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Article

On the q-Laplace Transform and Related Special Functions

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Abstract: Motivated by statistical mechanics contexts, we study the properties of the q-Laplace transform, which is an extension of the well-known Laplace transform. In many circumstances, the kernel function to evaluate certain integral forms has been studied. In this article, we establish relationships between q-exponential and other well-known functional forms, such as Mittag–Leffler functions, hypergeometric and H-function, by means of the kernel function of the integral. Traditionally, we have been applying the Laplace transform method to solve differential equations and boundary value problems. Here, we propose an alternative, the q-Laplace transform method, to solve differential equations, such as as the fractional space-time diffusion equation, the generalized kinetic equation and the time fractional heat equation.

Keywords: convolution property; G-transform; Gauss hypergeometric function; generalized kinetic equation; Laplace transform; Mittag–Leffler function; versatile integral

1. Introduction

The classical Laplace, Fourier and Mellin transforms have been widely used in mathematical physics and applied mathematics. The theory of the Laplace transform is well-known [1], and its generalization was considered by many authors [2–6]. Various existence conditions and detailed study about the range and invertibility were studied by Rooney [7]. The Laplace transform and Mellin transform are widely used together to solve the fractional kinetic equations and thermonuclear equations [8,9]. Different types of integral transforms, like the Hankel transform, Erdély–Kober type fractional integration operators, the Gauss hypergeometric function as a kernel, the Bessel-type integral transform, etc. [10], are introduced in the literature to solve the boundary value problems for models of ordinary and partial differential equations. In some situations, the solutions of the differential equation cannot be tractable using the classical integral transforms, but may be characterized by many integral transforms with various special functions as kernels. Many of the integral transforms can be interpreted in terms of the G-transform and H-transform [11–16].

In physical situations when an appropriate density is selected, the best practice is to maximize the entropy. Mathai and Rathie [17] considered various generalizations of the Shannon entropy measure and describe various properties, including additivity, the characterization theorem, etc. Mathai and Haubold [18] introduced a new generalized entropy measure, which is a generalization of the Shannon entropy measure. For a multinomial population $P = (p_1, \dots, p_k), p_i \geq 0,$

$i = 1, \ldots, k, p_1 + p_2 + \cdots + p_k = 1$, the Mathai's entropy measure (discrete case) is given by the relation:

$$M_{k,\alpha}(P) = \frac{\sum_{i=1}^{k} p_i^{2-\alpha} - 1}{\alpha - 1}, \ \alpha \neq 1, \ -\infty < \alpha < 2.$$

When $\alpha \to 1$, the above measure goes to the Shannon entropy measure, and this is a variant of Havrda–Charvat entropy and Tsallis entropy. One can derive Tsallis statistics and superstatistics [19–22] by using Mathai's entropy. By optimizing Mathai's entropy measure, a new pathway model, which consists of many of the standard distributions in the statistical literature as special cases (see [23]), is derived. The main idea behind the derivation of this model is the switching properties of the special functions, like ${}_1F_1$ and ${}_1F_0$, which means the binomial to exponential function. Thus, the pathway between the exponential function e^{-cx} and the binomial function $[1 - c(1-\alpha)x]^{\frac{1}{1-\alpha}}$ can be created with the parameter α named as the pathway parameter. For the real scalar case, the pathway density can be written in the form:

$$f_1(x) \quad = \quad c|x|^{\gamma}[1 - a(1-\alpha)|x|^{\delta}]^{\frac{\eta}{1-\alpha}}, a > 0, 1 - a(1-\alpha)|x|^{\delta} \geq 0, \eta > 0, \alpha < 1$$

where c is the normalizing constants. One can assume the Type 2 model by replacing $(1 - \alpha)$ by $-(\alpha - 1)$. These distributions include Type 1 beta, Type 2 beta, gamma, Weibull, Gaussian, Cauchy, exponential, Rayleigh, Student t, Fermi–Dirac, chi-square, logistic, etc. The corresponding asymmetric generalization was introduced and studied in the paper [24]. By representing the entropy function in terms of a density function $f(\cdot)$ for the continuous case and giving the suitable constraints therein, the generalized entropy is maximized. There are restrictions, such as the $[(\gamma - 1)(1-\alpha)]$-th moment, and the $[(\gamma-1)(1-\alpha)+\delta]$-th moments are constants for fixed $\gamma > 0$ and $\delta > 0$. Maximizing Mathai's entropy by using the calculus of variations, we get the basic function of the model, and when the range of x is restricted over the positive real line and by evaluating the normalizing constant, we get the pathway model introduced by Mathai [23]. As $q \to 1$, $f_1(x)$ tend to $f_2(x)$, which is the generalized gamma distribution, where $f_2(x)$ is given by:

$$f_2(x) = \frac{\delta(a\beta)^{\frac{\alpha}{\delta}}}{2\Gamma\left(\frac{\alpha}{\delta}\right)} \ \mid x \mid^{\alpha-1} \ \exp(-a\beta \mid x \mid^{\delta}); \ \ -\infty < x < \infty; a, \alpha, \beta, \delta > 0. \tag{1}$$

For different values of parameters in the pathway model, we get different distributions like Weibull, gamma, beta Type 1, beta Type 2, etc. By taking $\delta = \alpha$, $\beta = 1$, $a = \lambda^{\alpha}$ in $f_1(x)$, the pathway model reduces to the q-Weibull distribution, which facilitates a transition to the Weibull distribution [25]. The connection of pathway models and Tsallis statistics with the q-extended versions of various functions is also considered. To this extent, we generalize the Laplace transform using the switching property of ${}_0F_0$ to ${}_0F_1$. Here, the q-exponential function is the kernel, and we call the extension as the q-Laplace transform; as q approaches to unity, we get the Laplace transform of the original function.

The article is organized as follows. In Section 2, we introduce the q-Laplace transform and the obtained various properties of the transform. Section 3 deals with the q-Laplace transform of some basic functions, which includes special functions, like the hypergeometric function, the Mittag–Leffler function and the H-function. In Section 4, this transform is connected to other known integral transforms, like the Mellin transform, the G-transform and the Henkel transforms. In Section 5, we obtain the solution of the fractional space-time diffusion equation, the generalized kinetic equation and the time fractional heat equation through the q-Laplace transform in terms of the Mittag–Leffler function.

