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Operational Solution of Non-Integer Ordinary and Evolution-Type Partial Differential Equations

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Abstract: A method for the solution of linear differential equations (DE) of non-integer order and of partial differential equations (PDE) by means of inverse differential operators is proposed. The solutions of non-integer order ordinary differential equations are obtained with recourse to the integral transforms and the exponent operators. The generalized forms of Laguerre and Hermite orthogonal polynomials as members of more general Appèl polynomial family are used to find the solutions. Operational definitions of these polynomials are used in the context of the operational approach. Special functions are employed to write solutions of DE in convolution form. Some linear partial differential equations (PDE) are also explored by the operational method. The Schrödinger and the Black–Scholes-like evolution equations and solved with the help of the operational technique. Examples of the solution of DE of non-integer order and of PDE are considered with various initial functions, such as polynomial, exponential, and their combinations.

Keywords: inverse operator; derivative; differential equation; special functions; Hermite and Laguerre polynomials

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1. Introduction

Differential equations (DE) play an important role in pure mathematics and physics. They describe a broad range of physical processes and finding their solutions is of great importance. Only a few types of DE allow explicit analytical solutions. A vast literature is dedicated to the topic, and the contribution of scientists such as A.M. Mathai can hardly be overestimated (see, for example, [1,2]). Fractional calculus has rapidly drawn increasing attention from researchers in the last decade. They study the solutions of fractional reaction-diffusion, statistical, and other equations (see, for example, [3–6]. In many cases, expansion in series of orthogonal polynomials and their generalized forms with many indexes and variables as well as the usage of integral transforms are the most common tools to analytically solve DE.

The method of operational solution of DE demonstrated in [7–10] is applicable to a wide spectrum of physical problems, described by linear partial differential equations (PDE), such as propagation and radiation from charged particles [11–19], heat diffusion [20–22], including processes not described by Fourier law, and others [23–25]. In the context of the operational approach, the operational definitions for the polynomials through the operational exponent are very useful [26].The operational exponent is also applied when describing the fundamentals of structures in nature, including elementary particles and quarks [27–29]; such modern mathematical instruments are also used for the theoretical study of neutrino mixing [30–32] and for analysis of relevant experimental data [33–35]. The obtained solutions were formulated in terms of series of generalized forms of orthogonal polynomials of Hermite, Laguerre, more general Appèl, and some other polynomials [36,37], special functions of hyperbolic, elliptic Weierstrass and Jacobi-type, cylindrical Bessel-type, and generalized Airy-type functions.

While the role of various parameters in the solutions of DE and their physical meaning is most clear in the analytical form of the solutions, this last is not always available. Modern computer methods help to solve DE. The numerical approach is widely applied nowadays due to the revolutionary breakthrough in computational technique and technical support. Advanced numerical methods for the solution of fractional differential equations, formulated, for example, in [38–41], can be effectively executed with modern computers. In this context we note also semi-analytical models and numerical simulations of relaxation of hot electrons and holes [42], the diffusion of charge carriers, and the energy relaxation and transfer with respect to the electron excited states in crystals [43,44].

Different from these numerical computations, analytical solutions, when available, give clearer insight into the underlying physical processes. In the following we will apply the operational method to obtain exact solutions for some linear ordinary DE with non-integer derivatives and for evolution-type PDE, giving examples of solutions of Schrödinger-type and Black–Scholes-type equations, and their generalized forms with the Laguerre derivative operator.

The structure of the manuscript is as follows. In the first section we will explore generalized Hermite and Laguerre polynomials, the inverse derivative operator, the Laguerre derivative, and the relations between them; we will also touch on the Appèl polynomials. In the second section we will apply the orthogonal polynomials and inverse differential operators to find the solution of some non-integer order DE. In the third section we will construct convolution forms of solutions for DE with the help of special functions and integral transforms. In the fourth section we will consider the operational solutions for some PDE; in particular, we will consider the evolution partial differential equations of Schrödinger and Black–Scholes types. In every section we will consider examples of solutions with various initial functions, such as the functions $f(x) = x^n$, $f(x) = \sum_n c_n x^n$, $f(x) = e^{-x^2}$, $f(x) = e^{-\gamma x}$, $f(x) = \sum_k x^k e^{\gamma x}$, $f(x) = W_0(-x^2, 2)$. Eventually, we will provide the results and the conclusions.

2. Operational Approach and Orthogonal Polynomials

First of all, we note that an inverse function is one that undoes another function: For $f(x) = y$ the inverse is $g(y) = x$, $g(f(x)) = x$. The differential operators can be treated similarly. For the DE $\psi(D)F(x) = f(x)$, where $\psi(D)$ is a differential operator, the inverse differential operator $(\psi(D))^{-1}$ is defined, which undoes $\psi(D)$, $\psi(D)(\psi(D))^{-1}f(x) = f(x)$, so that $F(x) = (\psi(D))^{-1}f(x)$. Consider a common differential operator d_x: $F'(x) = f(x)$. Its inverse is $d_x{}^{-1}f(x) = F(x)$, which is an integral operator $\int f(x) = F(x) + C$, where C is the integration constant. The inverse derivative operator of the n-th order acts according to its definition:

$$D_x^{-n} f(x) = \frac{1}{(n-1)!}\int_0^x (x-\xi)^{n-1} f(\xi)d\xi, \; (n \in N = \{1, 2, 3, ...\}), \tag{1}$$

which is complemented by the definition for its zeroth order action:

$$D_x^0 f(x) = f(x) \tag{2}$$

and its action on the unity gives

$$D_x^{-n}\,\mathbf{1} = \frac{x^n}{n!}, \; (n \in N_0 = N \cup \{0\}). \tag{3}$$

It is elementary to demonstrate that, for example, the DE $\psi(D)F(x) = e^{\alpha x}$ has the following particular integral $F(x) = (\psi(D))^{-1}e^{\alpha x} = e^{\alpha x}(\psi(\alpha))^{-1}$, and to prove the following identity:

$$(\psi(D))^{-1}e^{\alpha x}f(x) = e^{\alpha x}(\psi(D+\alpha))^{-1}f(x). \tag{4}$$

With the help of the above identity the action of the shifted inverse differential operator $(\psi(D+\alpha))^{-1}$ on $f(x)$ can be expressed via the inverse differential operator $(\psi(D))^{-1}$, as follows:

$$F(x) = (\psi(D+\alpha))^{-1}f(x) = e^{-\alpha x}(\psi(D))^{-1}e^{\alpha x}f(x). \tag{5}$$

Equation (5) might seem trivial, but it is particularly useful for the solution of a broad class of DE with shifted differential operators.

