0$, $x_0\mu = \mu x_0$ *and* $x_1\mu = \mu x_1$.
(v) $x = (x_0, 0, x_2) \in J(A, \mu)$, $N_A(x_2) \neq 0$, $x_0\mu = \mu x_0$ *and* $x_2\mu = \mu x_2$.
Then, $(x^\sharp)^\sharp = N(x)x$ *for all* $x \in J(A, \mu)$ *if and only if* $N_A(\mu) = \mu^3$.

Proof. Let $x = (x_0, x_1, x_2) \in J(A, \mu)$ and $(x^\sharp)^\sharp = (a_0, a_1, a_2)$. By definition, $x^\sharp = (x_0^\sharp - x_1x_2, \mu^{-1}x_2^\sharp - x_0x_1, \mu x_1^\sharp - x_2x_0)$, so $a_0 = (x_0^\sharp - x_1x_2)^\sharp - (\mu^{-1}x_2^\sharp - x_0x_1)(\mu x_1^\sharp - x_2x_0)$. Now, using (9) and (10), it is easy to show that

$$\begin{aligned}(x_0^\sharp - x_1x_2)^\sharp &= (x_0^\sharp - x_1x_2)^2 - T_A(x_0^\sharp - x_1x_2)(x_0^\sharp - x_1x_2) + S_A(x_0^\sharp - x_1x_2) \\ &= (x_0^\sharp)^\sharp - x_0^\sharp\sharp(x_1x_2) + (x_1x_2)^\sharp.\end{aligned}$$

Hence,

$$\begin{aligned}a_0 &= (x_0^\sharp - x_1x_2)^\sharp - (\mu^{-1}x_2^\sharp - x_0x_1)(\mu x_1^\sharp - x_2x_0) \\ &= (x_0^\sharp)^\sharp - x_0^\sharp\sharp(x_1x_2) + (x_1x_2)^\sharp - \mu^{-1}x_2^\sharp\mu x_1^\sharp + \mu^{-1}x_2^\sharp x_2x_0 + x_0x_1\mu x_1^\sharp - x_0x_1x_2x_0. \quad (35)\end{aligned}$$

Similarly, we find that

$$\begin{aligned}a_1 &= \mu^{-1}(\mu x_1^\sharp - x_2x_0)^\sharp - (x_0^\sharp - x_1x_2)(\mu^{-1}x_2^\sharp - x_0x_1) \\ &= \mu^{-1}((\mu x_1^\sharp)^\sharp - (\mu x_1^\sharp)\sharp(x_2x_0) + (x_2x_0)^\sharp) \\ &\quad - x_0^\sharp\mu^{-1}x_2^\sharp + x_0^\sharp x_0x_1 + x_1x_2\mu^{-1}x_2^\sharp - x_1x_2x_0x_1 \quad (36)\end{aligned}$$

and

$$\begin{aligned}a_2 &= \mu(\mu^{-1}x_2^\sharp - x_0x_1)^\sharp - (\mu x_1^\sharp - x_2x_0)(x_0^\sharp - x_1x_2) \\ &= \mu((\mu^{-1}x_2^\sharp)^\sharp - (\mu^{-1}x_2^\sharp)\sharp(x_0x_1) + (x_0x_1)^\sharp) \\ &\quad - \mu x_1^\sharp x_0^\sharp + \mu x_1^\sharp x_1x_2 + x_2x_0x_0^\sharp - x_2x_0x_1x_2. \quad (37)\end{aligned}$$