2. The q-Laplace Transform and Basic Properties

The Laplace transform L of a function $f(\cdot)$ is given by:

$$L[f(x)](s) \equiv \int_0^\infty f(x)\, \mathrm{e}^{-sx}\, \mathrm{d}x$$

where $f(x)$ is defined over the positive real line and $s \in \mathbb{C}, \Re(s) > 0$, $\Re(\cdot)$ denotes the real part of $(\cdot)$. This Laplace transform plays a major role in pure and applied analysis, especially in solving differential equations. Now, we define the extended Laplace transform concept, namely the q-Laplace transform of a function, which will play a similar role in mathematical analysis, as well as mathematical physics. Instead of the exponential function, here, we consider the e_q^{-sx} the q-exponential defined as:

$$\mathrm{e}_q^{-x} \equiv c \begin{cases} [1-(1-q)x]^{\frac{1}{1-q}} \text{ for } 0 < x < \frac{1}{1-q}, q < 1 \\ [1+(q-1)x]^{-\frac{1}{q-1}} \text{ for } x \geq 0, q > 1 \end{cases} \tag{2}$$

with $\mathrm{e}_1^x \equiv \mathrm{e}^x$ and c is the normalizing constant. More precisely, for given function $f(\cdot)$ and for $s \in \mathbb{C}$ with support over $(0, \infty)$, we define its q-Laplace transform as:

$$L_q[f(x)](s) = \int_0^\infty [\mathrm{e}_q^{-sx}] f(x) \mathrm{d}x \text{ for }, \Re(s) > 0 \tag{3}$$

where e_q^{-x} is defined as in Equation (2). This Laplace transform can be written in the form,

$$L_q[f(x)](s) \quad = \quad \begin{cases} \int_0^{\frac{1}{(1-q)s}} [1-(1-q)sx]^{\frac{1}{1-q}} f(x) \mathrm{d}x & \text{for } \Re(1-(1-q)sx) > 0, \Re(s) > 0 \\ \int_0^\infty [1+(q-1)xs]^{-\frac{1}{q-1}} f(x) \mathrm{d}x & \text{for } \Re(s) > 0. \end{cases}$$

The q-Laplace transform of a function $f(\cdot)$ is valid at every point at which $f(\cdot)$ is continuous provided that the function is defined in $(0, \infty)$, is piecewise continuous and of bounded variation in every finite subinterval in $(0, \infty)$, and the integral is finite. Some basic properties of the q-Laplace transform are given below.

1. Scaling: For a real constant k, $L_q[kf(x)](s) = kL_q[f(x)](s)$.
2. Linearity : $L_q[mf(x) + ng(x)](s) = mL_q[f(x)](s) + nL_q[g(x)](s)$, where $m, n \in \Re$.
3. Transform of derivatives: For $\Re(s) > 0$, $L_q[\frac{\mathrm{d}}{\mathrm{d}x} f(x)](s) = sL_q(f)(sq)$ for all $q \in \Re/\{1\}$.

Proof. Let $g(x) = \frac{\mathrm{d}}{\mathrm{d}x} f(x)$. Then:

$$L_q\,[g(x)]\,(s) \quad = \quad \int_0^\infty [1+(q-1)xs]^{-\frac{1}{q-1}} \frac{\mathrm{d}}{\mathrm{d}x}[f(x)] \mathrm{d}x.$$

By applying integration by parts, we get:

$$\begin{aligned} \int_0^\infty [1+(q-1)xs]^{-\frac{1}{q-1}} \frac{\mathrm{d}}{\mathrm{d}x}[f(x)] \mathrm{d}x \quad &= \quad f(x)[1+(q-1)xs]^{-\frac{1}{q-1}} \Big|_0^\infty - \\ &\quad \int_0^\infty f(x)(-s)[1+(q-1)xs]^{-\frac{1}{q-1}-1} \mathrm{d}x \end{aligned}$$

which implies:

$$L_q[g(x)] = -f(0) + sL_{\frac{2q-1}{q}}(f)(sq).$$

As a consequence, we get:

$$D^n\{L_q\left[f(x)\right](s)\} = s^n L_q\left[f(x)\right](sA_n(q)) - \sum_{j=1}^{n} s^{n-j} D^j f(0) \tag{4}$$

where $A_n(q) = \prod_{j=1}^{n}(j+(j-1)q)$, for $q>1$, $D = \frac{\mathrm{d}}{\mathrm{d}x}f(x)$. □

4. Derivatives of transforms: The n^{th} derivative of the q-Laplace transform is given by $D^n\left[L_q(f)(s)\right] = A_{n-1}(q)\{L_q\left[(-x)^n \ {}_nA^{-1}(q)f(x\,{}_nA^{-1})\right](s)\}$ where ${}_nA^{-1}$ is the reciprocal of the n^{th} term of $A_n(q)$.

Proof. For $q>1$:

$$\begin{aligned} D^2(\mathrm{e}_q^{-sx}) &= qx^2[\mathrm{e}_q^{-sx}]^{1-2q} \\ D^3(\mathrm{e}_q^{-sx}) &= q(1-2q)(-x^3)[\mathrm{e}_q^{-sx}]^{2-3q} \\ &\vdots \\ D^n(\mathrm{e}_q^{-sx}) &= \prod_{j=1}^{n-1} A_j \int_0^\infty (-x)^n \mathrm{e}_q^{-s\,{}_nA^{-1}(q)x} f({}_nA^{-1}(q)x)\mathrm{d}x \\ &= A_{n-1}(q)\{L_q\left[(-x)^n \ {}_nA^{-1}(q)f(x\,{}_nA^{-1})\right]\}(s). \end{aligned}$$

□

5. Transforms of integrals: For $\Re(s)>0$, $L_q\left[\int_0^x f(t)\mathrm{d}t\right](s) = \frac{1}{s}L_q(f)(s)$.

Proof. For $q>1$, we have:

$$\begin{aligned} L_q\left[\int_0^x f(t)\mathrm{d}t\right](s) &= \int_0^\infty [\mathrm{e}_q^{-sx}]\{\int_0^x f(t)\mathrm{d}t\}\mathrm{d}x \\ &= -\frac{1}{s}\int_0^\infty \{\int_0^x f(t)\mathrm{d}t\}\frac{\mathrm{d}}{\mathrm{d}x}[\mathrm{e}_q^{-s(2-q)x}]\mathrm{d}x \\ &= -\frac{1}{s}\left\{\int_0^x f(t)\mathrm{d}t[\mathrm{e}_q^{-s(2-q)x}]\Big|_0^\infty - \int_0^\infty f(x)\mathrm{e}_q^{-sx}\mathrm{d}x\right\} \\ &= \frac{1}{s}L_q(f)(s) \text{ for } \Re(s)>0. \end{aligned}$$

□

6. Convolution property: Let $f_1(x)$ and $f_2(x)$ be two positive real scalar functions of x, and let $g_1(t)$ and $g_2(t)$ be their q-Laplace transform. Then,

$$L_q[f_1(x) * f_2(x)](s) = g_1(x)g_2(x)$$

where $f_1(x) * f_2(x) = \int_0^x f_1(t)f_2(x-t)\mathrm{d}t$.

Proof.

$$\begin{aligned} L_q[f_1(x) * f_2(x)](s) &= \int_0^\infty \mathrm{e}_q^{-sx}\left\{\int_0^x f_1(t)f_2(x-t)\mathrm{d}t\right\}\mathrm{d}x \\ &= \int_0^\infty \int_0^x [1+(q-1)xs]^{-\frac{1}{q-1}} f_1(t)f_2(x-t)\mathrm{d}t\mathrm{d}x \\ &= \int_{t=0}^\infty f_1(t)\left\{\int_{x=t}^\infty [1+(q-1)xs]^{-\frac{1}{q-1}} f_2(x-t)\mathrm{d}x\right\}\mathrm{d}t. \end{aligned}$$

Now, let us consider the integral $I = \int_{x=t}^{\infty}[1+(q-1)xs]^{-\frac{1}{q-1}}f_2(x-t)\mathrm{d}t$. Substitute $x - t = u$, and manipulate the integral; we get:

$$\begin{aligned} I = [1+(q-1)ts]^{-\frac{1}{q-1}} \quad \int_{x=0}^{\infty} \quad & [1+(q-1)us]^{-\frac{1}{q-1}}[1+(q-1)ts]^{\frac{1}{q-1}}[1+(q-1)us]^{\frac{1}{q-1}} \\ \times \quad & [1+(q-1)(t+u)s]^{-\frac{1}{q-1}}f_2(u)\mathrm{d}u. \end{aligned}$$