Traditionally, polynomial families are defined by their expansion in series. However, they can be defined operationally through the relationship with the exponential differential operators. We recall that, in general, an exponential of an operator can be viewed as the series expansion $e^{\hat{A}} = \sum_{n=0}^{\infty} \hat{A}^n/n!$. The Hermite polynomials of two variables [45], if considered in the context of the operational approach [37], can be explicitly defined by the following operational rule [36] in addition to their series expansion [46]:

$$H_n^{(m)}(x,y) = e^{y\frac{\partial^m}{\partial x^m}}[x^n],\ H_n^{(m)}(x,y) = n!\sum_{r=0}^{[n/m]} \frac{x^{n-mr}\,y^r}{(n-mr)!\,r!}. \tag{6}$$

For the first-order polynomial we obtain simply

$$H_n{}^{(1)}(x,y) = (x+y)^n, \tag{7}$$

and for the second-order polynomial we have the two-variable Hermite polynomials $H_n(x,y)$:

$$H_n{}^{(2)}(x,y) = H_n(x,y) = e^{y\frac{\partial^2}{\partial x^2}}[x^n],\ H_n(x,y) = n!\sum_{r=0}^{[n/2]} \frac{x^{n-2r}\,y^r}{(n-2r)!\,r!}. \tag{8}$$

Thus, the Hermite polynomials of two variables are defined through the action of the heat operator $\hat{S}$:

$$\hat{S} = e^{t\partial_x^2} \tag{9}$$

on the monomial x^n. The heat operator (9) was thoroughly studied, for example, in [47]. The Hermite polynomials of two variables have the following generating function:

$$e^{xt+yt^2} = \sum_{n=0}^{\infty} \frac{t^n}{n!}H_n(x,y) \tag{10}$$

and they actually represent another form of the common Hermite polynomials of one variable:

$$H_n(x,y) = (-i)^n y^{n/2} H_n\left(\frac{ix}{2\sqrt{y}}\right) = i^n(2y)^{n/2}He_n\left(\frac{x}{i\sqrt{2y}}\right). \tag{11}$$

Direct application of the operational definition (8) to the Hermite polynomials yields the following identity:

$$e^{t\frac{\partial^2}{\partial x^2}}H_n(x,y) = H_n(x,y+t), \tag{12}$$

which consists of a shift in the y variable.

With the help of the following relation for Hermite polynomials:

$$z^n H_n(x,y) = H_n(xz, yz^2) \tag{13}$$

and with the operational identity:

$$e^{y\frac{\partial^m}{\partial x^m}} f(x) = f\left(x + my\frac{\partial^{m-1}}{\partial x^{m-1}}\right)\{1\}, \tag{14}$$

applied together with the operational rule (5), we obtain for the action of the heat diffusion operator $\hat{S}$ on the polynomial-exponential function the following result:

$$e^{y\partial_x^2} x^k e^{\alpha x} = e^{(\alpha x + \alpha^2 y)} H_k(x + 2\alpha y, y). \tag{15}$$

The Hermite, Laguerre, and some other polynomials belong to a more general family of Appèl polynomials [48], if viewed in the framework of the operational approach. For example, the two-variable Hermite polynomials belong to the family of Appèl polynomials $a_n(x)$, which can be defined through the following generating function [47]:

$$\sum_{n=0}^{\infty} \frac{t^n}{n!} a_n(x) = A(t) e^{xt}, \tag{16}$$

where it is assumed that a finite region of t exists, in which $A(t)$ is expandable in Taylor series and this expansion converges. Then, with the help of the obvious identity: $te^{xt} = \hat{D}_x e^{xt}$, $\hat{D}_x = d/dx$, we can rewrite Equation (16) in the following operational form:

$$\sum_{n=0}^{\infty} \frac{t^n}{n!} a_n(x) = A(\hat{D}_x) e^{xt}. \tag{17}$$

Now, expanding the exponential in Equation (17) in series and equating the terms on the right- and left-hand sides of (17), we obtain the following definition for $a_n(x)$:

$$a_n(x) = A\left(\hat{D}_x\right) x^n, \tag{18}$$

where $A\left(\hat{D}_x\right)$ is the Appèl operator. In the case of the two-variable Hermite polynomials, the identity (18) becomes the operational definition (8) and Appèl operator for Hermite polynomials is realized by the exponential

$$A(\hat{D}_x)\big|_{a_n(x)=H_n(x,y)} = e^{y\hat{D}_x^2}. \tag{19}$$

Let us assume that the inverse of the Appèl operator $\left[A(\hat{D}_x)\right]^{-1}$ can be defined as $\left[A(\hat{D}_x)\right]^{-1} A(\hat{D}_x) = \hat{1}$. The main properties of the Appèl polynomials arise from the operational definition (18). For example, if the operators $A(\hat{D}_x)$ and $\hat{D}_x$ commute: $\left[A(\hat{D}_x), \hat{D}_x\right] = 0$, then, by acting with $\hat{D}_x$ on both sides of (18), we obtain the following relation for $a_n(x)$ and $a_{n-1}(x)$:

$$\hat{D}_x a_n(x) = n a_{n-1}(x). \tag{20}$$

Moreover, it follows from (18) that $a_{n+1}(x)$ and $a_n(x)$ are related to each other as follows:

$$a_{n+1}(x) = \left[A(\hat{D}_x) x\right] x^n. \tag{21}$$

This allows us to introduce the multiplicative operator $\hat{M}$ for Appèl polynomials:

$$a_{n+1} = \hat{M} a_n(x), \tag{22}$$

where $\hat{M}$ is given by the Appèl operator as follows:

$$\hat{M} = A\left(\hat{D}_x\right) x A\left(\hat{D}_x\right)^{-1}, \tag{23}$$

and on account of $\left[f(\hat{D}_x), x\right] = f'(\hat{D}_x)$, where f' is the derivative of f, we write:

$$\hat{M} = x + \left[A(\hat{D}_x)\right]^{-1} A'(\hat{D}_x). \tag{24}$$

For the Appèl polynomials $a_n(x)$ the operators $\hat{M}$ and $\hat{D}_x$ stand for the multiplicative and derivative operators and this set of operators: $\hat{M}, \hat{D}, \hat{1}$, realizes the Weyl–Heisenberg algebra. From the following relation for Appèl polynomials:

$$\hat{M}\hat{D}_x a_n(x) = n a_n(x) \tag{25}$$

it is easy to derive the following differential equation for Appèl polynomials:

$$x\hat{D}_x a_n(x) + \frac{A'(\hat{D}_x)}{A(\hat{D}_x)}\hat{D}_x a_n(x) = n a_n(x), \tag{26}$$

where A' is the derivative of A. This equation is valid for all of the polynomials belonging to the Appèl family. Moreover, it is easy to recognize that Appèl polynomials satisfy the following recurrence:

$$a_{n+1}(x) = \left(x + \frac{A'(\hat{D}_x)}{A(\hat{D}_x)}\right) a_n(x). \tag{27}$$

In the context of the Appèl polynomial family we obtain for the Hermite polynomials the following multiplicative operator $\hat{M}$:

$$\hat{M} = x + 2y\hat{D}_x. \tag{28}$$

The differential equation for the Hermite polynomials as part of the Appèl polynomial family reads as follows:

$$2y\hat{D}_x^2 H_n(x,y) + x\hat{D}_x H_n(x,y) = n H_n(x,y). \tag{29}$$

The Laguerre polynomials of two variables can be defined in the operational way or as a finite sum:

$$L_n(x,y) = e^{-y\frac{\partial}{\partial x}x\frac{\partial}{\partial x}} \left[\frac{(-x)^n}{n!}\right] = n!\sum_{r=0}^{n} \frac{(-1)^r y^{n-r} x^r}{(n-r)!(r!)^2}. \tag{30}$$

The Laguerre polynomials of two variables, as well as the Hermite polynomials of two variables, are just another way for writing proper polynomials of one variable [47]:

$$L_n(x,y) = y^n\, L_n\left(\frac{x}{y}\right),\ L_n(x) = y^{-n}\, L_n(xy,y) = L_n(x,1). \tag{31}$$

However, there is more than just another notation behind the introduction of this form with two variables in Hermite and Laguerre polynomials. It allows us to consider proper polynomials as solutions of partial differential equations (PDE) with proper initial conditions:

$$\partial_y L_n(x,y) = -\left(\partial_x x \partial_x\right) L_n(x,y) \text{ with } L_n(x,0) = \frac{(-x)^n}{n!} \tag{32}$$

for Laguerre polynomials $L_n(x,y)$ and

$$\partial_y H_n(x,y) = \partial_x^2 H_n(x,y) \text{ with } H_n(x,0) = x^n \tag{33}$$

for Hermite polynomials $H_n(x,y)$. We introduce the Laguerre derivative ${}_LD_x$ and then the two variable Laguerre polynomials can be operationally defined as follows:

$$L_n(x,y) = e^{y_L\hat{D}_x}\left[\frac{(-x)^n}{n!}\right], {}_L\hat{D}_x = -\hat{D}_x x\hat{D}_x. \tag{34}$$

