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Article

The Generalized Eta Transformation Formulas as the Hecke Modular Relation

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† Dedicated to Professor Dr. Masaaki Yoshida on his 75th birthday with friendship and great respect.
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Abstract: The transformation formula under the action of a general linear fractional transformation for a generalized Dedekind eta function has been the subject of intensive study since the works of Rademacher, Dieter, Meyer, and Schoenberg et al. However, the (Hecke) modular relation structure was not recognized until the work of Goldstein-de la Torre, where the modular relations mean equivalent assertions to the functional equation for the relevant zeta functions. The Hecke modular relation is a special case of this, with a single gamma factor and the corresponding modular form (or in the form of Lambert series). This has been the strongest motivation for research in the theory of modular forms since Hecke's work in the 1930s. Our main aim is to restore the fundamental work of Rademacher (1932) by locating the functional equation hidden in the argument and to reveal the Hecke correspondence in all subsequent works (which depend on the method of Rademacher) as well as in the work of Rademacher. By our elucidation many of the subsequent works will be made clear and put in their proper positions.

Keywords: RHB correspondence; transformation formula for Lambert series; Hurwitz zeta function; Lerch zeta function; vector space structure

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1. Hecke Modular Relation for Generalized Eta Functions

Rademacher's "Topics" [1], along with Siegel's "Advanced analytic number theory" [2], has been the masterpiece classic of the theory of algebraic aspects of analytic number theory and is widely read by researchers. Ref. [1] (Chapter 9) is devoted to the theory of the transformation formula for the Dedekind eta function $\eta(\tau)$; hereafter abbreviated as ETF. The main concern is about the ETF under a general Möbius transformation, not restricted to the Spiegelung $S : \tau \to \tau^{-1}$. The correspondence between the transformation formula under the Spiegelung and the functional equation for the associated zeta, L functions has been known as the Hecke correspondence, or more generally as the Riemann–Hecke–Bochner correspondence, RHB correspondence, also referred to as modular relation. This has been developed by many authors [3–12], culminating in the work of [13].

Rademacher [1] (Chapter 9), however, incorporates Iseki's paper [14] for the proof of ETF under a general substitution. Ref. [14] depends on the partial fraction expansion (PFE) for the cotangent function and [1] gives an impression that ETF must be proved by PFE. But, it is known that PFE is equivalent to the functional equation for the Riemann zeta function $\zeta(s)$, ref. [15], which naturally implies that ETF is also a consequence of the RHB correspondence. Indeed, Rademacher himself [16] developed the integral transform method to prove ETF prior to Hecke's discovery of the RHB correspondence, and Rademacher's method was used by many subsequent authors [17–21], all of whom used Rademcaher's method and not the RHB correspondence. Iseki [22] seems to be the first who revived Rademacher's method [16] to prove the functional equation, which was extended to the case of the Lambert series by Apostol [23]. Both of them used the gamma transform (56) of the Estermann-type zeta function, but the RHB correspondence does not seem to be perceived; for further reading, readers may refer to [24–26].

Thus, the real starter of the proper use of the RHB correspondence is [27],which cites [5] and proves the general ETF from the generating zeta function, satisfying the ramified (Hecke) functional equation. Ref. [28], a sequel to [27], treats a more general eta function on a totally real field of degree n via a similar argument based on the RHB correspondence. On the other hand, ref. [29] adopted the RHB correspondence, streamlining [20,21].

Our main aim is to elucidate the (Hecke) modular relation structure involved in earlier works by Rademacher, Dieter, and Schoenberg et al. and make further developments. In this paper we confine ourselves to the case of the Lambert series, but as we will see, there appear the Koshlyakov transforms, which are used recently, cf. [30].

Notation and symbols. Let

$$\ell_s(x) = \sum_{n=1}^{\infty} \frac{e^{2\pi i n x}}{n^s}, \quad \sigma > 1,\, x \in \mathbb{R} \quad \text{or } s = 1,\, 0 < x < 1$$

be the Lerch zeta function and

$$\zeta(x, x) = \sum_{n=0}^{\infty} \frac{1}{(n+x)^s} \quad 0 < x \le 1$$

be the Hurwitz zeta function, respectively. For $x = 1$ (and $\sigma > 1$), they reduce to the Riemann zeta function

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} \quad \sigma = \operatorname{Re} s > 1.$$

We make use of the vector space structures in the scone variable x of both these functions, for which we refer to [31–33]. Let $C(s) = \{a(n)\}$ be the vector space of all periodic arithmetic functions with period $c \in \mathbb{N}$ and let $D(c)$ be the corresponding space of the Dirichlet series $f(s) = \sum_{n=1}^{\infty} \frac{a(n)}{n^s}$, both with a dimension c. It is shown that one basis of $C(c)$ is the set of characters and the other is their orthogonality relation, which yields the bases of $D(c)$: $\{\ell_s(\frac{\nu}{c})|\nu = 1, \cdots, c\}$ and $\{\zeta(s, \frac{\nu}{c})|\nu = 1, \cdots, c\}$, respectively. One of the base change formulas

$$\ell_z\left(\frac{\nu}{c}\right) = c^{-z} \sum_{\lambda=1}^{c} e^{2\pi i \frac{\nu}{c}\lambda} \zeta\left(z, \frac{\lambda}{c}\right). \tag{1}$$

will play an important role.

$\ell_1(x)$ is not defined at integer points x and needs separate consideration. E.g., its odd part

$$\frac{1}{2}(\ell_1(x) - \ell_1(1-s)) = -\pi i \bar{B}_1(x) \tag{2}$$

is discontinuous at integer points x but has the value 0. The same applies to $\ell_0(x)$.