Let $[1+(q-1)ts]^{\frac{1}{q-1}}[1+(q-1)us]^{\frac{1}{q-1}}[1+(q-1)(t+u)s]^{-\frac{1}{q-1}}f_2(u) = f_2^*(u)$, then $f_2(x-t) = [1+(q-1)ts]^{-\frac{1}{q-1}}[1+(q-1)(x-t)s]^{-\frac{1}{q-1}}[1+(q-1)xs]^{\frac{1}{q-1}}f_2^*(x-t)$. Then:

$$\begin{aligned} L_q[f_1(x)*f_2(x)](s) \quad = \quad & \int_{t=0}^{\infty}[1+(q-1)ts]^{-\frac{1}{q-1}}f_1(t) \\ & \left\{\int_{x=t}^{\infty}[1+(q-1)(x-t)s]^{-\frac{1}{q-1}}f_2^*(x-t)\mathrm{d}x\right\}\mathrm{d}t. \end{aligned}$$

On substituting $x - t = u$, the integral can be separated, and hence, we have:

$$L_q[f_1(x)*f_2(x)](s) = L_q[f_1(x)]L_q[f_2^*(x)].$$

□

3. The q-Laplace Transform of Some Basic Functions

Let us introduce a new notation, $\Gamma_{(q)}(\alpha)$, such that:

$$\Gamma_{(q)}(\alpha) = \int_0^{\frac{1}{1-q}} x^{\alpha-1}[1-(1-q)x]^{\frac{1}{1-q}}\mathrm{d}x \text{ for } \Re(\alpha) > 0, q < 1.$$

If we replace $(1-q)$ by $-(q-1)$, then the function assumes the form:

$$\Gamma_{(q)}(\alpha) = \int_0^{\infty} x^{\alpha-1}[1+(q-1)x]^{-\frac{1}{q-1}}\mathrm{d}x \text{ for } \Re(\alpha) > 0, q > 1$$

and for $q = 1$ in the sense $q \to 1$, the q-gamma function is the usual classical gamma function defined as

$$\Gamma(\alpha) = \int_0^{\infty} x^{\alpha-1}\mathrm{e}^{-x}\mathrm{d}x.$$

Now, the q-gamma function can be explicitly written as:

$$\Gamma_{(q)}(\alpha) = \begin{cases} \frac{1}{(1-q)^{\alpha}} \frac{\Gamma(\alpha)\,\Gamma(\frac{1}{1-q}+1)}{\Gamma(\frac{1}{1-q}+\alpha+1)} & \text{for } q < 1 \\ \Gamma(\alpha) & \text{for } q = 1 \\ \frac{1}{(q-1)^{\alpha}} \frac{\Gamma(\alpha)\,\Gamma(\frac{1}{q-1}-\alpha)}{\Gamma(\frac{1}{q-1})} & \text{for } q > 1, \frac{1}{q-1} - \alpha > 0 \end{cases} \tag{5}$$

for $\Re(\alpha) > 0$. Here, $q = 1$ in the sense $q \to 1$ the q-gamma function $\Gamma_{(q)}(\cdot) \to \Gamma(\cdot)$, which can be easily proven using the asymptotic expansion of the gamma function:

$$\Gamma(z+a) \approx \sqrt{2\pi}z^{z+a-\frac{1}{2}}\mathrm{e}^{-z}.$$

Mathai [26] introduced a general class of integrals, known as the versatile integrals, which are connected to the reaction rate in kinetic theory. The integral is in the form:

$$I = \int_0^{\infty} x^{\gamma-1}[1+z_1^{\delta}(\alpha-1)x^{\delta}]^{-\frac{1}{\alpha-1}}[1+z_2^{\delta}(\beta-1)x^{-\rho}]^{-\frac{1}{\beta-1}} \tag{6}$$

for $\alpha, \beta > 1, z_1, z_2 \geq 0, \delta, \rho > 0, \Re(\gamma+1) > 0, \Re(\frac{1}{\alpha-1} - \frac{\gamma+1}{\delta}) > 0, \Re(\frac{1}{\beta-1} - \frac{1}{\rho}) > 0$, and the solution is obtained in terms of the H-function as follows:

$$I = c\ H^{2,2}_{2,2}\left[z_1 z_2 (\alpha-1)^{\frac{1}{\delta}} (\beta-1)^{\frac{1}{\rho}} \Big|^{(1-\frac{1}{\alpha-1}+\frac{\gamma}{\delta},\frac{1}{\delta}),(1-\frac{1}{\beta-1},\frac{1}{\rho})}_{(\frac{\gamma}{\delta},\frac{1}{\delta}),(0,\frac{1}{\rho})} \right]$$

where $c = \delta\rho z_1^{\gamma}(\alpha-1)^{\frac{\gamma}{\delta}}$ and $H^{m,n}_{p,r}$ is a H-function. Here, we provide the definition of H-function as follows:

$$H^{m,n}_{p,q}\left[z\Big|^{(a_1,\alpha_1),(a_2,\alpha_2),\cdots,(a_k,\alpha_k)}_{(b_1,\beta_1),(b_2,\beta_2),\cdots,(b_q,\beta_q)} \right] = \frac{1}{2\pi i}\int_L h(s) z^{-s} \mathrm{d}s$$

where:

$$h(s) = \frac{\left\{ \prod_{j=1}^{m} \Gamma(b_j+\beta_j s) \right\} \left\{ \prod_{j=1}^{n} \Gamma(1-a_j-\alpha_j s) \right\}}{\left\{ \prod_{j=m+1}^{q} \Gamma(1-b_j-\beta_j s) \right\} \left\{ \prod_{j=n+1}^{p} \Gamma(a_j+\alpha_j s) \right\}}$$

and L is a suitable path. An empty product is interpreted as unity, and it is assumed that the poles of $\Gamma(b_j+\beta_j s), j = 1,2,\ldots,m$ are separated from the poles of $\Gamma(1-a_j-\alpha_j s), j = 1,2,\ldots,n$. Here, $a_1, a_2, \ldots, a_p$; $b_1, b_2, \ldots, b_q$ are complex numbers and $\alpha_1, \alpha_2, \ldots, \alpha_p$, $\beta_1, \beta_2, \ldots, \beta_q$ are positive real numbers. The poles of $\Gamma(b_j+\beta_j s)$, $j = 1,2,\ldots,m$ are at the points $s = -\frac{b_j+\nu}{\beta_j}$ where $j = 1,2,\ldots,m, \nu = 0,1,\ldots$, and the poles of $\Gamma(1-a_j-\alpha_j s), j = 1,2,\ldots,n$ are at $s = \frac{1-a_k+\lambda}{\alpha_k}$ where $k = 1,2,\ldots,n, \lambda = 0,1,\ldots$. For more details about the theory and applications, refer to [27]. This integral includes the q-Laplace transform of gamma function and q-gamma function as special cases. Now, as $q \to 1$ in any of the functions $[1+z_1^{\delta}(\alpha-1)x^{\delta}]^{-\frac{1}{\alpha-1}}$ or $[1+z_2^{\delta}(\beta-1)x^{-\rho}]^{-\frac{1}{\beta-1}}$, we get the q-Laplace transform of some basic functions. The following table gives the q-Laplace transform of some basic functions with $q > 1$, which are special cases of the above integral. The results are obtained in terms of hypergeometric function. The Gaussian hyper geometric function is defined as:

$${}_mF_n\left[\begin{matrix} a_1,a_2,\cdots,a_m \\ b_1,b_2,\cdots,b_n \end{matrix} \Big| x \right] = \sum_{k=0}^{\infty} \frac{(a_1)_k (a_2)_k \cdots (a_m)_k}{(b_1)_k (b_2)_k \cdots (b_n)_k} \frac{x^k}{k!}$$

where $(a)_m$ denotes the Pochhammer symbol expressed in the form:

$$(a)_m = a(a+1)...(a+m-1),\ a \neq 0,\ m = 1,2... \tag{7}$$

Lemma 1. *For $\alpha, s \in C$, $\Re(s) > 0$ and for $q \neq 1$, the q- Laplace transform of $x^{\alpha-1}$ is given by $L_q[x^{\alpha-1}](s) = \frac{\Gamma_q(\alpha)}{s^{\alpha}}$ for $q \neq 1$.*

Proof. For $q > 1$,

$$\begin{aligned} L_q[x^{\alpha-1}](s) &= \int_0^{\infty} x^{\alpha-1} e_q^{-sx} \mathrm{d}x \\ &= \int_0^{\infty} x^{\alpha-1} [1+(q-1)sx]^{-\frac{1}{q-1}} \mathrm{d}x. \end{aligned}$$

Now, substitute $(q-1)sx = t$, and $\mathrm{d}x = \frac{1}{s(q-1)}\mathrm{d}u$. Then:

$$L_q[x^{\alpha-1}](s) = \frac{\Gamma_q(\alpha)}{s^{\alpha}},\ \ \alpha, s \in C, \Re(s) > 0.$$

□

Lemma 2. *For $s \in C$, $\Re(s) > 0$, there holds the formula:*

$$L_q[\mathrm{e}^{-ax}](s) = \frac{1}{(2-q)s}\ {}_1F_1\left[1; \frac{2q-3}{q-1}; \frac{a}{s(q-1)}\right]$$

for $a > 0$, $\frac{3}{2} < q < 2$.

Proof. For $q > 1$,

$$\begin{aligned} L_q[\mathrm{e}^{-ax}](s) &= \int_0^\infty [1+(q-1)sx]^{-\frac{1}{q-1}} \mathrm{e}^{-ax} \mathrm{d}x \\ &= \sum_{k=0}^{\infty} \frac{(-a)^k}{k!} \int_0^\infty x^k [1+(q-1)sx]^{-\frac{1}{q-1}} \mathrm{d}x \\ &= \frac{1}{s} \sum_{k=0}^{\infty} \frac{(-\frac{a}{s})^k}{k!} \frac{\Gamma_{(q)}(k+1)}{s^{k+1}}, \Re(\frac{1}{q-1} - k - 1) > 0 \\ &= \frac{1}{s(2-q)}\ {}_1F_1\left[1; \frac{2q-3}{q-1}; \frac{a}{s(q-1)}\right] \text{ for } \frac{3}{2} < q < 2, \Re(s) > 0, a > 0. \end{aligned}$$

□

Lemma 3. *For $a \in \Re, \Re(s) > 0$, the q-Laplace transform of the function e_q^{-ax} is given by $L_q[\mathrm{e}_q^{-ax}](s) = \frac{1}{(s+a)(2-q)} {}_2F_1\left[1, \frac{1}{2}; \frac{2q-3}{q-1}; -\frac{4as}{(a+s)^2}\right]$ for $\frac{3}{2} < q < 2$, $\left|\frac{4as}{(a+s)^2}\right| < 1$.*

Proof. For $q > 1$, the q-Laplace transform of the q-exponential function is given by:

$$\begin{aligned} L_q[\mathrm{e}_q^{-ax}](s) &= \int_0^\infty [1+(q-1)sx]^{-\frac{1}{q-1}} [1+(q-1)ax]^{-\frac{1}{q-1}} \mathrm{d}x \\ &= \int_0^\infty \{[1+(q-1)sx][1+(q-1)ax]\}^{-\frac{1}{q-1}} \mathrm{d}x \\ &= \int_0^\infty \left[1+(q-1)sx+(q-1)ax+(q-1)^2asx^2\right]^{-\frac{1}{q-1}} \mathrm{d}x \\ &= \frac{1}{(s+a)(2-q)} {}_2F_1\left[1, \frac{1}{2}; \frac{2q-3}{q-1}; -\frac{4as}{(a+s)^2}\right] \\ &\quad \text{provided } \frac{3}{2} < q < 2, \left|\frac{4as}{(a+s)^2}\right| < 1. \end{aligned}$$

□

Lemma 4. *For $\Re(s) > 0$ and for $\frac{3}{2} < q < 2$, $a \in \Re$, there holds the formula, $L_q[\cos(ax)](s) = \frac{1}{s(2-q)}\ {}_1F_2\left(1; \frac{2q-3}{2(q-1)}, \frac{3q-4}{2(q-1)}; -\frac{a^2}{[2s(q-1)]^2}\right)$.*

Proof. For $q > 1$, $\Re(s) > 0$, $a \in \Re$, the q-Laplace transform of the trigonometric function $\cos(ax)$ is given by:

$$\begin{aligned} L_q[\cos(ax)](s) &= \int_0^\infty [1+(q-1)sx]^{-\frac{1}{q-1}} \cos(ax) \mathrm{d}x \\ &= \sum_{k=0}^{\infty} \frac{(-1)^k\, a^{2k}}{(2k)!} \int_0^\infty x^{2k} [1+(q-1)sx]^{-\frac{1}{q-1}} \mathrm{d}x \end{aligned}$$

$$= \frac{1}{s(2-q)} \sum_{k=0}^{\infty} \frac{(a^2)^k}{[4(q-1)^2 s^2]^k} \frac{(1)_k}{k! \left(\frac{2q-3}{2(q-1)}\right)_k \left(\frac{3q-4}{2(q-1)}\right)_k} \text{ for } q > \frac{3}{2}.$$

By applying the properties of the beta function and integral evaluations, we get:

$$\begin{aligned} L_q[\cos(ax)](s) &= \frac{1}{s(2-q)} \, {}_1F_2\left(1; \frac{2q-3}{2(q-1)}, \frac{3q-4}{2(q-1)}; -\frac{a^2}{[2s(q-1)]^2}\right), \\ & \text{for } \frac{3}{2} < q < 2, q > 1, \Re(s) > 0, a \in \Re. \end{aligned}$$

□

One can easily check that as $q \to 1$, the above function gives a direct connection to the Laplace transforms of the original function simply by applying Sterling's approximation for the gamma function involved in the hypergeometric function involved in the equation.

Lemma 5. *The q-Laplace transform of the Gauss hypergeometric function is given by:*

$$L_q[{}_mF_n](s) = \frac{1}{s(2-q)} \quad {}_{m+1}F_{n+1}\left[\begin{matrix} a_1, a_2, \cdots, a_m, 1 \\ b_1, b_2, \cdots, b_n, \frac{2q-3}{q-1} \end{matrix} \middle| \frac{1}{(q-1)s}\right]$$

for $\Re(s) > 0, \frac{3}{2} < q < 2$.