This operational definition is equivalent to the summation definition (30), which can be easily proved by direct execution of the action of ${}_L\hat{D}_x$ on $(-x)^n/n!$:

$${}_L\hat{D}_x\left[\frac{(-x)^n}{n!}\right] = n\left[\frac{(-x)^{n-1}}{(n-1)!}\right]. \tag{35}$$

The differential and multiplicative operators are formed by the operators

$${}_LD_x = \partial_x x\partial_x = -\hat{P} \text{ and } \hat{M} = y - D_x^{-1}, \tag{36}$$

which do not commute:

$$\left[{}_LD_x, D_x^{-1}\right] = -1. \tag{37}$$

Moreover, in the framework of the inverse derivative (see (1)) the following operational relationship exists between them 10:

$${}_LD_x = \frac{\partial}{\partial D_x^{-1}}, \tag{38}$$

which immediately raises associations with the relationship between the momentum and the coordinate in quantum mechanics. This relationship allows us to solve operationally the differential equations with the Laguerre derivative operator $\partial_x x\partial_x$, as we will demonstrate in what follows. Directly from (36) and (38) we conclude that the Laguerre polynomials $L_n(x,y)$, (30) and (34), can be expressed in terms of the inverse derivative operator (1) as follows:

$$L_n(x,y) = n!\sum_{k=0}^{n}\frac{(-x)^k y^{n-k}}{(n-k)!(k!)^2} = \left(y - D_x^{-1}\right)^n\{1\}. \tag{39}$$

This relation is also particularly useful for solution of some types of DE, involving the Laguerre derivative. Moreover, the operational definition (34) and the relations (38) and (39) yield the following operational rule for the Laguerre polynomials:

$$\exp\left(\alpha\frac{\partial}{\partial D_x^{-1}}\right)L_n(x,y) = L_n(x,y-\alpha). \tag{40}$$

Framing classical polynomials in the Appèl family should be done with some caution. Strictly speaking, the two-variable Laguerre polynomials can be considered members of the Appèl family with respect to the y variable only. Indeed, they are not Appèl polynomials with respect to x. However, the Laguerre polynomial family can be introduced in the context of the Appèl family in a way similar to (16) and (18) by the following substitution:

$$x^n \to \frac{(-x)^n}{n!}, \quad \hat{D}_x \to {}_L\hat{D}_x. \tag{41}$$

In this way we obtain the following formula:

$$l_n(x) = A\left({}_L\hat{D}_x\right)\left[\frac{(-x)^n}{n!}\right]. \tag{42}$$

With respect to the y variable, the Laguerre polynomials, as defined in (30), certainly belong the Appèl family as they can be given by the following operational rule:

$$L_n(x,y) = C_0\left(x\hat{D}_y\right)[y^n], C_n(x) = \sum_{r=0}^{\infty} \frac{(-1)^r x^r}{r!(r+n)!} = x^{-\frac{n}{2}} J_n\left(2\sqrt{x}\right). \tag{43}$$

Moreover, a hybrid family of polynomials exists, defined by the Appèl operator $C_0(x\hat{D}_y^2)$ or, alternatively, defined by the following sum:

$$P_n(x,y) = n! \sum_{r=0}^{[n/2]} \frac{(-1)^r x^r y^{n-2r}}{(r!)^2 (n-2r)!}. \tag{44}$$

Further study of their properties is beyond the scope of the present paper, but they are quite interesting, being in between those of Laguerre and Hermite polynomials. Moreover, for $x = 1 - y^2/4$ these polynomials reduce to the Legendre family. Studies of these and relevant polynomials were recently performed in [49–51].

Eventually, let us note that umbral calculus can provide a common framework for known and new identities for orthogonal polynomials. Let us recall the identity [52]

$$\frac{1}{n+1} = \sum_{m=0}^{n} (-1)^m \binom{n}{m} \frac{1}{m+1}, \tag{45}$$

which, after defining the umbral variable $\hat{a}$ 53, reads in its terms as follows:

$$\hat{a}^m\, 1 = \frac{1}{m+1}, \quad \hat{a}^0 = 1. \tag{46}$$

Therefore, as a consequence of the binomial theorem and of definition (46), we can write Equation (45) in the following useful form:

$$\frac{1}{n+1} = (1-\hat{a})^n\, 1. \tag{47}$$

Now with the help of identity (47), it is easy to generate new identities, such as the obvious consequence of Equation (47):

$$\frac{1}{2n+1} = (1-\hat{a})^{2n}\, 1 = (1-\hat{a})^n\, (1-\hat{a})^n\, 1 == \sum_{r=0}^{n} (-1)^r \binom{n}{r} \hat{a}^r \sum_{s=0}^{n} (-1)^s \binom{n}{s} \hat{a}^s\, 1, \tag{48}$$

which, together with

$$\hat{a}^s \hat{a}^r\, 1 = \hat{a}^{s+r}\, 1 = \frac{1}{s+r+1}, \tag{49}$$

yields the relation

$$\frac{1}{2n+1} = \sum_{r=0}^{n} (-1)^r \binom{n}{r} \sum_{s=0}^{n} (-1)^s \frac{1}{s+r+1}. \tag{50}$$

Moreover, from (50) and (47) more identities follow:

$$\frac{1}{m+n+1} = \sum_{p=0}^{m} (-1)^p \binom{m}{p} \sum_{s=0}^{n} (-1)^s \binom{n}{s} \frac{1}{p+s+1}, \tag{51}$$

$$\frac{1}{m\,n+1} = \sum_{s_m=0}^{n} (-1)^{s_m} \binom{n}{s_m} \ldots \sum_{s_1=0}^{n} (-1)^{s_1} \binom{n}{s_1} \frac{1}{1+\sum_{r=1}^{m} s_r}, \quad m \in \mathrm{Z} \cup m > 0. \tag{52}$$

We can define the operator of umbral derivative $\hat{\Delta}_a$ [53] by the following rule:

$$\hat{\Delta}_a \hat{a}^n = n\, \hat{a}^{n-1}, \tag{53}$$

which, together with the multiplication condition:

$$\hat{a} \cdot \hat{a}^n = \hat{a}^{n+1} \tag{54}$$

yields the following result for the commutator of the two operators [53]:

$$\left[\hat{\Delta}_a, \hat{a}\right] = \hat{\Delta}_a \hat{a} - \hat{a}\, \hat{\Delta}_a = 1, \tag{55}$$

Equation (55) allows us to use Weyl–Heisenberg algebra when needed. We can define the associated Hermite polynomials of two variables, with the operator $\hat{a}$ as one of the variables, and thus we come to the following sum:

$$H_n(\hat{a}, y)1 = n! \sum_{r=0}^{\left[\frac{n}{2}\right]} \frac{\hat{a}^{n-2r} y^r}{(n-2r)!r!} 1 = n! \sum_{r=0}^{\left[\frac{n}{2}\right]} \frac{y^r}{(n-2r)!(n-2r+1)r!}. \tag{56}$$

The multiplication condition (54) does not define any new polynomial family and such Hermite polynomials $H_n(\hat{a}, y)$ satisfy the following relation

$$H_n(\hat{a}, y)\, 1 = \frac{1}{n+1} H_{n+1}(1, y) - y^{\left[\frac{n+1}{2}\right]}, \tag{57}$$

and the following recurrences:

$$\left(\hat{\Delta}_a H_n(\hat{a}, y)\right) 1 = n H_{n-1}(\hat{a}, y)\, 1, \tag{58}$$

$$\left(\partial_y H_n(\hat{a}, y)\right) 1 = \hat{\Delta}_a^2 H_n(\hat{a}, y)\, 1, \tag{59}$$

which are direct generalizations of the relevant terms for the two-variable Hermite polynomials $H_n(x, y)$ equations. Indeed, Equation (59) is the umbral heat equation—the direct generalization of the heat Equation (33). It can be used to define the associated polynomials (56) in terms of the following operational equation:

$$H_n(\hat{a}, y)\, 1 = e^{y \hat{\Delta}^2{}_a}\, \hat{a}^n\, 1, \tag{60}$$

which is the generalization of definition (8). Further study of this topic represents stand-alone research; it will be addressed elsewhere.