Another important vector space is the space $\mathcal{K}_s$ of the Kubert functions, which are periodic functions with period 1, satisfying the Kubert relation:

$$\sum_{r=0}^{m-1} f\left(\frac{x+r}{m}\right) = m^{1-s} f(x).$$

Cf., Milnor [34]. $\mathcal{K}_s$ is of dimension 2 and is spanned by $\ell_s(x)$ and $\ell_s(1-x)$ for $s \neq$ negative integers while it is spanned by $\zeta(s,x)$ and $\zeta(s,1-x)$ for $s \neq$ non-negative integers. The Kubert relations

$$\sum_{\mu=1}^{c} \ell_s\left(\frac{x+\mu}{c}\right) = c^{1-s}\ell_s(x), \quad 0 < x < 1 \tag{3}$$
$$\sum_{\mu=1}^{c} \zeta\left(s, \frac{x+\mu}{c}\right) = c^{1-s}\zeta(s,x), \quad 0 < x \leq 1$$

hold for $s \in \mathbb{C}$ except for singularities.

Since every element of $\mathcal{K}_s$ is a linear combination of these two zeta functions, we write

$$f(s,x) \leftrightarrow \zeta(s,x), \quad g(s,x) \leftrightarrow \ell_s(x)$$

to mean that $f(s,x)$ is of the Hurwitz zeta-type resp. $g(s,x)$ of the Lerch zeta-type, satisfying the same conditions as $\zeta(s,x)$ resp. $\ell_s(x)$ does. This in particular applies to their even and odd parts.

Define

$$\mathcal{E}_c^{a,b}(f,g,w,z) = \sum_{\lambda=1}^{c} f\left(w, 1-\left\{\frac{a\lambda}{c}\right\}\right) g\left(z, 1-\left\{\frac{b\lambda}{c}\right\}\right). \tag{4}$$

Equation (4) is Estermann's type of Dedekind sum whose concrete case will appear in the second proof of Theorem 1. We substitute the functional equation

$$f(1-w,x) = \frac{\Gamma(w)}{(2\pi)^w}\left(e^{-\frac{\pi i}{2}w} g(w,x) + e^{\frac{\pi i}{2}w} g(w,1-x)\right)$$

or

$$g(1-z,x) = \frac{\Gamma(z)}{(2\pi)^z}\left(e^{-\frac{\pi i}{2}z} f(z,1-x) + e^{\frac{\pi i}{2}z} f(z,x)\right).$$

as the case may be to deduce

$$\begin{aligned} f(1-w,x)g(1-z,y) &= \frac{\Gamma(w)\Gamma(z)}{(2\pi)^{w+z}}\big(e^{-\frac{\pi i}{2}(w+z)} f(w,1-x)g(z,y) \\ &+ e^{\frac{\pi i}{2}(w+z)} f(w,x)g(z,1-y) + e^{\frac{\pi i}{2}(w-z)} f(w,x)g(z,y) \\ &+ e^{-\frac{\pi i}{2}(w-z)} f(w,1-x)g(z,1-y)\big) \end{aligned} \tag{5}$$

This will appear in Section 5.

It was Mikolás [35] who first introduced the transcendental generalization of the Dedekind sums in which, instead of (4), the f, f-type zeta functions are considered as with almost all preceding papers. In the second proof of Theorem 1, we will reveal that the Estermann-type zeta functions makes things simpler.

2. The Rademacher–Apostol Case

In this section we display the elucidation of Rademacher's integral transform method by showing the functional equation for the zeta function and the general ETF as developed in Rademacher [16] (for eta function) and also by Apostol [17] (for the Lambert series). The residual function in Theorem 1 is the corrected form of that of [17] in the form nearest to Apostol's. This corrected form was first proved by Mikolás [36] (p. 106) and shortly thereafter by Iseki [14], both of whom treated the case $p \geq 1$. Then, as stated above, ref. [22] proved the Hecke functional equation in the case $p = 1$ and Apostol [23] used the same method to treat the case $p > 1$, without mentioning the RHB correspondence.

Toward the end we shall briefly explain the case of Krätzel [37].

Let $c \in \mathbb{N}$, $p \geq 1$ be an odd integer and let h be an integer such that $(h, c) = 1$. Define the Rademacher–Apostol zeta function

$$Z_p(s, h) = \sum_{\mu, \nu = 1}^{c} e^{\frac{2\pi i h \mu \nu}{c}} \zeta\left(s, \frac{\mu}{c}\right) \zeta\left(s + p, \frac{\nu}{c}\right). \tag{6}$$

Let

$$g_p(x) = g_p\left(e^{2\pi i \frac{iz+h}{c}}\right) = \frac{1}{2\pi i} \int_{(\gamma)} \Gamma(s) Z_p(s, h) c^{-1} (2\pi c z)^{-s} \, \mathrm{d}s, \tag{7}$$

be the Hecke gamma transform of $Z_p(s, h)$ as in [16] (1.14), where $\gamma > 1$.

Theorem 1. *The zeta function $Z_p(s, h)$ satisfies the Hecke functional equation*

$$(2\pi c)^{-s - \frac{p-1}{2}} \Gamma(s) Z_p(s, h) = (2\pi c)^{s + \frac{p-1}{2}} (-1)^{\frac{p-1}{2}} \Gamma(-s) Z_p(1 - p - s, H), \tag{8}$$

where H is an integer such that

$$hH \equiv -1 \bmod c. \tag{9}$$

The Lambert series (7) satisfies the transformation formula

$$g_p\left(e^{2\pi i \frac{iz+h}{c}}\right) = g_p\left(e^{2\pi i \frac{iz^{-1}+H}{c}}\right) + \mathrm{P}_p(z), \tag{10}$$