(i) Here, $x_0 = x_2 = 0$; therefore, (35) implies $a_0 = 0$ and (36) gives that $a_1 = \mu^{-1}(\mu x_1^\sharp)^\sharp = \mu^{-1}N_A(x_1)x_1\mu^\sharp = 0 = N(x)x_1$. Finally, (37) gives that $a_2 = 0$ as required.
(ii) Since $x_2 = 0$, $x_1\mu = \mu x_1$ and $N_A(x_1) = 0$, we find by (35) that $a_0 = (x_0^\sharp)^\sharp + x_0\mu x_1 x_1^\sharp = N_A(x_0)x_0 + x_0\mu N_A(x_1) = N(x)x_0$. Now, (36) gives that $a_1 = \mu^{-1}(\mu x_1^\sharp)^\sharp + x_0^\sharp x_0x_1 = \mu^{-1}N_A(x_1)x_1\mu^\sharp + N_A(x_0)x_1 = N(x)x_1$, and by (37), we obtain $a_2 = \mu(x_0x_1)^\sharp - \mu x_1^\sharp x_0^\sharp = 0 = N(x)0$.
(iii) Since $x_1 = 0$ and $N_A(x_2) = 0$, (35) yields $a_0 = (x_0^\sharp)^\sharp + \mu^{-1}x_2^\sharp x_2x_0 = N_A(x_0)x_0 + \mu^{-1}N_A(x_2)x_0 = N(x)x_0$. Now, since $x_0\mu^{-1} = \mu^{-1}x_0$, we have that $x_0^\sharp\mu^{-1} = \mu^{-1}x_0^\sharp$, so (36) gives that $a_1 = \mu^{-1}(x_2x_0)^\sharp - \mu^{-1}x_0^\sharp x_2^\sharp = 0 = N(x)0$. Finally, (37) gives that $a_2 = \mu(\mu^{-1}x_2^\sharp)^\sharp + x_2x_0x_0^\sharp = \mu N_A(x_2)x_2(\mu^{-1})^\sharp + N_A(x_0)x_2 = N(x)x_2$.
(iv) Since $x_2 = 0$, $x_0\mu = \mu x_0$ and $x_1\mu = \mu x_1$, (35) yields $a_0 = (x_0^\sharp)^\sharp + \mu x_0x_1x_1^\sharp = N_A(x_0)x_0 + \mu N_A(x_1)x_0 = N(x)x_0$. Now, (37) gives that $a_2 = \mu(x_0x_1)^\sharp - \mu x_1^\sharp x_0^\sharp = 0$. Finally, (36) gives that $a_1 = \mu^{-1}(\mu x_1^\sharp)^\sharp + x_0^\sharp x_0x_1 = \mu^{-1}N_A(x_1)x_1\mu^\sharp + N_A(x_0)x_1$. Thus, $a_1 = N(x)x_1$ if and only if $\mu^{-1}N_A(x_1)x_1\mu^\sharp + N_A(x_0)x_1 = N(x)x_1$, which occurs if and only if $\mu^{-1}N_A(x_1)x_1\mu^\sharp = \mu N_A(x_1)x_1$. Since $N_A(x_1) \neq 0$ and $x_1\mu^\sharp = \mu^\sharp x_1$, this occurs if and only if $\mu^\sharp x_1 = \mu^2 x_1$. Finally, $N_A(x_1) \neq 0$ implies that x_1 is invertible, so $\mu^\sharp x_1 = \mu^2 x_1$ if and only if $N_A(\mu) = \mu\mu^\sharp = \mu^3$.

(v) Since $x_1 = 0$, (35) yields $a_0 = (x_0^\sharp)^\sharp + \mu^{-1}x_2^\sharp x_2 x_0 = N(x)x_0$. Furthermore, since x_0 commutes with μ, $x_0^\sharp$ commutes with μ. So, $x_0^\sharp \mu^{-1} = \mu^{-1}x_0^\sharp$. Hence, (36) gives that $a_1 = \mu^{-1}(x_2x_0)^\sharp - \mu^{-1}x_0^\sharp x_2^\sharp = 0 = N(x)0$. Finally, (37) yields $a_2 = \mu(\mu^{-1}x_2^\sharp)^\sharp + x_2x_0x_0^\sharp = \mu N_A(x_2)x_2(\mu^{-1})^\sharp + N_A(x_0)x_2$. Thus, $a_2 = N(x)x_2$ if and only if $\mu N_A(x_2)x_2(\mu^{-1})^\sharp + N_A(x_0)x_2 = N(x)x_2$, which occurs if and only if $\mu N_A(x_2)x_2(\mu^{-1})^\sharp = \mu^{-1}N_A(x_2)x_2$. Since $N_A(x_2) \neq 0$ and $x_2(\mu^{-1})^\sharp = (\mu^{-1})^\sharp x_2$, this occurs if and only if $(\mu^{-1})^\sharp x_2 = \mu^{-2}x_2$. Finally, $N_A(x_2) \neq 0$ implies that x_2 is invertible, so $(\mu^{-1})^\sharp x_2 = \mu^{-2}x_2$ if and only if $N_A(\mu^{-1}) = \mu^{-1}(\mu^{-1})^\sharp = \mu^{-3}$. This is equivalent to $N_A(\mu) = \mu^3$.