3. Operational Solution of Some Non-Integer Ordinary DE

Let us consider a differential equation where ν is not necessarily an integer, shifted by the constant α derivative d_x:

$$\left(\beta^2 - \widetilde{D}^2\right)^\nu F(x) = f(x),\ \widetilde{D} \equiv d_x + \alpha,\ \alpha,\ \beta = \text{constant}. \tag{61}$$

Its particular integral formally reads

$$F(x) = \left(\beta^2 - \widetilde{D}^2\right)^{-\nu} f(x), \tag{62}$$

and it can be found in the form of the integral if the well-known operational identity [26,47] is applied:

$$\hat{q}^{-\nu} = \frac{1}{\Gamma(\nu)} \int_0^\infty e^{-\hat{q}t} t^{\nu-1} dt,\ \min\{\mathrm{Re}(q), \mathrm{Re}(\nu)\} > 0. \tag{63}$$

For $\hat{q} = \beta^2 - \widetilde{D}^2$ we obtain the following particular solution, involving the integrated weighted action of the operator $e^{t\widetilde{D}^2}$ on the initial function $f(x)$:

$$F(x) = \left(\beta^2 - \widetilde{D}^2\right)^{-\nu} f(x) = \frac{1}{\Gamma(\nu)} \int_0^\infty e^{-\beta^2 t} t^{\nu-1} e^{t\widetilde{D}^2} f(x) dt. \tag{64}$$

Example 1. *It is inviting to choose the initial function for Equation (61) in the form of the monomial $f(x) = x^n$. The action of the heat diffusion operator $\hat{S}$ on the monomial gives the Hermite polynomials according to their operational definition (30); the action of $\hat{S}$ on the polynomial-exponential function is given in (15). With account for the generating function (10), we directly write the particular integral (62) for $f(x) = x^n$ as follows:*

$$F(x)|_{f(x)=x^n} = \left(\beta^2 - (d_x + \alpha)^2\right)^{-\nu} x^n = \frac{1}{\Gamma(\nu)} \int_0^\infty e^{-t((\beta^2 - \alpha^2)} t^{\nu-1} H_n(x + 2\alpha t, t) dt. \tag{65}$$

The resulting function (65) with the Hermite polynomial of two variables is characterized by the shift of the argument $x \to x + 2\alpha t$. Evidently, for $\alpha = 0$ we have Equation (66):

$$\left(\beta^2 - d_x^2\right)^{\nu} Q(x) = f(x), \tag{66}$$

whose solution in the integral form is nothing but a particular case of (64) with $\widetilde{D} \to D$:

$$Q(x) = \frac{1}{\Gamma(\nu)} \int_0^\infty e^{-\beta^2 t} t^{\nu-1} \hat{S} f(x) dt \tag{67}$$

and involves the action of the heat diffusion operator $\hat{S} = e^{t\partial_x^2}$ (9).

Example 2. *Let us choose the Gaussian initial function $f(x) = e^{-x^2}$. Then, by means of the operational rule 47,*

$$\hat{S} f(x) = e^{y\partial_x^2} e^{-x^2} = e^{-\frac{x^2}{1+4y}} \Big/ \sqrt{1+4y}, \tag{68}$$

we immediately get the desired solution:

$$Q(x)|_{f(x)=e^{-x^2}} = \frac{1}{\Gamma(\nu)} \int_0^\infty \frac{dt}{\sqrt{1+4t}} e^{-\beta^2 t} t^{\nu-1} e^{-\frac{x^2}{1+4t}}. \tag{69}$$

Now, let us consider the following equation with the Laguerre derivative ${}_L D_x$, where ν is not necessarily an integer:

$$(\beta - {}_L D_x)^{\nu} Y(x) = f(x),\ \nu \in \text{Reals}. \tag{70}$$

Its operational solution is:

$$Y(x) = (\beta - {}_L D_x)^{-\nu} f(x) = \frac{1}{\Gamma(\nu)} \int_0^\infty e^{-\beta t} t^{\nu-1} e^{t_L D_x} f(x) dt. \tag{71}$$

The common change of variables $t \to e^t$ in such cases transforms the solution of the fractional Laguerre Equation (70) into

$$Y(x) = (\beta - {}_L D_x)^{-\nu} f(x) = \frac{1}{\Gamma(\nu)} \int_{-\infty}^{\infty} e^{t\nu} e^{-\beta e^t} e^{e^t {}_L D_x} f(x) dt, \tag{72}$$

and in the particular case of $\beta = 1,\ \nu = 1$ we obtain the Laplace transforms $1/(1 - \hat{a}) = \int_0^\infty e^{-s(1-\hat{a})} ds$ for the Laguerre derivative operator ${}_L D_x$, where the substitution $a \to {}_L D_x$ has been performed.

Example 3. *For Equation (71) with the initial monomial $f(x) = x^n$ the following particular integral arises:*

$$Y(x)|_{f(x)=x^n} = (\beta - {}_L D_x)^{-\nu} x^n = \frac{(-1)^n n!}{\Gamma(\nu)} \int_0^\infty e^{-\beta t} t^{n+\nu-1} L_n(x/t) dt. \tag{73}$$

Suppose the initial function $f(x)$ is expandable in series of the Laguerre polynomials $L_n(x)$:

$$f(x) = \sum_{n=0}^{\infty} c_n L_n(x). \tag{74}$$

Then with the help of (40), we readily write the solution (71) of Equation (70) in the integral form:

$$Y(x)\Big|_{f(x)=\sum\limits_{n=0}^{\infty} c_n L_n(x)} = \frac{1}{\Gamma(\nu)} \int_0^\infty e^{-\beta t} t^{\nu-1} \sum_{n=0}^{\infty} c_n L_n(x, 1-t) dt. \tag{75}$$

Example 4. *Let us consider the initial exponential function $f(x) = e^{-\gamma x}$. Then, the usage of the generalized Gleisher operational rule 10,*

$$e^{-t_L D_x} e^{-\gamma x} = e^{-\frac{\gamma x}{1-\gamma t}} \Big/ (1 - \gamma t), \tag{76}$$

gives the solution:

$$Y(x) = (\beta - {}_L D_x)^{-\nu} e^{-\gamma x} = \frac{1}{\Gamma(\nu)} \int_0^\infty e^{-\beta t} t^{\nu-1} e^{-\frac{\gamma x}{1+\gamma t}} \frac{dt}{1+\gamma t}. \tag{77}$$

Now, let us consider an ordinary DE like (61), with shifted Laguerre derivative ${}_L D_x$ instead of the common derivative d_x:

$$\left(\beta^2 - ({}_L D_x + \alpha)^2\right)^{\nu} Z(x) = f(x). \tag{78}$$

Let us choose the initial function for (78) in the form of the particular case of the Bessel–Wright function:

$$f(x) = W_0(-x^2, 2), \tag{79}$$

where

$$W_n(x, m) = \sum_{s}^{\infty} \frac{x^s}{s!(ms+n)!}, \tag{80}$$

is the particular case of the Bessel–Wright function [47]. In complete analogy with (63) we readily write the operational integral solution:

$$Z(x) = \left(\beta^2 - {}_L {D_x}^2\right)^{-\nu} W_0(-x^2, 2) = \frac{1}{\Gamma(\nu)} \int_0^\infty e^{-\beta^2 t} t^{\nu-1} e^{t_L D_x^2} f(x) dt. \tag{81}$$