where

$$\begin{aligned} \mathrm{P}_p(z) &= \mathrm{Res}_{s=-p,\cdots,0,1} \Gamma(s) Z_p(s, h) c^{-p} (2\pi c z)^{-s} \\ &= \frac{-1}{2(p+1)!} \left(\frac{2\pi z}{c}\right)^p B_{p+1} + \frac{(-1)^{\frac{p-1}{2}}}{2(p+1)!} \left(\frac{2\pi}{c}\right)^p z^{-1} B_{p+1} \\ &+ \frac{-i(2\pi i)^p}{2(p)!} s_{p,1}(c, h) + \frac{1}{2} \delta_{p,1} \log a + \frac{1}{2}\left(1 - (-1)^{\frac{p-1}{2}}\right) \zeta(p) \\ &+ \sum_{r=2}^{p} \frac{(-1)^r}{r!} (2\pi z)^{r-1} \frac{-(2\pi i)^{p+1-r}}{2(p+1-r)!} s_{p,r}(c, h), \end{aligned} \tag{11}$$

and where $\delta_{p,1}$ is the Kronecker symbol.

Proof. We combine the Hurwitz Formula (12) and the base change Formula (13) with $f = \chi_\mu$ to deduce (14): The Hurwitz formula (i.e., the functional equation for the Hurwitz zeta function): for $\sigma > 1, 0 < x \leq 1$,

$$\zeta(1 - s, x) = \frac{\Gamma(s)}{(2\pi)^s} \left(e^{-\frac{\pi i s}{2}} \ell_s(x) + e^{\frac{\pi i s}{2}} \ell_s(1 - x)\right). \tag{12}$$

The base change—linear combination expression—formula reads

$$\begin{aligned} \frac{1}{c^s} \sum_{a=1}^{c} a(n) \zeta\left(s, \frac{n}{c}\right) &= D(s, a) = \frac{1}{\sqrt{c}} \sum_{n=1}^{c} \hat{a}(n) \ell_s\left(\frac{n}{c}\right) \\ &= \frac{1}{\sqrt{c}} \sum_{n=1}^{c-1} \hat{a}(n) \ell_s\left(\frac{n}{c}\right) + \frac{\hat{a}(c)}{\sqrt{c}} \zeta(s), \end{aligned} \tag{13}$$

where $\hat{a}(n)$ is the DFT (discrete Fourier transform) of $a(n)$. Choosing $a(n) = \chi_\mu(n)$, $\chi\mu$ being the characteristic function of μ, we see that its DFT is the character, which implies (1).

Combining (12) and (1), we deduce

$$\zeta\left(s,\frac{\mu}{c}\right)=\Gamma(1-s)\frac{2}{(2\pi c)^{1-s}}\tag{14}$$
$$\times\left(\sin\frac{\pi}{2}s\sum_{\lambda=1}^{c}\cos\frac{2\pi\lambda\mu}{c}\zeta\left(1-s,\frac{\lambda}{c}\right)+\cos\frac{\pi}{2}s\sum_{\lambda=1}^{c}\sin\frac{2\pi\lambda\mu}{c}\zeta\left(1-s,\frac{\lambda}{c}\right)\right).$$

Substituting (14) in (6) and using

$$\sum_{\mu=1}^{c}e^{\frac{2\pi ih\mu\nu}{c}}\cos\frac{2\pi\lambda\mu}{c}=\sum_{\mu=1}^{c}\cos\frac{2\pi h\mu\nu}{c}\cos\frac{2\pi\lambda\mu}{c}\tag{15}$$
$$\sum_{\mu=1}^{c}e^{\frac{2\pi ih\mu\nu}{c}}\sin\frac{2\pi\lambda\mu}{c}=\sum_{\mu=1}^{c}\sin\frac{2\pi h\mu\nu}{c}\sin\frac{2\pi\lambda\mu}{c},$$

we conclude that

$$Z_p(s,h)=c^{-1}(2\pi c)^s\left(\sum_{\lambda,\mu,\nu=1}^{c}\cos\frac{2\pi h\mu\nu}{c}\cos\frac{2\pi\lambda\mu}{c}\frac{1}{\cos\frac{\pi}{2}s}\zeta\left(1-s,\frac{\lambda}{c}\right)\zeta\left(p+s,\frac{\nu}{c}\right)\right.\tag{16}$$
$$\left.+\sum_{\lambda,\mu,\nu=1}^{c}\sin\frac{2\pi h\mu\nu}{c}\sin\frac{2\pi\lambda\mu}{c}\frac{1}{\sin\frac{\pi}{2}s}\zeta\left(1-s,\frac{\lambda}{c}\right)\zeta\left(p+s,\frac{\nu}{c}\right)\right).$$

Changing s by $1-p-s$ and μ by $H\mu$, where H is as in (9), then the second factor remains unchanged up to the additional factor $(-1)^{\frac{p-1}{2}}$. Hence,

$$Z_p(1-p-s,H)=(2\pi c)^{1-p-2s}(-1)^{\frac{p-1}{2}}Z_p(s,h),$$

which is (8).

Substituting (16) in (7), we derive that

$$g_p(x)=\frac{1}{2c^{p+1}}\tag{17}$$
$$\times\left(\sum_{\lambda,\mu,\nu=1}^{c}\cos\frac{2\pi h\mu\nu}{c}\cos\frac{2\pi\lambda\mu}{c}\frac{1}{2\pi i}\int_{(\gamma)}\frac{1}{\cos\frac{\pi}{2}s}\zeta\left(1-s,\frac{\lambda}{c}\right)\zeta\left(s+p,\frac{\nu}{c}\right)z^{-s}\,ds\right.$$
$$\left.+\sum_{\lambda,\mu,\nu=1}^{c}\sin\frac{2\pi h\mu\nu}{c}\sin\frac{2\pi\lambda\mu}{c}\frac{1}{2\pi i}\int_{(\gamma)}\frac{1}{\sin\frac{\pi}{2}s}\zeta\left(1-s,\frac{\lambda}{c}\right)\zeta\left(s+p,\frac{\nu}{c}\right)z^{-s}\,ds\right),$$

which is [16] (1.27).