□

Proposition 2. *Let A be a central simple algebra over F. Then, $(x^\sharp)^\sharp = N(x)$ for all $x \in J(A,\mu)$ if and only if $\mu \in F^\times$.*

Proof. Let $\mu \in F^\times$ then by Lemma 4, the adjoint identity holds for all $x \in J(A,\mu)$. Suppose now that the adjoint identity holds for all $x \in J(A,\mu)$. Let $x = (x_0, 1, 0) \in J(A,\mu)$ for some $x_0 \in A$. Then, $x^\sharp = (x_0^\sharp, -x_0, \mu)$ and so

$$(x^\sharp)^\sharp = ((x_0^\sharp)^\sharp + x_0\mu, \mu^{-1}\mu^\sharp + (x_0^\sharp)^\sharp x_0, 0). \tag{38}$$

Furthermore, $N(x) = N_A(x_0)1 + \mu$. Since the adjoint identity holds by assumption, we see that by using (38),

$$((x_0^\sharp)^\sharp + x_0\mu, \mu^{-1}\mu^\sharp + (x_0^\sharp)^\sharp x_0, 0) = (N_A(x_0)x_0 + \mu x_0, N_A(x_0) + \mu, 0). \tag{39}$$

We know that $(x_0^\sharp)^\sharp = N_A(x_0)x_0$ for all $x_0 \in A$ by (12), and so by comparing the first components of (39), we find that $x_0\mu = \mu x_0$ for all $x_0 \in A$. Hence, $\mu \in C(A)$, and since A is a central simple algebra by assumption, $\mu \in F^\times$. □

If $\mu \in F^\times$, then the norm N *permits Jordan composition*, i.e. $N(U_x y) = N_A(x)^2 N(y)$ for all $x, y \in J(A,a)$. The following result is a corrected version of [11] (Theorem 5.2.5), and a weak generalization of the Jordan composition for $\mu \in A^\times \setminus F$:

Theorem 8. *Let $x = (x_0, 0, 0) \in A$, $y = (y_0, y_1, y_2) \in J(A,\mu)$ and suppose that one of the following holds:*

(i) $T_A(y_0y_1y_2) = T_A(N_A(y_0)x_0^\sharp y_1 y_2 x_0^\sharp)$.

(ii) $y_0y_1y_2 = N_A(y_0)x_0^\sharp y_1 y_2 x_0^\sharp$.

(iii) $y_i = 0$ *for some* $i = 0, 1, 2$.

Then, $N(U_x(y)) = N(x)^2 N(y)$.

Proof. Using the definitions, we see that $T(x,y) = T_A(x_0y_0)$ and $x^\sharp \sharp y = (x_0^\sharp \sharp y_0, -x_0^\sharp y_1, -y_2 x_0^\sharp)$. So $U_x(y) = T(x,y)x - x^\sharp \sharp y = (U_{x_0}(y_0), x_0^\sharp y_1, y_2 x_0^\sharp)$. This yields

$$\begin{aligned} N(U_x(y)) &= N_A(U_{x_0}(y_0))1 + \mu N_A(x_0^\sharp y_1) + \mu^{-1}N_A(y_2 x_0^\sharp) - T_A(U_{x_0}(y_0)x_0^\sharp y_1 y_2 x_0^\sharp)1 \\ &= N_A(x_0)^2(N_A(y_0)1 + \mu N_A(y_1) + \mu^{-1}N_A(y_2) - T_A(N_A(y_0)x_0^\sharp y_1 y_2 x_0^\sharp)1) \\ &= N(x)^2(N(y) + T_A(y_0y_1y_2)1 - T_A(N_A(y_0)x_0^\sharp y_1 y_2 x_0^\sharp)1), \end{aligned}$$

where in the second equality we have used the fact that $N_A(x_0^\sharp) = N_A(x_0)^2$, and that $N_A(U_{x_0}(y_0)) = N_A(x_0)^2 N_A(y_0)$. Therefore, $N(U_x(y)) = N(x)^2 N(y)$, if and only if $T_A(y_0y_1y_2)1 = T_A(N_A(y_0)x_0^\sharp y_1 y_2 x_0^\sharp)$.
(ii) and (iii) are examples where this is the case. □