Proof. For $q > 1$, the q-Laplace transform of the Gauss hyper geometric function is given by:

$$\begin{aligned} L_q[{}_mF_n](s) &= \sum_{k=0}^{\infty} \frac{(a_1)_k (a_2)_k \cdots (a_m)_k}{(b_1)_k (b_2)_k \cdots (b_n)_k} \frac{1}{k!} \int_0^{\infty} x^k [1 + (q-1)sx]^{-\frac{1}{q-1}} \\ &= \frac{1}{(2-q)s} \sum_{k=0}^{\infty} \frac{(a_1)_k (a_2)_k \cdots (a_m)_k, (1)_k}{(b_1)_k (b_2)_k \cdots (b_n)_k, (\frac{2q-3}{q-1})_k} \frac{(\frac{1}{(q-1)s})^k}{k!} \\ &= \frac{1}{s(2-q)} \quad {}_{m+1}F_{n+1}\left[\begin{matrix} a_1, a_2, \cdots, a_m, 1 \\ b_1, b_2, \cdots, b_n, \frac{2q-3}{q-1} \end{matrix} \middle| \frac{1}{(q-1)s}\right] \text{ for } \Re(s) > 0, \frac{3}{2} < q < 2. \end{aligned}$$

□

Corollary: When $m = n = 0$, we get the exponential function, and the q-Laplace transform is the confluent hypergeometric function ${}_1F_1$.

3.1. The q-Laplace Transform of the Mittag–Leffler Function

The single parameter Mittag–Leffler function is defined as follows:

$$E_\alpha(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(1+\alpha k)}, \text{for } \alpha \in \mathcal{C}, \Re(\alpha) > 0.$$

Lemma 6. *For* $q > 1, \Re(s) > 0$, *the q-Laplace transform of* $E_\alpha(x^\alpha)$ *is given by:*

$$L_q[E_\alpha(x^\alpha)](s) = \frac{1}{s(q-1)\Gamma(\frac{1}{q-1})} \quad H_{1\,2}^{2\,1}\left[\frac{1}{s^\alpha (q-1)^\alpha} \middle|_{(0,1),(\frac{2-q}{q-1},\alpha)}^{(0,1)}\right]$$

with suitable restrictions for the existence of Mittag–Leffler function.

Proof. For $q > 1, \Re(s) > 0$, the q-Laplace transform of the Mittag–Leffler function is given by:

$$\begin{aligned} L_q[E_\alpha(x^\alpha)](s) &= \sum_{k=o}^{\infty} \frac{1}{\Gamma(1+\alpha k)} \int_0^\infty x^{\alpha k}[1+(q-1)sx]^{-\frac{1}{q-1}} \mathrm{d}x \\ &= \sum_{k=o}^{\infty} \frac{\Gamma\left(\frac{2-q}{q-1} - \alpha k\right)}{\Gamma(\frac{1}{q-1})} \left(\frac{1}{s(q-1)}\right)^{\alpha k+1} \\ &= \frac{1}{s(q-1)\Gamma(\frac{1}{q-1})} \; H_{1\,2}^{2\,1}\left[\frac{1}{s^\alpha (q-1)^\alpha} \Bigg|_{(0,1),(\frac{2-q}{q-1},\alpha)}^{(0,1)} \right]. \end{aligned}$$

□

The generalized Mittag–Leffler function introduced by Prabhakar is defined as follows:

$$E_{\beta,\gamma}^{\delta}(z) = \sum_{k=0}^{\infty} \frac{(\delta)_n z^k}{\Gamma(\beta k+\gamma)}, \text{for } \beta,\gamma,\delta \in \mathcal{C}, \Re(\gamma) > 0, \Re(\delta) > 0.$$

Lemma 7. *Let $\beta,\gamma,\delta \in C, \Re(\beta) > 0, \Re(\gamma) > 0, \Re(\delta) > 0, \Re(\frac{1}{q-1} - \gamma) > 0$, and for $1 < q < 2$, there holds the formula:*

$$L_q[E_{\beta,\gamma}^{\delta}(ax^\beta)](s) = \frac{1}{s^\gamma (q-1)^\gamma \Gamma(\delta)\Gamma(\frac{1}{q-1})} \; H_{1\,2}^{2\,1}\left[{}^{(1-\delta,1)}_{(0,1),(\frac{1}{q-1}-\gamma,\beta)} \Bigg| \frac{1}{s^\beta (q-1)^\beta} \right]$$

for $1 < q < 2, \Re(\beta) > 0, \Re(\gamma) > 0, \Re(\delta) > 0, \Re(\frac{1}{q-1} - \gamma) > 0$.

The proof is similar to Lemma (7).

The details of the existence conditions, various properties and applications of H-functions are available in [27].

3.2. The q-Laplace Transform of the Fox H Function

Lemma 8. *For $q < 1$, consider the following restrictions. Let $a^* = \sum_{i=1}^{n} a_i - \sum_{i=n+1}^{p} \alpha_i + \sum_{j=1}^{m} \beta_j - \sum_{j=m+1}^{r} \beta_j$, $\Delta = \sum_{j=1}^{r} \beta_j - \sum_{i=1}^{p} \alpha_i$ and $\mu = \sum_{j=1}^{r} b_j - \sum_{i=1}^{p} a_i + \frac{p-r}{2}$ from the basic definition of the H-function.*

If either $a^* > 0, a^* = 0, \Re(\mu) < -1, \min\limits_{1\le j\le m} \dfrac{\Re(b_j)}{\beta_j} > -1$

when $a^* > 0, a^* = 0, \Delta \ge 0, \min\limits_{1\le j\le m} \left[\dfrac{\Re(b_j)}{\beta_j}, \dfrac{\Re(\mu)+\frac{1}{2}}{\Delta}\right] > -1$

when $a^* = 0, \Delta < 0$, then for $1 < q < 2$, the q-Laplace transform of the H-function exists, and the formula:

$$L_q[H_{p,r}^{m,n}](s) = \frac{1}{s(q-1)\Gamma(\frac{1}{q-1})} H_{p+1,r+1}^{m+1,n+1}\left[\frac{1}{s(q-1)} \Bigg|_{(b_1,\beta_1),(b_2,\beta_2),\cdots,(b_r,\beta_r),(\frac{2-q}{q-1},1)}^{(a_1,\alpha_1),(a_2,\alpha_2),\cdots,(a_k,\alpha_k),(0,1)} \right]$$

holds for $s \in \mathbb{C}, \Re(s) > 0$.

Proof. For $q > 1$,

$$\begin{aligned} L_q[H_{p,r}^{m,n}](s) &= \frac{1}{2\pi i} \int_L h(t) \int_0^\infty x^{-t}[1+(q-1)sx]^{-\frac{1}{q-1}} \mathrm{d}x \wedge \mathrm{d}t \\ &= \frac{1}{s(q-1)\Gamma(\frac{1}{q-1})} \frac{1}{2\pi i} \int_L h(t)\, \Gamma(1-t)\, \Gamma\left(\frac{2-q}{q-1} + t\right) \mathrm{d}t \end{aligned}$$

$$= \frac{1}{s(q-1)\Gamma(\frac{1}{q-1})} H^{m+1,n+1}_{p+1,r+1}\left[\frac{1}{s(q-1)}\Big|^{(a_1,\alpha_1),(a_2,\alpha_2),\cdots,(a_k,\alpha_k),(0,1)}_{(b_1,\beta_1),(b_2,\beta_2),\cdots,(b_r,\beta_r),(\frac{2-q}{q-1},1)}\right]$$

with suitable existing conditions. □

4. Connection to Other Integral Transforms

In this section, we consider connections of the q-Laplace transform of a function $f(\cdot)$ to other integral transforms. The following theorem gives a relation between the Mellin transform of the q-Laplace transform of a function, where the Mellin transform of the function $f(x)$ for $x > 0$ is defined by $(Mf)(t) = \int_0^\infty x^{t-1} f(x)\mathrm{d}x,\ t \in \mathcal{C}$.