Shifting the integration path to $\sigma=1-p-\gamma$ and applying (8), we conclude [16] (1.29), which is (10).

Incorporating the residual function found in [17] with the correction calculated in [30], we arrive at the general transformation formula, entailing ETF [16] (1.45), completing the proof. □

Proof. We may give a more lucid proof of (8) using the Estermann-type Dedekind sum

$$\mathcal{E}_c^{a,b}(w,z)=\sum_{\lambda \bmod c}\zeta\left(w,1-\left\{\frac{a\lambda}{c}\right\}\right)\ell_z\left(1-\left\{\frac{b\lambda}{c}\right\}\right)\tag{18}$$
$$=\sum_{\lambda=1}^{c-1}\zeta\left(w,\left\{\frac{a\lambda}{c}\right\}\right)\ell_z\left(\frac{b\lambda}{c}\right)+\zeta(w)\zeta(z).$$

Estermann [38] (19) established the functional equation

$$\mathcal{E}_c^{a,1}(s,s) = -2(2\pi)^{2s-2}\Gamma^2(1-s)\Big(\cos(\pi s)\mathcal{E}_c^{1,-a}(1-s,1-s) - \mathcal{E}_c^{1,a}(1-s,1-s)\Big), \quad (19)$$

which is a special case of the more general functional equation

$$\mathcal{E}_c^{a,b}(1-w,1-z) = \frac{2\Gamma(w)\Gamma(z)}{(2\pi)^{w+z}} \times \Big(\cos\frac{\pi}{2}(w+s)\mathcal{E}_c^{b,-a}(z,w) + \cos\frac{\pi}{2}(w-s)\mathcal{E}_c^{b,a}(z,w)\Big). \quad (20)$$

We consider the sum slightly more general than (6):

$$I_p(w,z,h) := \sum_{\mu,\nu=1}^{c} e^{\frac{2\pi i h\mu\nu}{c}}\zeta\Big(w,\frac{\mu}{c}\Big)\zeta\Big(z,\frac{\nu}{c}\Big) = \sum_{\mu=1}^{c}\zeta\Big(w,\frac{\mu}{c}\Big)\sum_{\nu=1}^{c} e^{\frac{2\pi i h\mu\nu}{c}}\zeta\Big(z,\frac{\nu}{c}\Big). \quad (21)$$

The inner sum on the right of (21) is $c^z\ell_z\Big(\frac{h\mu}{c}\Big)$ in view of the base change Formula (1) becomes

$$I_p(w,z,h) = c^z\sum_{\mu=1}^{c}\zeta\Big(w,\frac{\mu}{c}\Big)\ell_z\Big(\frac{h\mu}{c}\Big) = c^z\mathcal{E}_c^{1,h}(w,z), \quad (22)$$

which becomes

$$Z_p(s,h) = I_p(s,s+p,h) = c^{s+p}\mathcal{E}_c^{1,h}(s,s+p), \quad (23)$$

on specifying $w = s$, $z - p + s$. Hence, substituting (20) in (22), we deduce that

$$I_p(w,z,h) = c^z\frac{2\Gamma(1-w)\Gamma(1-z)}{(2\pi)^{2-w-z}} \times \Big(-\cos\frac{\pi}{2}(w+s)\mathcal{E}_c^{-h,1}(1-z,1-w) + \cos\frac{\pi}{2}(w-s)\mathcal{E}_c^{h,1}(1-z,1-w)\Big). \quad (24)$$

Specifying $w = s$, $z - p + s$, (24) reads

$$Z_p(s,h) = I_p(s,s+p,h) = c^{s+p}\frac{2\Gamma(1-s)\Gamma(1-p-s)}{(2\pi)^{2-2s-p}} \times \Big(-\cos\frac{\pi}{2}(2s+p)\mathcal{E}_c^{-h,1}(1-p-s,1-s) + \cos\frac{\pi}{2}p\mathcal{E}_c^{h,1}(1-p-s,1-s)\Big). \quad (25)$$

Taking the oddness of p into account, this reduces to

$$Z_p(s,h) = c^{s+p}\frac{2\Gamma(1-s)\Gamma(1-p-s)}{(2\pi)^{2-p-2s}}(-1)^{\frac{p-1}{s}}\sin\pi s\mathcal{E}_c^{-h,1}(1-p-s,1-s),$$

whence

$$\Gamma(s)Z_p(s,h) = c^{s+p}\frac{\Gamma(1-p-s)}{(2\pi)^{1-p-2s}}(-1)^{\frac{p-1}{s}}\mathcal{E}_c^{-h,1}(1-p-s,1-s). \quad (26)$$

Now, let H be as in (9). Then,

$$\mathcal{E}_c^{-h,1}(1-p-s,1-s) = \mathcal{E}_c^{1,H}(1-p-s,1-s) = c^{1-s}Z_p(1-p-s,H)$$

by (23). Substituting this in (26) proves (8). □

Proof. We may restore the argument of [16,17] to prove (10) and the proof entails the proof of (8), cf. [30]. □

3. The Krätzel Case

Ref. [37] deals with a generalization (38) of the eta function which depends on the Hecke gamma transform of the zeta function

$$Z_{a.b}(s) := \frac{1}{\Gamma(s+1)\sin\frac{\pi}{2ab}s}\zeta\left(\frac{1}{a}s\right)\zeta\left(-\frac{1}{b}s\right), \tag{27}$$

where a, b are natural numbers, $(a, b) = 1$. $Z_{a.b}(s)$ satisfies the Hecke functional equation

$$\Gamma(s)Z_{a.b}(s) = \Gamma(-s)Z_{b,a}(-s). \tag{28}$$

Krätzel's method is essentially that of Rademacher, although he does not refer to [16], and we give a brief account on this point.