Now we should compute the action of the heat operator with Laguerre derivative ${}_LD_x$ on the initial function $f(x) = W_0(-x^2, 2)$. With the help of the operational definition of Laguerre polynomials (30) and of the Gleisher operational rule [10],

$$e^{{}_LD_x^2}\, W_0(-x^2, 2) = \, W_0(-1/(1+4t),\, 2)\Big/\sqrt{1+4t}, \tag{82}$$

we obtain the particular integral as follows:

$$Z(x)|_{f(x)=W_0(-x^2,2)} = \frac{1}{\Gamma(\nu)}\int_0^\infty e^{-\beta^2 t} t^{\nu-1} W_0(-1/(1+4t),\, 2)\frac{dt}{\sqrt{1+4t}}. \tag{83}$$

The operational definitions of the polynomials and relevant operational rules allow writing solutions with ease for other types of equations too. For example, consider the following fractional order DE:

$$\left(xd_x^2 + (\alpha+1)d_x\right)^\nu V(x) = f(x) \tag{84}$$

Usage of the operational rule (63) immediately yields the integral solution for (84):

$$V(x) = \overline{D}_x^{-\nu} f(x) = \frac{1}{\Gamma(\nu)}\int_0^\infty e^{-\beta t} t^{\nu-1} e^{t\overline{D}_x} f(x)dt, \tag{85}$$

which involves the operational exponent action: $e^{t\overline{D}_x} f(x)$, where we denoted the differential operator

$$\overline{D}_x = x{d^2}_x + (\alpha+1)d_x. \tag{86}$$

Consider the initial condition $f(x) = x^n$. Direct application of the operational definition of the generalized Laguerre polynomials $L_n^{(\alpha)}(x, y)$,

$$L_n^{(\alpha)}(x, y) = \exp\left[-y\overline{D}_x\right]\left\{\frac{(-x)^n}{n!}\right\}, \tag{87}$$

immediately gives results in the particular integral of the generalized Laguerre polynomials with the exponential power weight:

$$V(x)|_{f(x)=x^n} = \overline{D}_x^{-\nu} x^n = \frac{(-1)^n n!}{\Gamma(\nu)}\int_0^\infty e^{-\beta t} t^{\nu-1} L_n^{(\alpha)}(x, -t)dt. \tag{88}$$

Now consider the other initial condition function: $f(x) = e^{-\gamma x}$. To obtain the solution we exploit the generalized form of the Gleisher operational rule 47, which yields the solution

$$V(x)|_{f(x)=e^{-\gamma x}} = \overline{D}_x^{-\nu} e^{-\gamma x} = \frac{1}{\Gamma(\nu)}\int_0^\infty \frac{dt}{(1+\gamma t)^{\alpha+1}} e^{-\beta t} t^{\nu-1} e^{-\frac{\gamma x}{1+\gamma t}}. \tag{89}$$

We have demonstrated that the usage of the inverse derivative, combined with the operational formalism, provides a straightforward and easy way of solving some classes of linear DE. In what follows we will demonstrate how this technique allows solutions of partial differential equations (PDE).

4. Convolution Forms for Solution of DE

In what follows, we will apply the inverse differential operators in order to obtain the convolution forms of solution for Equation (61). The operational approach to the solution of Equation (61) involves the exponential operator technique, the inverse derivative formalism, and integral transforms. In general, for solution of equations with $D+\alpha$ operational rule (5) can be applied, where

$$\psi^{-1}(D)=\left(\beta^2-d_x^2\right)^{-\nu}. \tag{90}$$

We continue with account for (66), (67) and (9), and make use of the action of the heat diffusion operator $\hat{S}$ (9) on $e^{\alpha x}g(x)$ with the help of the following chain rule:

$$e^{y\partial_x^2}e^{\alpha x}g(x)=e^{\alpha\, x}e^{\alpha^2 y}\,e^{2\alpha y\partial_x}e^{y\partial_x^2}g(x), \tag{91}$$

where y and α are the parameters. This results in the following particular solution for Equation (61), expressed as the integral:

$$F(x)=\frac{1}{\Gamma(\nu)}\int_0^\infty t^{\nu-1}e^{-(\beta^2-\alpha^2)t}\hat{\Theta}\hat{S}f(x)dt, \tag{92}$$

where $\hat{S}$ is the heat operator (9) and $\hat{\Theta}$ is the well-known operator of translation:

$$\hat{\Theta}=e^{2\alpha t\partial_x},\;\; \hat{\Theta}f(x)=f(x+2\alpha t). \tag{93}$$

The action of the operator $\hat{S}=e^{t\partial_x^2}$ can be written in the form of the Gaussian integral transform:

$$\Omega(x,t)\equiv\hat{S}f(x)=\frac{1}{2\sqrt{\pi t}}\int_{-\infty}^{\infty}\exp\left\{-\frac{(x-\xi)^2}{4t}\right\}f(\xi)d\xi. \tag{94}$$

Therefore, apart from the phase factor, the solution (92) of Equation (61) consists of the integrated action of the heat operator $\hat{S}$ and in the consequent translation by $\hat{\Theta}$ of the initial function $f(x)$:

$$F(x)=\frac{1}{\Gamma(\nu)}\int_0^\infty t^{\nu-1}e^{-t(\beta^2-\alpha^2)}U(x,t)dt, \tag{95}$$

where the integrand function $U(x,t)$ is (94), shifted by $\hat{\Theta}$:

$$U(x,t)\equiv\hat{\Theta}\hat{S}f(x)=\Omega(x+2\alpha t,t)=\frac{1}{2\sqrt{\pi t}}\int_{-\infty}^{\infty}\exp\left\{-\frac{(x+2\alpha t-\xi)^2}{4t}\right\}f(\xi)d\xi. \tag{96}$$

Example 5. *The example of the Gaussian initial condition $f(x)=\exp(-x^2)$ can be the illustration of the operational solution, described above. Accounting for (68), we directly write the solution of Equation (61), which is in turn a Gaussian:*

$$F(x)|_{f(x)=\exp(-x^2)}=\frac{1}{\Gamma(\nu)}\int_0^\infty\frac{t^{\nu-1}e^{-(\beta^2-\alpha^2)t}}{\sqrt{1+4t}}e^{-\frac{(x+2\alpha t)^2}{1+4t}}dt. \tag{97}$$

Note that from the general form of the solution (92), using the operational definition of the Hermite polynomials (8), we can directly obtain the solution (65) of the DE (61).