Theorem 1. *For $t \in \mathcal{C}$, $\Re(t) < \frac{1}{q-1}, q > 1$, the Mellin transform $L_q[x^{\gamma-1}f(x)](s)$ is given by:*

$$\mathfrak{M}(L_q(x^{\gamma-1}f);t) = \frac{\Gamma(t)\Gamma(\frac{1}{q-1}-t)}{(q-1)^t\Gamma(\frac{1}{q-1})}\mathfrak{M}(f;\gamma-t).$$

Proof. For $q > 1$:

$$\begin{aligned}
\mathfrak{M}L_q[x^{\gamma-1}f(x)](s) &= \int_0^\infty s^{t-1}\int_0^\infty x^{\gamma-1}\mathrm{e}_q^{-sx} f(x)\mathrm{d}x \\
&= \int_0^\infty x^{\gamma-1} f(x)\frac{1}{(q-1)^t x^t}\frac{\Gamma(t)\Gamma(\frac{1}{q-1}-t)}{\Gamma(\frac{1}{q-1})};\ \Re(\frac{1}{q-1}-t) > 0 \\
&= \frac{\Gamma(t)\Gamma(\frac{1}{q-1}-t)}{\Gamma(\frac{1}{q-1})}\mathfrak{M}(f;\gamma-t);\ \Re(t) < \frac{1}{q-1}
\end{aligned}$$

hence the result. □

Remark 1. *For $\gamma = 1$ and $t \in \mathcal{C}$, it directly implies that the Mellin transform of the q-Laplace transform is given by:*

$$\mathfrak{M}(L_q(f);t) = \frac{\Gamma(t)\Gamma(\frac{1}{q-1}-t)}{(q-1)^t\Gamma(\frac{1}{q-1})}\mathfrak{M}(f;1-t) \text{ for } q > 1, \Re(t) < \frac{1}{q-1}.$$

The G-transform of the function $f(x)$ is given in the form:

$$(Gf)(t) = \int_0^\infty G^{m,n}_{p,r}\left[xt\Big|^{(a_i)_{1,p}}_{(b_i)_{1,r}}\right] f(x)\mathrm{d}x$$

where the Meijers G-function is considered as the kernel, with suitable existence conditions. The following theorem helps to evaluate the G-transform of $L_q(f(x))$.

Theorem 2. *The G-transform of $L_q(f(x))$ is given by the following relation:*

$$\mathbb{G}^{m,n}_{p,r}\{L_q[f(x)]\}(t) = \frac{1}{(q-1)\Gamma(\frac{1}{q-1})}\mathbb{G}^{m+1,n+1}_{p+1,r+1}[f(x)](t)$$

with suitable existing conditions.

Proof.

$$\mathbb{G}^{m,n}_{p,r}\{L_q[f(x)]\}(t) = \int_0^\infty G^{m,n}_{p,r}\left[st\Big|^{(a_i)_{1,p}}_{(b_j)_{1,r}}\right] f(s)\mathrm{d}s$$

$$= \frac{1}{2\pi i}\int_L \int_0^\infty h(\omega)(st)^{-\omega} L_q[f(x)](s)\mathrm{d}s\mathrm{d}\omega$$

$$\begin{aligned}
&= \frac{1}{2\pi i}\int_L \int_0^\infty \int_0^\infty h(\omega)(st)^{-\omega}[1+(q-1)sx]^{-\frac{1}{q-1}} f(x)\mathrm{d}x\mathrm{d}s\mathrm{d}\omega \\
&= \frac{1}{2\pi i}\int_L \int_0^\infty h(\omega)x^{\omega-1}\frac{\Gamma(1-\omega)\Gamma(\frac{1}{q-1}-1+\omega)}{[x(q-1)]^{1-\omega}\Gamma(\frac{1}{q-1})} f(x)\mathrm{d}x\mathrm{d}\omega \\
&= \frac{1}{(q-1)\Gamma(\frac{1}{q-1})}\int_0^\infty G^{m+1,n+1}_{p+1,r+1}\left[st\Big|^{(a_i)_{1,p},\frac{1}{q-1}}_{(b_j)_{1,r},1}\right] f(s)\mathrm{d}s \\
&= \frac{1}{(q-1)\Gamma(\frac{1}{q-1})}\mathbb{G}^{m+1,n+1}_{p+1,r+1}[f(x)](t).
\end{aligned}$$

□

Remark 2. *The q-Laplace transform can be converted in terms of the G-transform in the sense that the q-exponential can be converted as* $\frac{1}{\Gamma(\frac{1}{q-1})}G^{1,0}_{0,1}[-(q-1)sx|^{1-\frac{1}{q-1}}]$ *for* $|(q-1)sx|\le 1$. *That is:*

$$L_q[f(x)](s) = \int_0^\infty G^{1,0}_{0,1}[-(q-1)sx|_0^{1-\frac{1}{q-1}}]f(x)\mathrm{d}x \text{ for } s>0, |(q-1)sx|\le 1. \tag{8}$$

Now, the integral transforms is of the form:

$$(Hf)(t) = \int_0^\infty H^{m,n}_{p,r}\left[xt\Big|^{(a_i,\alpha_i)_{1,p}}_{(b_i,\beta_i)_{1,r}}\right]f(x)\mathrm{d}x$$

which is known as the H-transform with suitable existence conditions.

The Hankel transform of a function $f(x)$ for $x>0$ is defined by:

$$(H_n f)(t) = \int_0^\infty (xt)^{\frac{1}{2}}J_n(xt)f(x)\mathrm{d}x$$

where $J_n(z)$ is the Bessel function of the first kind of order $\eta\in\mathbb{C}$, such that $\Re(\eta)>-1$, which is given by:

$$J_n(z) = \sum_{k=0}^\infty \frac{(-1)^k}{\Gamma(\eta+k+1)k!}\left(\frac{z}{2}\right)^{2k+\eta}.$$

Theorem 3. *The Hankel transform of the q-Laplace transform* $(H_nL_q(f))(t)$ *can be expressed in terms of the H-transform.*