Theorem 2. *The Krätzel–Rademacher method yields the modular relation* (28) *as well as the transformation formula*

$$\eta_{a,b}(x) = x^{-\frac{ab}{2}}\eta_{b,a}\left(\frac{1}{x}\right). \tag{29}$$

Proof. For the moment, we work with ($\mathrm{Re}\, x > 0$ and $|\arg z| < \frac{\pi}{2ab}$)

$$\tilde{\eta}_{a,b}(x) := \prod_{m=1}^{\infty}\prod_{\nu=0}^{a-1}\left(1 - e^{2\pi i \varepsilon_{2\nu+1}(4a) n^{\frac{b}{a}} x^b}\right), \tag{30}$$

where $\varepsilon_{2\nu+1}(4a) = e^{2\pi i \frac{2\nu+1}{4a}}$. Then, for $\varkappa > \frac{a}{b}$, we have by the Hecke gamma transform

$$\log\tilde{\eta}_{a,b}(x) = -\frac{1}{2\pi i}\int_{(\varkappa)}\Gamma(s)\zeta(s+1)\zeta\left(\frac{b}{a}s\right)\sum_{\nu=0}^{a-1}\left(e^{2\pi i\frac{2\nu+1}{4a}}\right)^{-s}\left(2\pi x^b\right)^{-s} ds. \tag{31}$$

Now the sum becomes

$$\sum_{\nu=0}^{a-1}\left(e^{2\pi i\frac{2\nu+1}{4a}}\right)^{-s} = \sum_{\nu=1}^{a}\left(e^{2\pi i\frac{2\nu-1}{4a}}\right)^{-s} = \frac{\sin\frac{\pi}{2}s}{sin\frac{\pi}{2a}s}.$$

Hence, (31) becomes

$$\log\tilde{\eta}_{a,b}(x) = -\frac{1}{2\pi i}\int_{(\varkappa)}\Gamma(s)\frac{\sin\frac{\pi}{2}s}{\sin\frac{\pi}{2a}s}\zeta(s+1)\zeta\left(\frac{b}{a}s\right)\left(2\pi x^b\right)^{-s} ds. \tag{32}$$

Now we apply the functional equation *only to one factor* $\zeta(s+1)$:

$$\zeta(s+1) = -(2\pi)^s\frac{\pi}{\Gamma(s+1)\sin\frac{\pi}{2}s}\zeta(-s). \tag{33}$$

Substituting (33) in (32), we obtain

$$\log\tilde{\eta}_{a,b}(x) = \frac{1}{2\pi i}\int_{(\varkappa)}\frac{\Gamma(s)}{\Gamma(s+1)\sin\frac{\pi}{2a}s}\pi\zeta(-s)\zeta\left(\frac{b}{a}s\right)\left(x^b\right)^{-s} ds. \tag{34}$$

Note that the factor $\frac{\Gamma(s)}{\Gamma(s+1)}ds$ being $\frac{1}{s}ds$ remains invariant under the change of variable $s \to as$, so that (34) becomes as in Krätzel,

$$\log\tilde{\eta}_{a,b}(x) = \frac{1}{2\pi i}\int_{(\varkappa_1)}\frac{\Gamma(s)}{\Gamma(s+1)\sin\frac{\pi}{2}s}\pi\zeta(-as)\zeta(bs)\left(x^{ab}\right)^{-s} ds, \tag{35}$$

where $\varkappa_1 > \frac{1}{b}$. These two are the main ingredients of Krätzel and correspond to Rademacher's (17).

Changing the variable $s \to abs$, (35) becomes

$$\log \tilde{\eta}_{a,b}(x) = \frac{1}{2\pi i}\int_{(\varkappa_2)} \Gamma(s) Z_{a,b}(s) x^{-s}\, \mathrm{d}s, \tag{36}$$

i.e., the Hecke gamma transform of $Z_{a,b}(s)$, where $\varkappa_2 > a$. As usual, shifting the integration path to $\sigma = -\varkappa_2 < -\frac{1}{a}$, we encounter poles and we are to find residues. The resulting integral is the same as (36), with x changed by $\frac{1}{x}$. Krätzel writes [37] (p. 116), "Then under the substitution $s \to -s$, the functional Equation (28) follows on symmetry grounds", meaning that he proves (28) at this stage.

Krätzel treats (35) and shifts the line to $-\varkappa_2 < -\frac{1}{a}$, finding the sum of residues

$$-\gamma_{a,b}(x) + \gamma_{b,a}\left(\frac{1}{x}\right) + \frac{1}{2}(b-a)\log 2\pi - \frac{ab}{2}x, \tag{37}$$

where

$$\gamma_{a,b}(x) = \frac{\pi}{\sin\frac{\pi}{2a}} \zeta\left(-\frac{b}{a}\right) x^b.$$

Hence, defining

$$\eta_{a,b}(x) = (2\pi)^{\frac{1-b}{2}} e^{\gamma_{a,b}(x)} \tilde{\eta}_{a,b}(x), \tag{38}$$

we conclude (29). □

4. Unification of Rademacher and Dieter Cases

In this section we prove the modular relation structure of the zeta functions and the general ETFs contained in [16–18]. We work in the framework of Dieter with slight modifications. Let $p, d, f, \alpha, and \beta$ be integers satisfying the conditions $p \geq 1$ being odd, $(h, c) = 1$, $f \geq 1$, $0 < \alpha \leq f$. f works as a fixed aixiliary modulus and $d = -h$ in Section 2. In Dieter's case, $\alpha, \beta \not\equiv 0 \bmod f$ is also assumed. Then, the Dieter zeta function is defined by

$$f_{\alpha,\beta}(s,x) = f_{p,\alpha,\beta}\left(s, e^{2\pi i \frac{iz+h}{c}}\right) = \sum_{\mu=0}^{c-1} \sum_{\nu=1}^{fc} e^{2\pi i \frac{h\mu\nu + \gamma\nu}{c}} \zeta\left(s, \frac{\mu}{c} + \frac{\alpha}{cf}\right) \zeta\left(s+p, \frac{\nu}{cf}\right), \tag{39}$$

where

$$\gamma(-\alpha,-\beta) = -\gamma(\alpha,\beta), \quad \gamma = \gamma(\alpha,\beta) = \frac{-h\alpha - c\beta}{f}. \tag{40}$$