Remark 2. *Let $f : J(A,\mu) \to J(A,\mu)$ be an automorphism. Then,*

$$f((x_0,x_1,x_2)) = f((x_0,0,0)) + f((0,x_1,0)) + f((0,0,x_2)). \tag{40}$$

Now, for each $x \in A$, we have $f((0,\overline{x},0)) = f((x,0,0))f((0,1,0)$, and $f((0,0,\overline{x})) = f((x,0,0))f((0,0,1))$. On the other hand, by using the definition of $\overline{x}$,

$$f((0,\overline{x},0)) = \frac{1}{2}\operatorname{Tr}_A(x)f((0,1,0)) - \frac{1}{2}f((0,x,0)),$$
$$f((0,0,\overline{x})) = \frac{1}{2}\operatorname{Tr}_A(x)f((0,0,1)) - \frac{1}{2}f((0,0,x)).$$

Hence,

$$f((0,x,0)) = f((0,1,0))(\operatorname{Tr}_A(x) - 2f((x,0,0))), \tag{41}$$
$$f((0,0,x)) = f((0,0,1))(\operatorname{Tr}_A(x) - 2f((x,0,0))). \tag{42}$$

So, by (40)–(42), we see that any automorphism of $J(A,\mu)$ is determined by its restriction on A^+, and its value on $(0,1,0)$ and $(0,0,1)$. Let $f : J(A,\mu) \to J(A,\mu)$ be an automorphism that fixes A^+; then, $f|_{A^+} = \tau$ is either an automorphism or an anti-automorphism of A. Moreover, clearly $f((1,0,0)) = (1,0,0)$, so

$$f((x_0,x_1,x_2)) = (\tau(x_0),0,0) + (\tau(x_1),0,0)f((0,1,0)) + (\tau(x_2),0,0)f((0,0,1)).$$

Calculation to try gain some deeper understanding on the automorphisms are tedious and did not lead us anywhere so far.

5. The Nine-Dimensional Non-Associative Algebras $J(K,\mu)$

Let K/F be a separable cubic field extension with $\operatorname{Gal}(K/F) = \langle\sigma\rangle$, norm N_K, and trace T_K. For all $x_0 \in K$, we have $x_0^\sharp = \sigma(x_0)\sigma^2(x_0)$ and $\overline{x_0} = \frac{1}{2}(\sigma(x_0) + \sigma^2(x_0))$. Assume $\mu \in K^\times$.

Let us compare the first Tits construction $J(K,\mu)$ with the algebra D^+ for a (perhaps non-associative) cyclic algebra $D = (K/F,\sigma,\mu)$ over F of degree three. Consider D as a left K-vector space with basis $\{1,z,z^2\}$. Write R_x for the matrix of right multiplication by $x = x_0 + x_1 z + x_2 z^2$, $x_i \in K$, with respect to the basis $\{1,z,z^2\}$, then the cubic map $N_D : D \to K$, $N_D(x) = \det(R_x)$ (which is the reduced norm of the central simple algebra D if $\mu \in F^\times$), is given by

$$N_D(x) = N_K(x_0) + \mu N_K(x_1) + \mu^2 N_K(x_2) - \mu T_K(x_0\sigma(x_1)\sigma^2(x_2)).$$

If N_D is anisotropic then D is a division algebra over F. If $\mu \in K \setminus F$, we obtain $N_D(lx) = N_K(l)N_D(x)$ for all $x \in D$, $l \in K$ [11] (Propositions 4.2.2 and 4.2.3).