Proof. The integral transform with the Hankel kernel, which is operated on the q-Laplace transform, is given by:

$$\begin{aligned}
H_nL_q(f)(t) &= \int_0^\infty (st)^{\frac{1}{2}}J_n(st)\int_0^\infty \mathrm{e}_q^{-sx}f(x)\mathrm{d}x\mathrm{d}s \\
&= \int_0^\infty t^{\frac{1}{2}}\sum_{k=0}^\infty \frac{(-1)^k}{\Gamma(\eta+k+1)k!}\left(\frac{t}{2}\right)^{2k+\eta}\frac{\Gamma(2k+\eta+\frac{3}{2})\Gamma(\frac{1}{q-1}-2k-\eta-\frac{3}{2})}{[(q-1)x]^{2k+\eta+\frac{3}{2}}\Gamma(\frac{1}{q-1})}f(x)\mathrm{d}x \\
&= \frac{t^{\eta+\frac{1}{2}}}{2^\eta(q-1)^{\eta+\frac{3}{2}}\Gamma(\frac{1}{q-1})}\int_0^\infty H^{2\,1}_{1\,3}\left[\left(\frac{t}{2(q-1)x}\right)^2\Big|^{(0,1),(\frac{1}{q-1}-\eta-\frac{3}{2}),(\eta,1)}_{(-\eta-\frac{1}{2},2)}\right]x^{-\eta-\frac{3}{2}}f(x)\mathrm{d}x
\end{aligned}$$

which is the H-transform of $x^{-\eta-\frac{3}{2}}f(x)$. $\square$

Remark 3. *The q-Laplace transform of $f(\cdot)$ for $q < 1$ can be considered as a general case of the Riemann–Liouville integral operator, since for $q = 0$ and for $x = \frac{u}{t}$, we get the general form of the Riemann–Liouville operator.*

Remark 4. *We can extend the q-Laplace transform to its generalized version by considering the function $f(\cdot)$ with support over $(0, \infty)$ with:*

$$L_q[f(x)](s) = \int_0^\infty (xs)^{-\alpha}[\mathrm{e}_q^{-sx}]f(x)\mathrm{d}x \text{ for } \Re(s) > 0, \Re(\alpha) > 0 \tag{9}$$

where e_q^{-x} is defined as in 2. Now, as $q \to 1$, we get the generalized Laplace transform of the function f, with support over the positive real line defined as:

$$(Lf)(t) = \int_0^\infty (xt)^{-\alpha}\mathrm{e}^{-(tx)^k}f(x)\mathrm{d}x$$

that has interesting application in various fields.

5. Differential Equations by Means of the q-Laplace Transform

In this section, we apply the properties of the q-Laplace transform to solve the fractional space-time diffusion equation, the kinetic equation and the time-fractional heat equation.

5.1. Fractional Space-Time Diffusion: Laplace Transform and H-Function

We consider the following diffusion model with fractional-order spatial and temporal derivatives:

$$_0D_t^\beta N(x,t) = \eta \, _xD_\theta^\alpha N(x,t), \tag{10}$$

with the initial conditions $_0D_t^{\beta-1}N(x,0) = \sigma(x), 0 \leq \beta \leq 1, \lim_{x\to\pm\infty} N(x,t) = 0$, where η is a diffusion constant; $\eta, t > 0, x \in R; \alpha, \theta, \beta$ are real parameters with the constraints:

$$0 < \alpha \leq 2, |\theta| \leq min(\alpha, 2-\alpha),$$

and $\delta(x)$ is the Dirac-delta function. Then, for the fundamental solution of (1) with initial conditions, there holds the formula:

$$N(x,t) = \frac{t^{\beta-1}}{\alpha|x|} H_{3,3}^{2,1}\left[\frac{|x|}{(\eta t^\beta)^{1/\alpha}} \left|\begin{smallmatrix}(1,1/\alpha),(\beta,\beta/\alpha),(1,\rho)\\(1,1/\alpha),(1,1),(1,\rho)\end{smallmatrix}\right.\right], \alpha > 0 \tag{11}$$

where $\rho = \frac{\alpha-\theta}{2\alpha}$. The following special cases of (1) are of special interest for fractional diffusion models:

(i) For $\alpha = \beta$, the corresponding solution of (1), denoted by N_α^θ, can be expressed in terms of the H-function as given below and can be defined for $x > 0$:

Non-diffusion: $0 < \alpha = \beta < 2; \theta \leq \min\{\alpha, 2-\alpha\}$,

$$N_\alpha^\theta(x) = \frac{t^{\alpha-1}}{\alpha|x|} H_{3,3}^{2,1}\left[\frac{|x|}{t\eta^{1/\alpha}} \left|\begin{smallmatrix}(1,1/\alpha),(\alpha,1),(1,\rho)\\(1,1/\alpha),(1,1),(1,\rho)\end{smallmatrix}\right.\right], \ \rho = \frac{\alpha-\theta}{2\alpha}. \tag{12}$$

(ii) When $\beta = 1, 0 < \alpha \leq 2; \theta \leq \min\{\alpha, 2-\alpha\}$, then (1) reduces to the space-fractional diffusion equation, which is the fundamental solution of the following space-time fractional diffusion model:

$$\frac{\partial N(x,t)}{\partial t} = \eta \, _xD_\theta^\alpha N(x,t), \eta > 0, x \in R, \tag{13}$$

with the initial conditions $N(x,t=0)=\sigma(x), \lim_{x\to\pm\infty} N(x,t)=0$, where η is a diffusion constant and $\sigma(x)$ is the Dirac-delta function. Hence, for the solution of (1), there holds the formula:

$$L_\alpha^\theta(x)=\frac{1}{\alpha(\eta t)^{1/\alpha}}\, H_{2,2}^{1,1}\left[\frac{(\eta t)^{1/\alpha}}{|x|}\left|\begin{matrix}(1,1),(\rho,\rho)\\ (\frac{1}{\alpha},\frac{1}{\alpha}),(\rho,\rho)\end{matrix}\right.\right],\ 0<\alpha<1, |\theta|\leq\alpha, \tag{14}$$

where $\rho=\frac{\alpha-\theta}{2\alpha}$. The density represented by the above expression is known as α-stable Lévy density. Another form of this density is given by:

$$L_\alpha^\theta(x)=\frac{1}{\alpha(\eta t)^{1/\alpha}}\, H_{2,2}^{1,1}\left[\frac{|x|}{(\eta t)^{1/\alpha}}\left|\begin{matrix}(1-\frac{1}{\alpha},\frac{1}{\alpha}),(1-\rho,\rho)\\ (0,1),(1-\rho,\rho)\end{matrix}\right.\right], 1<\alpha<2, |\theta|\leq 2-\alpha. \tag{15}$$

(iii) Next, if we take $\alpha=2, 0<\beta<2; \theta=0$, then we obtain the time-fractional diffusion, which is governed by the following time-fractional diffusion model:

$$\frac{\partial^\beta N(x,t)}{\partial t^\beta}=\eta\frac{\partial^2}{\partial x^2}N(x,t), \eta>0, x\in R,\ 0<\beta\leq 2, \tag{16}$$

with the initial conditions ${}_0D_t^{\beta-1}N(x,0)=\sigma(x), {}_0D_t^{\beta-2}N(x,0)=0,$ for $x\in r$, $\lim_{x\to\pm\infty}N(x,t)=0$, where η is a diffusion constant and $\sigma(x)$ is the Dirac-delta function, whose fundamental solution is given by the equation:

$$N(x,t)=\frac{t^{\beta-1}}{2|x|}\, H_{1,1}^{1,0}\left[\frac{|x|}{(\eta t^\beta)^{1/2}}\left|\begin{matrix}(\beta,\beta/2)\\ (1,1)\end{matrix}\right.\right]. \tag{17}$$

(iv) If we set $\alpha=2, \beta=1$ and $\theta\to 0$, then for the fundamental solution of the standard diffusion equation:

$$\frac{\partial}{\partial t}N(x,t)=\eta\frac{\partial^2}{\partial x^2}N(x,t), \tag{18}$$

with initial condition:

$$N(x,t=0)=\sigma(x), \lim_{x\to\pm\infty}N(x,t)=0, \tag{19}$$

there holds the formula:

$$N(x,t)=\frac{1}{2|x|}H_{1,1}^{1,0}\left[\frac{|x|}{\eta^{1/2}t^{1/2}}\left|\begin{matrix}(1,1/2)\\ (1,1)\end{matrix}\right.\right]=(4\pi\eta t)^{-1/2}\exp[-\frac{|x|^2}{4\eta t}], \tag{20}$$

which is the classical Gaussian density.