We assume $\gamma(-\alpha, -\beta) = \gamma(\alpha, \beta)$ for $\alpha, \beta \equiv 0 \bmod f$, which we to $\gamma(0,0)$. We also assume that μ varies $1, \cdots, c$ in the case of $\gamma(0,0)$. Then, (39) with $p = 1$ amounts to (6). In almost all subsequent studies after Rademacher, it is necessary to consider the even part [18] (2,11), which is

$$g_{\alpha,\beta}(s,x) := f_{\alpha,\beta}(s,x) + f_{-\alpha,-\beta}(s,x). \tag{41}$$

One speculated reason for this is stated in [30].

Let

$$G_p(x) = G_p\left(e^{2\pi i \frac{iz+h}{c}}\right) = \frac{1}{2\pi i}\int_{(\gamma)} \Gamma(s) g_{\alpha,\beta}(s,x)(cf)^{-1}(2\pi c z)^{-s}\, \mathrm{d}s, \tag{42}$$

be the Hecke gamma transform, where $\gamma > 1$.

Theorem 3. *Rademacher's transform yields the transformation formula*

$$G_{p,\alpha,\beta}\left(e^{2\pi i \frac{iz+h}{c}}\right) = G_{p,\alpha,\beta}\left(e^{2\pi i \frac{iz^{-1}+H}{c}}\right) + \mathrm{P}(z), \tag{43}$$

where

$$\mathrm{P}(z) = \sum_{s=p,\cdots,0,1} \mathrm{Res}\Gamma(s) g_{\alpha,\beta}(s,x)(cf)^{-1}(2\pi cz)^{-s}. \tag{44}$$

as well as the Hecke functional equation for the even part $g_{\alpha,\beta}(s,x)$ of the Dieter zeta function

$$(2\pi cf)^{-s-\frac{p-1}{2}}\Gamma(s) g_{p,\alpha,\beta}(s,x) = (2\pi cf)^{s+\frac{p-1}{2}}(-1)^{\frac{p-1}{2}}\Gamma(1-p-s) g_{p,\alpha',\beta'}(1-p-s,x), \tag{45}$$

where H is an integer as in (9) and

$$\begin{pmatrix}\alpha\\ \beta\end{pmatrix} = \begin{pmatrix} H & c\\ b & -h\end{pmatrix}\begin{pmatrix}\alpha'\\ \beta'\end{pmatrix}. \tag{46}$$

The theorem also covers Theorem 1.

Proof. We give a proof verbatim to that of Theorem 1. We employ (14) as

$$\zeta\left(s, \frac{\mu}{c} + \frac{\alpha}{cf}\right) = \Gamma(1-s)\frac{2}{(2\pi cf)^{1-s}}\left(\sin\frac{\pi}{2}s\sum_{\lambda=1}^{c}\cos 2\pi\lambda\left(\frac{\mu}{c}+\frac{\alpha}{cf}\right)\zeta\left(1-s,\frac{\lambda}{cf}\right)\right. \tag{47}$$
$$\left. + \cos\frac{\pi}{2}s\sum_{\lambda=1}^{c}\sin 2\pi\lambda\left(\frac{\mu}{c}+\frac{\alpha}{cf}\right)\zeta\left(1-s,\frac{\lambda}{cf}\right)\right).$$

Substituting (14) in (8), we find that

$$\begin{aligned} &c(2\pi cf)^{-s}\Gamma(s) f_{\alpha,\beta}(s,x) \\ &= \sum_{\lambda,\nu=1}^{fc}\left(\sum_{\mu=0}^{c-1} e^{2\pi i\frac{h\mu\nu+\gamma\nu}{c}}\cos 2\pi\lambda\left(\frac{\mu}{c}+\frac{\alpha}{cf}\right)\frac{1}{\cos\frac{\pi}{2}s}\zeta\left(1-s,\frac{\lambda}{cf}\right)\zeta\left(p+s,\frac{\nu}{cf}\right)\right. \\ &\left. + \sum_{\mu=0}^{c-1} e^{2\pi i\frac{h\mu\nu+\gamma\nu}{c}}\sin 2\pi\lambda\left(\frac{\mu}{c}+\frac{\alpha}{cf}\right)\frac{1}{\sin\frac{\pi}{2}s}\zeta\left(1-s,\frac{\lambda}{cf}\right)\zeta\left(p+s,\frac{\nu}{cf}\right)\right). \end{aligned} \tag{48}$$

To proceed further with the non-degenerated (48) we need a counterpart of (15) and for this we need to consider the even part [18] (2,11), which is (41).