For the solution (64) of DE (61) with given initial function $f(x)$ we have to calculate the action of the exponential differential operator $e^{t\widetilde{D}^2}$ in the exponential. This can be performed in a number of different ways. One of them consists in direct application of operational definitions, as we did in the case of the initial monomial x^n. However, this is a rare case. The exponential operator of the second-order derivative can be reduced to the exponential of the first-order derivative if we apply the integral presentation for the exponential of a square of an operator $\hat{q}$ [54]:

$$e^{\hat{q}^2} = \int_{-\infty}^{\infty} e^{-\xi^2+2\xi\hat{q}} d\xi \Big/ \sqrt{\pi}, \tag{98}$$

in our case $\hat{q} = \sqrt{t}\widetilde{D}$. The above formula then reads as follows:

$$e^{t\widetilde{D}^2} f(x) = \int_{-\infty}^{\infty} e^{-\xi^2+2\xi\sqrt{t}\widetilde{D}} f(x) d\xi \Big/ \sqrt{\pi}. \tag{99}$$

Accounting for the action of the translation operator $e^{\eta(\partial_x+\alpha)} f(x) = e^{\eta\alpha} f(x+\eta)$, we obtain the following particular integral (62) for the DE on non-integer order (61):

$$F(x) = \frac{1}{\sqrt{\pi}\Gamma(\nu)} \int_0^{\infty} t^{\nu-1} \exp\left((\alpha^2-\beta^2)t\right) \int_{-\infty}^{\infty} \exp\left(-(\xi-\sqrt{t}\alpha)^2\right) f(x+2\xi\sqrt{t}) d\xi dt \tag{100}$$

Now, upon subject to the change of variables

$$\eta = x + 2\xi\sqrt{t} \text{ and } t = \tau^2, \tag{101}$$

we end up with the following form of the particular solution for Equation (61):

$$F(x) = \frac{1}{\sqrt{\pi}\Gamma(\nu)} \int_0^{\infty} \tau^{2(\nu-1)} \exp\left(-(\beta\tau)^2\right) \int_{-\infty}^{\infty} \exp\left(-\left(\frac{\eta-x}{2\tau}\right)^2 + \alpha(\eta-x)\right) f(\eta) d\eta d\tau \tag{102}$$

Several convolution forms are possible for the solution of (61). Indeed, for an arbitrary function $f(x)$ in the r.h.s. of (61) and the real values of α and $\nu > 0$ we can involve the generating function for Hermite polynomials (10) to disentangle two integrals in (102):

$$F(x) = \frac{1}{\sqrt{\pi}\Gamma(\nu)} \sum_{n=0}^{\infty} \int_0^{\infty} \tau^{2(\nu-1)} \exp(-\beta^2\tau^2) H_n\left(\alpha, -\frac{1}{4\tau^2}\right) d\tau \frac{1}{n!} \int_{-\infty}^{\infty} (\eta-x)^n f(\eta) d\eta \tag{103}$$

It follows from Equation (103) that the solution of DE (61) can be written in the form of series

$$F(x) = \sum_{n=0}^{\infty} C_n \phi(x), \tag{104}$$

involving the convolution $\int_{-\infty}^{\infty} \Phi(x-\eta) f(\eta) d\eta \equiv \Phi(x) * f(x)$, $\Phi(x-\eta) = (\eta-x)^n$ with the power kernel:

$$\phi(x) = \Phi(x) * f(\eta),\ \Phi(x) = x^n. \tag{105}$$

The respective coefficients in the series depend on the order of the equation, which can be a non-integer, and on the constants α, β as follows:

$$C_n = \frac{(-1)^n}{n!\sqrt{\pi}\Gamma(\nu)} \int_0^{\infty} \tau^{2(\nu-1)} e^{-\beta^2\tau^2} H_n\left(\alpha, -\frac{1}{4\tau^2}\right) d\tau. \tag{106}$$

Other convolution forms with different kernels are possible. Among them the Gaussian frequency kernel form is, perhaps, the most compact. Indeed, the integral form (102) of the solution of DE (61) can be viewed as the integral of the following convolution:

$$F(x) = \frac{1}{\sqrt{\pi}\,\Gamma(\nu)} \int_0^\infty \tau^{2(\nu-1)} e^{-(\beta\tau)^2} \varphi(x,\tau) d\tau = \frac{1}{\sqrt{\pi}\,\Gamma(\nu)} \int_{-\infty}^{\infty} e^{2\nu\tau - \beta^2 e^{2\tau}} \varphi(x, e^\tau) d\tau, \tag{107}$$

where $\varphi = \int_{-\infty}^{\infty} G(x-\eta) f(\eta) d\eta$ has the kernel

$$\varphi(x,\tau) = G(x,\tau) * f(x),\ G(x,\tau) = e^{-(x/2\tau)^2 - \alpha x}. \tag{108}$$

The above expression involves the convolution with the Gauss frequency function kernel. Furthermore, the remaining integral can be taken, and it gives the Bessel function of the second kind $K_\kappa(x)$:

$$\int_0^\infty \tau^{2(\nu-1)} e^{-(\beta\tau)^2 - \frac{(x-\eta)^2}{4\tau^2}} d\tau = \left(\frac{|x-\eta|}{2\beta}\right)^{\nu-1/2} K_{\nu-\frac{1}{2}}(\beta|x-\eta|). \tag{109}$$

Note that for the integer order of the equation, $\nu \in Z$, we have semi-integer index of the Bessel function of the second kind, $K_{n-1/2}(x)$, the latter easily expressed in elementary functions, for example: $K_{1/2}(x) = \sqrt{\frac{\pi}{2x}} e^{-x}$, etc.

Thus, we have obtained the particular solution $F(x) = \left(\beta^2 - (D+\alpha)^2\right)^{-\nu} f(x)$ for DE (61) in the form of the integral, which appears in the form of the convolution with the initial function $f(x)$:

$$\begin{aligned} F(x) &= \frac{1}{\sqrt{\pi}\Gamma(\nu)} \int_{-\infty}^{\infty} \left(\frac{|x-\eta|}{2\beta}\right)^{\nu-1/2} e^{-\alpha(x-\eta)} K_{\nu-\frac{1}{2}}(\beta|x-\eta|) f(\eta) d\eta \\ &= \frac{1}{\sqrt{\pi}\Gamma(\nu)} \int_{-\infty}^{\infty} \chi(x-\eta) f(\eta) d\eta \end{aligned} \tag{110}$$

with the kernel, containing the Bessel function of the second kind $K_{\nu-1/2}$, the exponential, and the n power of x:

$$\chi(x-\eta) = \left(\frac{|x-\eta|}{2\beta}\right)^{\nu-1/2} e^{-\alpha(x-\eta)} K_{\nu-\frac{1}{2}}(\beta|x-\eta|) \tag{111}$$

Finally can we write the compact convolution form of the solution of DE (61) as follows:

$$\begin{aligned} F(x) &= \frac{1}{\sqrt{\pi}\Gamma(\nu)} \chi * f, \\ \chi &= \left(\frac{|x|}{2\beta}\right)^{\nu-1/2} e^{-\alpha x} K_{\nu-\frac{1}{2}}(\beta|x|). \end{aligned} \tag{112}$$

So far we have demonstrated that the usage of the inverse derivative and of the inverse differential operators constitutes a straightforward and easy way to solve some classes of linear DE. In what follows we will apply this concept to solve more complicated problems, formulated in terms of PDE.

5. Operational Solution for Evolution-Type Partial Differential Equations

The technique of the inverse differential and exponential operators is useful for finding solutions to a broad range of mathematical and physical problems. In what follows we shall demonstrate the solution of the evolution-type DE by the operational approach. Let us consider the Schrödinger equation for an electric charge in a constant electric field in imaginary time. It effectively corresponds to the case when the charge diffuses under a potential barrier in the electric field, so that the charge energy is lower than the height of the barrier. This process is governed by the Schrödinger equation upon the $t \to i\tau$, $\beta \to -\beta$ change:

$$\partial_t F(x,t) = \alpha \partial_x^2 F(x,t) + \beta\, x\, F(x,t),\ F(x,0) = f(x), \tag{113}$$

which is the common heat equation $\partial_t F(x,t) = \partial_x^2 F(x,t)$ with the linear term βx in the r.h.s. The solution of DE (113) can be obtained operationally:

$$\begin{aligned} F(x,t) &= e^{\Phi(x,t;\beta)}\, \hat{\Theta}\hat{S} f(x) = e^{\Phi(x,t;\beta)} f(x + \beta\, t^2, t), \\ \Phi(x,t;\beta) &= \tfrac{1}{3}\alpha t (\beta t)^2 + \beta t x, \end{aligned} \tag{114}$$

and consists in the transform of the initial function $F(x,0) = f(x)$ by the operators $\hat{S} = e^{\alpha t \partial_x^2}$ and $\hat{\Theta} = e^{\alpha\beta t^2 \partial_x}$. Note that, although the solution for the Schrödinger equation in the electric field in real and in imaginary time, i.e., over and under the barrier, has the same structure (114), there is a fundamental difference between them. Indeed, the $F(x,t)$ function for a particle in quantum mechanics is the amplitude of the probability of finding it at point x at moment t: $F(x,t) = \Psi(x,t)$. For the charge over the barrier, the solution $F(x,t) \to \Psi(x,\tau)$ of the Schrödinger equation is complex due to the complex phase $\Phi(x,\tau;\beta)$; this does not trouble the probability $|\Psi(x,\tau)|^2$ over the barrier for $t \to \infty$, which regularly converges.