5.2. Solution of the Generalized Kinetic Equation

Consider the generalized kinetic equation derived by Haubold and Mathai [8],

$$N(t)-N_0=-c_0\, {}_0D_t^{-\alpha}N(t) \text{ for } \alpha>0, \tag{21}$$

where ${}_0D_t^{-\alpha}N(t)$ is the Riemann–Liouville integral operator, in the form:

$${}_0D_t^{-\alpha}N(t)=\frac{1}{\Gamma(\alpha)}\int_0^t (t-u)^{\alpha-1}f(u)\mathrm{d}u$$

with the assumption that ${}_aD_t^0 g(t)=g(t)$.

Lemma 9. *The solution of the kinetic Equation (21) is given by:*

$$N(t) = \frac{N(0)}{2-q} E_{\alpha}\left(\frac{c_0 \Gamma_{(q)}(\alpha) t^{\alpha}}{\Gamma(\alpha)(2-q)}\right)$$

where $E_{\alpha}(\cdot)$ represents the two parameter Mittag–Leffler function.

Proof. The q-Laplace transform of the Riemann–Liouville integral operator is given by $L_q[f(s)] = \frac{\Gamma_{(q)}(\alpha)\tilde{f}(u)}{s^{\alpha}(2-q)\Gamma(\alpha)}$ using the convolution property of the q-Laplace transform, and $\tilde{f}(u)$ is the q-Laplace transform of $f(u)$. Now, by applying the q-Laplace transform on both sides of (21), we get:

$$\tilde{N}(t) - \frac{N(0)}{s(2-q)} = -c_0 \frac{\Gamma_{(q)}(\alpha)}{s^{\alpha}(2-q)\Gamma(\alpha)} \tilde{N}(t)$$

where $\tilde{N}(t) = L_q[N(t)]$, the q-Laplace transform of $N(t)$. Simplifying the equation we get

$$\tilde{N}(t) = \frac{N(0)}{s(2-q)} \left\{1 + \frac{c_0 \Gamma_{(q)}(\alpha)}{s^{\alpha}(2-q)\Gamma(\alpha)}\right\}^{-1}$$

This can be expanded as an infinite sum, and on finding the inverse q-Laplace transform, we get:

$$\begin{aligned} N(t) &= \frac{N(0)}{(2-q)} \sum_{k=0}^{\infty} \left[\frac{c_0 \Gamma_{(q)}(\alpha)}{\Gamma(\alpha)(2-q)}\right]^k \frac{t^{\alpha k}}{\Gamma(\alpha k+1)} \\ &= \frac{N(0)}{2-q} E_{\alpha}\left(\frac{c_0 \Gamma_{(q)}(\alpha) t^{\alpha}}{\Gamma(\alpha)(2-q)}\right) \end{aligned}$$

for $\left|\frac{c_0\Gamma_{(q)}(\alpha)}{\Gamma(\alpha)(2-q)}\right| < 1$ where $E_{\alpha}(\cdot)$ represents the two-parameter Mittag–Leffler function. □

5.3. Solution of the Time-Fractional Heat Equation

The standard heat equation is:

$$\frac{\partial u(x,t)}{\partial t} = \frac{\partial^2 u(x,t)}{\partial x^2}$$

where $u(x,t)$ represents the temperature, which is a function of time t and space x. Let us write the equation in terms of the derivative operator D, such as:

$$D_t(u) = D_x^2(u) \tag{22}$$

where $u = u(x,t)$. Then, for $t \geq 0$, the boundary conditions are that $u(t,0) = u(t,L) = 0$ where L represents the length of a heating rod and an initial condition:

$$u(0,x) = -\frac{4a}{L^2}x^2 + \frac{4a}{L}x$$

where $a = u(0, \frac{L}{2})$. The general solution for Equation (22) assumed to be in the form $u(t,x) = w(t)v(x)$ yields:

$$D(w(t))v(x) = w(t)D^2(v(x)) \Rightarrow \frac{D_t(w(t))}{w(t)} = \frac{D_x^2(v(x))}{w(x)} = K(say)$$

obtained from the general Equation (22). Let θ be the temperature decaying rate, and let $K = -\theta^2$ for $\theta \in \Re$; then, the ordinary differential equations $D(w(t)) = -\theta^2 w(t)$ and $D^2(v(x)) = -\theta^2 v(x)$ provide the general solution of Equation (22) of the form:

$$u(t,x) = K_1 \cos(\theta x) \mathrm{e}^{-\theta^2 t} + K_2 \sin(\theta x) \mathrm{e}^{-\theta^2 t}.$$

Now, let us consider the time fractional heat equation of the form:

$$D_t^{\alpha}(u) = D_x^2(u) \; 0 \leq \alpha < 2. \tag{23}$$

By considering similar steps as in the general solution and using the Laplace transform method to solve the differential equation $D_t^{\alpha}(w(t)) = -\theta^2 w(t)$, this yields the Mittag–Leffler function (similar steps as in Section 5.) as in the form:

$$w(t) = \sum_{k=0}^{\infty} \frac{(-\theta^2 t^{\alpha})^k}{\Gamma(\alpha k + 1)}. \tag{24}$$

Now, motivated from the same, we apply the q-Laplace transform for Equation (23) to obtain the solution for $D_t^{\alpha}(w(t)) = -\theta^2 w(t)$. The solution turns out to be:

$$w(t) = \frac{1}{(2-q)} \sum_{k=0}^{\infty} \left[\frac{-\theta^2 \Gamma_{(q)}(\alpha)}{\Gamma(\alpha)(2-q)} \right]^k \frac{t^{\alpha k}}{\Gamma(\alpha k + 1)} = \frac{1}{2-q} E_{\alpha} \left(\frac{-\theta^2 \Gamma_{(q)}(\alpha) t^{\alpha}}{\Gamma(\alpha)(2-q)} \right) \tag{25}$$

and hence, the general solution can be derived accordingly. Throughout the derivation, we consider the Laplace transformation for $q > 1$. Similar derivation exists, when $q < 1$.

6. Conclusions

In this article, we have proposed the q-Laplace transform as a suitable extension of the well-known Laplace transform. Despite the fact that it is difficult to evaluate some of the H-function numerically due to the constraints, the proposed method is an improvement over the regular practice of evaluating the Laplace transform within boundary values. The numerical illustration is not incorporated in this article; however, the methodology proposed here would be to generalize the result obtained in the regular sense of the Laplace transform. Another enhancement in this theory is that we applied the method of q-Laplace transforms in the generalized functional forms, such as Mittag–Leffler, hyper geometric, etc., so that applicability for particular functions, such as exponential, gamma, etc., can be easily deductible. The natural extension of the existing methodology explained in this article would further be considered for its generalized form, and it is an avenue for further research that could flow from this work.

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