Then, we are to incorporate

$$\begin{aligned} &\sum_{\mu=0}^{c-1} e^{2\pi i\frac{h\mu\nu+\gamma\nu}{c}}\cos 2\pi\lambda\left(\frac{\mu}{c}+\frac{\alpha}{cf}\right) + \sum_{\mu=0}^{c-1} e^{2\pi i\frac{h\mu\nu-\gamma\nu}{c}}\cos 2\pi\lambda\left(\frac{\mu}{c}-\frac{\alpha}{cf}\right) \\ &= \sum_{\mu=0}^{c-1} e^{2\pi i\frac{h\mu\nu+\gamma\nu}{c}}\cos 2\pi\lambda\left(\frac{\mu}{c}+\frac{\alpha}{cf}\right) + \sum_{\mu=0}^{c-1} e^{2\pi i\frac{-h\mu\nu-\gamma\nu}{c}}\cos 2\pi\lambda\left(\frac{\mu}{c}+\frac{\alpha}{cf}\right) \\ &= 2\left(\sum_{\mu=0}^{c-1}\mathrm{Re}\left(e^{2\pi i\frac{h\mu\nu+\gamma\nu}{c}}\right)\cos 2\pi\lambda\left(\frac{\mu}{c}+\frac{\alpha}{cf}\right)\right) \\ &= 2\sum_{\mu=0}^{c-1}\cos 2\pi\frac{h\mu\nu+\gamma\nu}{c}\cos 2\pi\left(\lambda\frac{\mu}{c}+\frac{\alpha}{cf}\right) \end{aligned} \tag{49}$$

and

$$\begin{aligned} &\sum_{\mu=0}^{c-1} e^{2\pi i\frac{h\mu\nu+\gamma\nu}{c}}\sin 2\pi\lambda\left(\frac{\mu}{c}+\frac{\alpha}{cf}\right) + \sum_{\mu=0}^{c-1} e^{2\pi i\frac{h\mu\nu-\gamma\nu}{c}}\sin 2\pi\lambda\left(\frac{\mu}{c}-\frac{\alpha}{cf}\right) \\ &= -2i\sum_{\mu=0}^{c-1}\sin 2\pi\frac{h\mu\nu+\gamma\nu}{c}\sin 2\pi\left(\lambda\frac{\mu}{c}+\frac{\alpha}{cf}\right). \end{aligned} \tag{50}$$

Substituting in (48), we obtain

$$\begin{aligned} &c(2\pi cf)^{-s}\Gamma(s)g_{\alpha,\beta}(s,x) \qquad (51)\\ &=2\sum_{\lambda,\nu=1}^{fc}\sum_{\mu=0}^{c-1}\cos 2\pi\frac{h\mu\nu+\gamma\nu}{c}\cos 2\pi\left(\lambda\frac{\mu}{c}+\frac{\alpha}{cf}\right)\frac{1}{\cos\frac{\pi}{2}s}\zeta\left(1-s,\frac{\lambda}{cf}\right)\zeta\left(p+s,\frac{\nu}{cf}\right)\\ &-2i\sum_{\lambda,\nu=1}^{fc}\sum_{\mu=0}^{c-1}\sin 2\pi\frac{h\mu\nu+\gamma\nu}{c}\sin 2\pi\left(\lambda\frac{\mu}{c}+\frac{\alpha}{cf}\right)\frac{1}{\sin\frac{\pi}{2}s}\zeta\left(1-s,\frac{\lambda}{cf}\right)\zeta\left(p+s,\frac{\nu}{cf}\right). \end{aligned}$$

Changing s by $1-p-s$ and μ by $H\mu$, where $hH\equiv -1 \bmod c$, then the right-hand side of (51) is changed into the one with the factor $(-1)^{\frac{p-1}{2}}$ and with the new pair of parameters α',β'. Hence,

$$c(2\pi cf)^{s+p-1}(-1)^{\frac{p-1}{2}}\Gamma(1-p-s)g_{\alpha',\beta'}(s,x)=c(2\pi cf)^{-s}\Gamma(s)g_{\alpha,\beta}(s,x),$$

which is (45).

Shifting the integration path in (42) to $\sigma=1-p-\gamma$ and applying (8) establishes the assertion. The residual function (44) may be found in [18] (p. 48).

The degenerate case of (48) leads to a generalization of Rademacher's functional equation. Indeed, (48) with $f=1,\gamma(0,0)$ reads

$$\begin{aligned} &c(2\pi c)^{-s}\Gamma(s)f_{0,0}(s,x) \qquad (52)\\ &=\sum_{\lambda,\nu=1}^{c}\left(\sum_{\mu=1}^{c}e^{\frac{2\pi ih\mu\nu}{c}}\cos\frac{2\pi\lambda\mu}{c}\frac{1}{\cos\frac{\pi}{2}s}\zeta\left(1-s,\frac{\lambda}{c}\right)\zeta\left(p+s,\frac{\nu}{c}\right)\right.\\ &\left.+\sum_{\mu=1}^{c}e^{\frac{2\pi ih\mu\nu}{c}}\sin\frac{2\pi\lambda\mu}{c}\frac{1}{\sin\frac{\pi}{2}s}\zeta\left(1-s,\frac{\lambda}{c}\right)\zeta\left(p+s,\frac{\nu}{c}\right)\right). \end{aligned}$$

Substituting (15) in (52) proves the Rademacher–Apostol case [39]:

$$(2\pi c)^{-s-\frac{p-1}{2}}\Gamma(s)Z_p(s,h)=(2\pi c)^{s+\frac{p-1}{2}}(-1)^{\frac{p-1}{2}}\Gamma(-s)Z_p(1-p-s,H), \qquad (53)$$

where

$$Z_p(s,h)=\sum_{\mu,\nu=1}^{c}e^{\frac{2\pi ih\mu\nu}{c}}\zeta\left(s,\frac{\mu}{c}\right)\zeta\left(s+p,\frac{\nu}{c}\right).$$

Equation (53) reduces to (8) for $p=1$. □

Other papers dealing with generalizations of the eta function use

$$\mathcal{E}(s,h)=\sum_{\mu,\nu=1}^{c}e^{\frac{2\pi ih\mu\nu}{c}}\zeta\left(s,\frac{\mu}{c}\right)\ell_{s+1}\left(\frac{\nu}{c}\right).$$

instead of (6) and are feasible for descriptions in the form of the Hecke correspondence. We hope to return to the study of this aspect and more general Dedekind sums including one with Kubert functions elsewhere. But, we shall mention one type of Estermann-type in the next section.