Example 6. *Let us consider the initial polynomial $f(x) = \sum_n c_n x^n$ in the context of the Fourier heat conduction of DE (113). The operational definition of the Hermite polynomials (8) gives $e^{a\partial_x} x^n = H_n(x,a)$, and the operator $\hat{\Theta} = e^{b\partial_x}$ gives the shift: $F(x) \propto H_n(x+b,a)$. The solution immediately appears in terms of the sum of the Hermite polynomials:*

$$F(x,t) = e^{\Phi} \sum_n c_n H_n\left(x + \alpha\beta t^2, \alpha t\right). \tag{115}$$

Example 7. *Now let us choose the initial condition $f(x) = \sum_k x^k e^{\gamma x}$. This function for $\gamma < 0$ represents a pulse, the shape of which depends on the values of k and γ, and varies from a sudden surge to a flat, smooth spatial wave. This choice of the initial function allows for modeling heat pulses for experimental tests (see, for example, [55]). Now applying the operational rule (15), where, in our case $y \to \alpha t$, and the shift by the translation operator $\hat{\Theta} = e^{\alpha\beta t^2 \partial_x}$, we obtain the solution in the form of the Hermite polynomials*

$$F(x,t) = e^{\Phi + \Delta_1} \sum_k H_k\left(x + 2t\alpha\gamma + t^2\alpha\beta, \alpha t\right) \tag{116}$$

with the common phase Φ written in the solution (114) and $\Delta_1 = \gamma\,(x + \gamma\alpha\, t + \alpha\beta t^2)$. For $\gamma = 0$ it immediately returns the result (115). For pure Fourier heat conduction $\beta = 0$ and the solution further simplifies:

$$F(x,t)|_{\beta=0, f(x)=x^k e^{\gamma x}} = e^{\gamma x + \gamma^2 \alpha t} \sum_k H_k(x + 2t\alpha\gamma, \alpha t). \tag{117}$$

It is easy to follow its evolution in time: for $t >> x/\alpha\gamma$ the coordinate dependence fades out: $F(x,t)|_{t>>x/\alpha\gamma} \cong H_k(2t\alpha\gamma, \alpha t)\exp\{\gamma^2 \alpha t\}$, and the time dependence prevails. For relatively short times of the evolution of the initial heat pulse $f(x) = \sum_k x^k e^{\gamma x}$, such that $t << x/\alpha\gamma$, the solution is approximated by $F(x,t)|_{t<<x/\alpha\gamma} \cong e^{x\gamma}(1 + t\alpha\gamma^2) H_k(x,\alpha t)$ and for very short times $\alpha t \to 0$ the Hermite polynomials tend to $H_k(x,0) = x^k$, which is in perfect agreement with our initial condition $f(x) = x^k e^{\gamma x}$.

Deeper consideration of the above topic is beyond the scope of the present paper. In forthcoming publications we will apply the operational method to explore and solve relativistic heat equations and other non-local extensions of the heat conduction.

Let us consider the following modification of the common Black–Scholes differential equation with the Laguerre derivative (see (30) and (36)) and the initial function $g(x) = A(x,0)$:

$$\frac{1}{\rho}\frac{\partial}{\partial t}A(x,t) = (\partial_x x \partial_x)^2 A(x,t) + \lambda(\partial_x x \partial_x)A(x,t) - \mu A(x,t),\ A(x,0) = g(x), \tag{118}$$

where ρ, λ, and μ are constants. Equation (118) is in fact the general form of the equation, which unifies the Laguerre heat equation and the matter diffusion equation with Laguerre derivative, as previously explored in [10,37]. In order to solve Equation (118), we employ the operational method. As usually in Black–Scholes DE, we distinguish the perfect square of the derivative, in this case of ${}_L D_x = \partial_x x \partial_x$. Then the solution takes the form of the exponential

$$A(x,t) = \exp\left\{\rho t\left(({}_L D_L + \lambda/2)^2 - \varepsilon\right)\right\}g(x),\ \varepsilon = \mu + (\lambda/2)^2. \tag{119}$$

With the help of the operational identity (98), we reduce $e^{(a_L D_x)^2}$ to the first-order Laguerre derivative in the exponential and thus the following solution for $A(x,t)$ arises:

$$\begin{gathered} A(x,t) = \frac{\exp(-\varepsilon\alpha^2)}{\sqrt{\pi}} \int_{-\infty}^{\infty} \exp(-\sigma^2 - \sigma\alpha\lambda - 2\sigma\alpha_L D_x) g(x) d\sigma, \\ \alpha = \alpha(t) = \sqrt{\rho t}. \end{gathered} \tag{120}$$

The above integral form of the solution, provided the integral converges, contains the exponential Laguerre derivative, which acts on the initial function: $e^{-a_L D_x} g(x)$.

Example 8. *Let us consider the example of the polynomial initial function $A(x,0) = g(x) = \sum_n c_n x^n$. Following the operational definition of the Laguerre polynomials (30), we directly write the solution for DE (118):*

$$A(x,t)|_{g(x)=\sum_n c_n x^n} = \sum_n (-1)^n \frac{n!}{\sqrt{\pi}} e^{-\varepsilon\alpha^2/4} \int_{-\infty}^{\infty} e^{-\sigma^2 - \sigma\alpha\lambda} L_n(x, 2\sigma\alpha) d\sigma. \tag{121}$$

Consequent integration results in the finite sum, involving gamma function Γ and hypergeometric function ${}_1F_1$:

$$\begin{gathered} A(x,t)|_{g(x)=\sum_n c_n x^n} = \frac{e^{-\alpha^2\mu}}{\sqrt{\pi}} \sum_n (-1)^n (n!)^2 \sum_{r=0}^{n} \frac{(-x)^r (2\alpha)^{n-r}}{(n-r)!(r!)^2} \times \\ \left(\frac{\alpha\lambda}{2}\left(e^{i(n-r)\pi} - 1\right) I + \frac{1}{2}\left(e^{i(n-r)\pi} + 1\right) J\right), \\ I = \Gamma(1 + \tfrac{n-r}{2})\, {}_1F_1\left(\tfrac{1-(n-r)}{2}, \tfrac{3}{2}, -\left(\tfrac{\alpha\lambda}{2}\right)^2\right), \\ J = \Gamma\left(\tfrac{1+n-r}{2}\right) {}_1F_1\left(-\tfrac{n-r}{2}, \tfrac{1}{2}, -\left(\tfrac{\alpha\lambda}{2}\right)^2\right). \end{gathered} \tag{122}$$