On the other hand, $J(K,\mu)$ is a nine-dimensional non-associative unital algebra over F with multiplication

$$xy = (x_0 \cdot y_0 + \overline{x_1 y_2} + \overline{x_2 y_1}, \overline{x_0}y_1 + \overline{y_0}x_1 + \mu^{-1}(x_2 \times y_2), \overline{x_0}y_2 + \overline{y_0}x_2 + \mu(x_1 \times y_1))$$

for $x = (x_0,x_1,x_2), y = (y_0,y_1,y_2) \in J(K,\mu)$, cubic norm map

$$N((x_0,x_1,x_2)) = N_K(x_0) + \mu N_K(x_1) + \mu^{-1}N_K(x_2) - T_K(x_0x_1x_2),$$

and trace $T(x) = T_K(x_0)$. Moreover, we have

$$x^\sharp = (\sigma(x_0)\sigma^2(x_0) - x_1x_2, \mu^{-1}\sigma(x_2)\sigma^2(x_2) - x_0x_1, \mu\sigma(x_1)\sigma^2(x_1) - x_2x_0).$$

If $\mu \in F^\times$, then $D = (K/F, \sigma, \mu)$ is an associative cyclic algebra over F of degree three and $J(K, \mu) \cong D^+$ is a special cubic Jordan algebra. It is well known that the isomorphism $G : D^+ = (K/F, \sigma, \mu)^+ \to J(K, \mu)$ is given by

$$x_0 + x_1 z + x_2 z^2 \mapsto (x_0, \sigma(x_1), \mu\sigma^2(x_2)).$$

However, if $\mu \in K \setminus F$, then the map $G : D^+ \to J(K, \mu)$ is not an algebra isomorphism between $(K/F, \sigma, \mu)^+$ and $J(K, \mu)$, where now $(K/F, \sigma, \mu)$ is a non-associative cyclic algebra, since $\sigma(\mu) \neq \mu$. However, for $\mu \in K \setminus F$, the map $G : D^+ \to J(K, \mu)$ still yields an isometry of norms, since

$$\begin{aligned} N((x_0, \sigma(x_1), \mu\sigma^2(x_2)) &= N_K(x_0) + \mu N_K(x_1) + \mu^{-1} N_K(x_2) - \mu T_K(x_0 \sigma(x_1) \sigma(x_2)) \\ &= N_D((x_0, x_1, x_2)); \end{aligned}$$

hence, the norms of the two nonisomorphic non-associative algebras $D^+ = (K/F, \sigma, \mu)^+$ and $J(K, \mu)$ are isometric.

6. Conclusions

We looked at the following canonical question: "what happens if we choose the element μ that is used in the first Tits construction $J(A, \mu)$ in $A^\times$ instead of in $F^\times$?" We showed that the basic ingredients for an interesting theory are in place: our new algebras $J(A, \mu)$ carry maps that can be understood as generalizations of the classical norms and traces, and that behave surprisingly similar to the norms and traces of their classical counterparts; we have a function N on $J(A, \mu)$ that extends the cubic norm of A (however, it has values in A), a trace function $T : J(A, \mu) \to F$, and a quadratic map $\sharp : J(A, \mu) \to J(A, \mu)$. Operations like $x \sharp y$ can easily be defined. Some of the main identities from the classical setup hold (Theorems 1 and 2), some others hold only for some elements, e.g., Lemmas 2 and 3, Corollaries 1 and 2, but not in general, and some hold if—and only if—$\mu \in F^\times$ (Proposition 2, Theorem 7), i.e., they hold only in the classical case.

It seems a hard problem to check when the algebras $J(A, \mu)$ are division algebras. It would also be interesting to compute their automorphisms; however, we expect the automorphism group to be "small". Here is one indication as to why this is the case: For Albert algebras over fields F of characteristic not 2 or 3, we know that the similarities of their norms are given either by scalar multiplications or the U operators [4]. Using Theorem 8 (iii), we see that for $J(A, \mu)$ with $\mu \in A^\times \setminus F$, scalar multiplications still give similarities; the U-operators, however, do not.

Even partial results on automorphisms or similarities could give an insight on what is happening in this general context, and it would be interesting to address questions of whether there are inner automorphisms, whether there are cubic subfields fixed by automorphisms like in the classical case [2], etc.

The fact that two nonisomorphic algebras $D^+ = (K/F, \sigma, \mu)^+$ and $J(K, \mu)$ have isometric norms is an example of how rich the structure theory for non-associative algebras really is (Section 4).

This is an exploratory paper, but our results show that the algebras $J(A, \mu)$ obtained via a generalized first Tits construction merit a closer look. As one referee pointed out, they also show the weaknesses of the language that we have at our disposal, which describes highly non-associative structures.

7. Materials and Methods

We used classical methods from algebra.

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