5. The Schoenberg Case

This section is concerned with [20], which is reproduced in [21] (pp. 184–202, Chapter VIII). In [21] (p. 184) it is stated that the transition is made from Hecke's Eisenstein series of weight -2 [21] (p. 164) to a linearly equivalent system containing non-analytic function G_2.

We stick to [20] (p. 5), which is directly related to (5).

In particular,

$$\zeta(s,\alpha)\ell_{s+1}(\beta) = \frac{-i(2\pi)^{2s}}{\sin \pi s}\big(- e^{-\pi i s}\zeta(-s, 1-\beta)\ell_{1-s}(\alpha) \tag{54}$$
$$+ e^{\pi i s}\zeta(-s,\beta)\ell_{1-s}(1-\alpha) + \zeta(-s,1-\beta)\ell_{1-s}(1-\alpha)$$
$$- \zeta(-s,\beta)\ell_{1-s}(\alpha)\big).$$

We write $\xi = e^{2\pi i\beta}$ and define the Lambert series [20] (20)

$$U(x;\alpha,\beta) = \sum_{\substack{n>0 \\ m>-\alpha}} \frac{\xi^n}{n} e^{-(m+\alpha)nx}, \quad x > 0. \tag{55}$$

Then, [20] (26) considered the gamma transform of the Estermann-type zeta function

$$U(x;\alpha,\beta) = \frac{1}{2\pi i}\int_{(\varkappa)} \Gamma(s)\zeta(s,\alpha)\ell_{s+1}(\beta)c^{-x}\,\mathrm{d}s, \tag{56}$$

where $\varkappa > 1$. If we substitute (54) into (56), then the integral is hardly tractable. This is why Schoenberg deduced only an asymptotic formula for $U(x;\alpha,\beta)$.

Let

$$\boldsymbol{a} = (a_1, a_2) \in \mathbb{Z}^2, \quad \alpha = \alpha(\boldsymbol{a}) = \frac{a_1}{cN} + \frac{r}{c}, \quad \beta = \beta(\boldsymbol{a}) = \xi_r, \tag{57}$$

where

$$\xi_r = e^{2\pi i\left(\frac{a_1'}{cN} + \frac{qr}{c}\right)}, \quad a_1' = aa_1 + ca_2. \tag{58}$$

Then, we consider

$$X(\boldsymbol{a}) = X(a_1,a_2) = U(x;\alpha,\beta) = U\Big(2\pi cx; \frac{a_1}{cN} + \frac{r}{c}, \xi_r\Big). \tag{59}$$

But, what is needed eventually is an expression for the even part $X(a_1,a_2) + X(-a_1,-a_2)$ ([20] (p. 8)) and we prove the following theorem for the zeta function of the even part.

Theorem 4. *For*

$$Z(s,\alpha,\beta) = \zeta(s,\alpha)\ell_{s+1}(\beta) + \zeta(s,1-\alpha)\ell_{s+1}(1-\beta)$$

and

$$\tilde{Z}(s,\alpha,\beta) = \zeta(s,1-\beta)\ell_{s+1}(\alpha) + \zeta(-s,\beta)\ell_{1-s}(1-\alpha)$$

the functional equation

$$Z(s,\alpha,\beta) = 2(2\pi)^{2s}\tilde{Z}(-s,\alpha,\beta) \tag{60}$$

holds.

Proof. In [20] (p. 7), Schoenberg defined

$$\xi_r' = e^{2\pi i\left(-\frac{-a_1'}{cN} + \frac{qr}{c}\right)} \tag{61}$$

and noted

$$\xi_r' = \xi_{c-r}^{-1}, \tag{62}$$

Hence,

$$\alpha(-\boldsymbol{a}) = 1 - \alpha(\boldsymbol{a}), \quad \beta(-\boldsymbol{a}) = 1 - \beta(\boldsymbol{a}).$$

$$X(-\boldsymbol{a}) = U(x; 1-\alpha, 1-\beta). \tag{63}$$

It follows that when substituting from (54) in $X(\boldsymbol{a}) + X(-\boldsymbol{a})$, the sums with the third and the fourth terms vanish and we sum only the first two terms of (54) and the sine function cancels. Hence, the zeta function $Z(s, \alpha, \beta)$ of $X(\boldsymbol{a}) + X(-\boldsymbol{a})$ is

$$\begin{aligned} &Z(s,\alpha,\beta) \\ &= \frac{-i(2\pi)^{2s}}{\sin \pi s}\left(-e^{-\pi i s} + e^{\pi i s}\right) \\ &\Big(\zeta(-s, 1-\alpha(\boldsymbol{a}))\ell_{1-s}(-\alpha(\boldsymbol{a})) + \zeta(-s, \beta(\boldsymbol{a}))\ell_{1-s}(\alpha(\boldsymbol{a}))\Big) \\ &= 2(2\pi)^{2s}(\zeta(-s, 1-\beta)\ell_{1-s}(\alpha) + \zeta(-s, \beta)\ell_{1-s}(1-\alpha)), \end{aligned} \tag{64}$$

which proves (60). □

Hence, what comes out is the Hecke gamma transform of a tractable function and the process onwards is verbatim to that of the preceding sections and we do not go into details.

Remark 1. *By taking up the Dedekind eta function, one of the most famous example of a half-integral weight modular form, we have made clear how deeply the RHB correspondence lies in the general transformation formula, not restricted to the functional equation. We have restored Rademacher's opus [16] by streamlining the history that it is his own method of using integral transforms preceding Hecke to deduce the general transformation formula rather than the PFE of the cotangent function. We have also clarified Koshlyakov's intervention using the Fourier–Bessel expansion and thus PFE.*

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