Now suppose the initial function $A(x,0)$ can be expanded in series of the Laguerre polynomials: $g(x) = \sum_n a_n L_n(x)$. The operational relationships (40) and (31) in this case immediately propose the solution of DE (118) in the following form:

$$A(x,t)|_{g(x)=\sum_n a_n L_n(x)} = \frac{1}{\sqrt{\pi}} e^{-\varepsilon\alpha^2/4} \sum_n a_n \int_{-\infty}^{\infty} e^{(-\sigma^2 - \sigma\alpha\lambda)} L_n(x, 2\sigma\alpha + 1) d\sigma. \tag{123}$$

Now let us consider the general case of the initial function $A(x,0) = g(x)$. Then the solution of DE (118), $A(x,t)$ can be obtained in the following steps. With the help of the operational definitions

(36) and with the inverse derivative Formula (1), we write the solution in terms of the operator of the inverse derivative D_x^{-1} and of the function φ:

$$A(x,t)=\frac{1}{\sqrt{\pi}}e^{-\varepsilon\alpha^2}\int_{-\infty}^{\infty}e^{-\sigma^2-\sigma\alpha\lambda}e^{-2\sigma\alpha\frac{\partial}{\partial D_x^{-1}}}\varphi(D_x^{-1})\mathbf{1}d\sigma,\ \varphi(D_x^{-1})\,\mathbf{1}=g(x), \tag{124}$$

where φ is the image function, determined by the integral: $\varphi(x)=\int_0^{\infty}\exp(-\kappa)g(x\kappa)d\kappa$. The exponential of the Laguerre derivative, acting on the initial function, yields the solution of the Laguerre diffusion equation 10:

$$\partial_t f(x,t)=-{}_LD_x f(x,t),\ f(x,0)=g(x),\ f(x,t)=e^{-t_LD_x}g(x). \tag{125}$$

Hence, by applying the exponential differential operator to the function $g(x)=\varphi(D_x^{-1})\mathbf{1}$: $e^{-t\frac{\partial}{\partial D_x^{-1}}}\varphi(D_x^{-1})\mathbf{1}$, we obtain the solution of the Laguerre diffusion equation:

$$f(x,t)=e^{-t\frac{\partial}{\partial D_x^{-1}}}g(x)=\varphi\left(D_x^{-1}-t\right)\mathbf{1}. \tag{126}$$

With account for the above relation (126) the solution of DE (118) becomes:

$$A(x,t)=\frac{1}{\sqrt{\pi}}e^{-\varepsilon\alpha^2}\int_{-\infty}^{\infty}e^{-\sigma^2-\sigma\alpha\lambda}g(x,t)d\sigma, \tag{127}$$

$$g(x,t)=\varphi(D_x^{-1}-2\sigma\alpha)\,\mathbf{1}=e^{-2\sigma\alpha\frac{\partial}{\partial D_x^{-1}}}\varphi\left(D_x^{-1}\right)\mathbf{1}. \tag{128}$$

Example 9. *Let us consider the particular case of the Bessel–Wright function [47]* $W_n(x,m)=\sum_{r=0}^{\infty}\frac{x^r}{r!(mr+n)!}$, $m\in$, $n\in_0$, *for* $m=2$, $n=0$ *as the initial function:* $g(x)=W_0(-x^2,2)=\sum_{r=0}^{\infty}\frac{(-1)^r x^{2r}}{r!(2r)!}$. *Its image is* $\varphi(x)=e^{-x^2}$. *The operational identity (98) and the function (128) together yield, in accordance with the previously computed in [10], the result:*

$$g(x,t)=\frac{1}{\sqrt{\pi}}\int_{-\infty}^{\infty}e^{-\xi^2+4i\sigma\alpha\xi}C_0(2i\xi x)d\xi \tag{129}$$

where C_0 *is the particular case of the Bessel–Tricomi function* [56]: $C_n(x)=\sum_{r=0}^{\infty}\frac{(-x)^r}{r!(r+n)!}$, $n\in_0$. *A relationship with the Bessel functions exists:* $C_n(x)=x^{-n/2}J_n(2\sqrt{x})$. *Finally, we have obtained the solution of DE (118) with the initial condition* $g(x)=W_0(-x^2,2)$ *in the form of the integral (127) of the exponentially weighted function (129).*

6. Results

We have obtained solutions for some ordinary DE of non-integer order with shifted derivatives. In particular, we derived the integral form of the particular solution $F(x)=\left(\beta^2-(\partial_x+\alpha)^2\right)^{-\nu}f(x)$ for real values of ν. The integrand involves the operators of heat propagation $\hat{S}$ and translation $\hat{\Theta}$, which act on the function $f(x)$. Moreover, the convolution form of these solutions $\phi(x,\tau)=G(x,\tau)*f(\eta)$ and the integrals of other convolutions with several kernels different from each other are obtained. The comprehensive solution with the kernel, involving the Bessel function of the second kind with power-exponential weight, is obtained. Other integral forms of the solution with the convolutions with the Gaussian frequency kernel and with the monomial kernel are also obtained. We considered the examples of the Gaussian distribution $f(x) = e^{-x^2}$ and of the monomial $f(x) = x^k$ and found explicit solutions for them in terms of integrals and series of Hermite polynomials. We operationally solved the DE with Laguerre derivatives: $(x\partial^2{}_x + (\alpha+1)\partial_x)^\nu F(x) = f(x)$ and demonstrated the examples of solutions for the functions $f(x) = \exp(-\gamma x)$, $f(x) = x^k$ and for the Bessel–Wright function $f(x) = W_0(-x^2, 2)$. The obtained operational solutions are expressed in terms of the integrals of generalized Laguerre polynomials and Bessel functions.

The linear evolution-type PDEs were solved by the operational technique. In particular, the Black–Scholes equation with the Laguerre derivative ${}_LD_x = \partial_x x \partial_x$ was solved operationally. The example of the initial polynomial was considered. By using the operational definitions for Hermite polynomials we obtained explicit solutions in the form of the polynomials of x with the coefficients, given by Γ and ${}_1F_1$ functions. The solution of the Black–Scholes type equation with Laguerre derivative ${}_LD_x$ for the Bessel–Wright function $f(x) = W_0(-x^2, 2)$ is obtained in the integral form, involving Bessel–Tricomi function $C_n(x)$. We have obtained the operational solution of a Fourier-type heat equation with an additional term, describing the heat exchange with the environment, for the initial distribution $f(x) = \sum_k x^k e^{\gamma x}$, which describes a heat pulse for $\gamma < 0$. We also obtained the solution of the Schrödinger equation for a charge in electric field in real and in imaginary time, i.e., over and under the potential barrier, and demonstrated that in real time, i.e. under the barrier, the solution is purely real, contrary to that over the barrier. Thus $|F|^2$ diverges for $t \to \infty$ in the case of the real solution and $\beta \neq 0$, but converges otherwise, as a square of the amplitude $|\Psi(x,t)|^2$ of the probability function should behave (see also [57]).

7. Conclusions

In the present work we advocate the operational approach for solution of linear DE and the use of inverse differential operators, which allow direct and straightforward finding of solutions. The latter include the action of the operator of heat conduction and the operator of shift and dilatation. The operational approach involves operational definitions for Hermite and Laguerre orthogonal polynomials. In this way, we avoid cumbersome calculations and directly obtain the results of the action of proper exponential differential operators on the initial functions. If the DE contains the Laguerre derivatives, the commutation relationship between the inverse derivative operator and the Laguerre derivative operator helps in solving DE. Complemented by the usage of the integral transforms where needed, the operational technique yielded solutions of relatively complicated DE, such as Black–Scholes-type DE with Laguerre derivative, etc. Thus, our research demonstrates that the operational approach for solution of linear DEs is advantageous for its ease. The solutions are derived directly based on the operational definitions and on commutation relationships. Operational study of more complicated equations, describing heat propagation accounting for wave and ballistic heat transfer and, for equations, modeling other physical processes, is possible. It will be performed in forthcoming publications.

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