Mild Solutions to the Cauchy Problem for Some Fractional Differential Equations with Delay

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Abstract: In this paper, we present new existence theorems of mild solutions to Cauchy problem for some fractional differential equations with delay. Our main tools to obtain our results are the theory of analytic semigroups and compact semigroups, the Kuratowski measure of non-compactness, and fixed point theorems, with the help of some estimations. Examples are also given to illustrate the applicability of our results.

Keywords: fractional differential equations; analytic semigroup; compact semigroup; fixed point; mild solution

1. Introduction

In this paper, we consider the following Cauchy problem for fractional differential equations with delay in a Banach space $X$ which could be an infinite dimensional space:

$$
\begin{cases}
^cD_q^t u(t) = Au(t) + f(t, u_t), & t \in [0, T],

u(t) = \phi(t), & t \in [-\varpi, 0],
\end{cases}
$$

(1)

where $T, \varpi > 0, D_q^t, q \in (0, 1)$, is the Liouville-Caputo fractional derivative of order $q$, $A$ is the infinitesimal generator of an analytic semigroup $B(\cdot)$ of uniformly bounded linear operator on $X$, $f$ is a given function, $u_t : [-\varpi, 0] \to X$ is defined by

$$u_t(\theta) = u(t + \theta), \quad \theta \in [-\varpi, 0],$$

and $\phi \in C([-\varpi, 0], X)$.

As shown in [1–19] and the references therein, differential equations with delay or differential equations of fractional order have appeared in many branches of science and technology. They have received a lot of attention in all these years.

The paper is organized as follows. In Section 2, we first recall and give some basic facts or results about semigroup theory and related tools which will be used in our investigation. Then, we study the existence of mild solutions to the Cauchy Problem (1) and prove our main results. In Section 3, we give some examples to illustrate our abstract results.

2. Results and Proofs

Beta function:

$$B(p, q) = \int_0^1 t^{p-1}(1 - t)^{q-1} dt, \quad p, q > 0.$$  

Gamma function:

$$\Gamma(p) = \int_0^\infty t^{p-1}e^{-t} dt, \quad p > 0.$$
It is well known that
\[ B(p, q) = \frac{\Gamma(p)\Gamma(q)}{\Gamma(p + q)}, \quad \Gamma(p + 1) = p\Gamma(p). \]

Throughout this paper, \((X, \|\cdot\|)\) is a Banach space, \(C([a, b], X)\) denotes the space of the continuous functions from \([a, b]\) to \(X\) with the norm
\[ \|x\|_{[a,b]} = \max_{t \in [a,b]} \|x(t)\|. \]

Set
\[ C_0(X) := \{ x(t); \quad x(t) \in C([-\omega, T], X) \text{ and } x(t) \equiv 0, -\omega \leq t \leq 0 \} \]
with the norm
\[ \|x\|_{C_0(X)} = \max_{t \in [0,T]} \|x(t)\|. \]

**Definition 1.** (cf., e.g., [19]) The Liouville-Caputo derivative of order \( q \) for a function \( f \in C^1[0,\infty) \) can be written as
\[ ^cD^q f(t) = \frac{1}{\Gamma(1-q)} \int_0^t \frac{f'(s)}{(t-s)^q} ds, t > 0, 0 < q < 1. \]

Since \( A : D(A) \subset X \rightarrow X \) is the infinitesimal generator of an analytic semigroup \( B(t) \) of uniformly bounded operators, we know from [20] that, there exists \( M \geq 1 \) such that \( \|B(t)\| \leq M \) for all \( t \geq 0 \). Moreover, \( B(t) \) is continuous in the uniform operator topology for all \( t \geq 0 \), i.e.,
\[ \lim_{\eta \to 0} \|B(t + \eta) - B(t)\| = 0, \forall t \geq 0. \]

As in many papers on fractional differential equations, for \( x \in X \), we define two operators \( \{\Phi(t)\}_{t \geq 0} \) and \( \{\Psi(t)\}_{t \geq 0} \) by
\[ \Phi(t)x := \int_0^{\infty} \eta_q(\theta)B(t^q \theta)x d\theta, \quad \Psi(t)x := q \int_0^{\infty} \rho_q(\theta)B(t^q \theta)x d\theta, 0 < q < 1, \]
where
\[ \eta_q(\theta) = \frac{1}{q} \theta^{-\frac{1}{q}} \rho_q(\theta^{-\frac{1}{q}}), \]
\[ \rho_q(\theta) = \frac{1}{\pi} \sum_{n=1}^{\infty} (-1)^{n-1} \theta^{-qn-1} \frac{\Gamma(nq + 1)}{n!} \sin(n\pi q), \]
\( \theta \in (0, \infty) \), and \( \eta_q \) is a probability density function defined on \((0, \infty)\) and satisfies
\[ \eta_q(\theta) \geq 0 \text{ for all } \theta \in (0, \infty) \]
and
\[ \int_0^{\infty} \eta_q(\theta) d\theta = 1, \quad \int_0^{\infty} \theta \eta_q(\theta) d\theta = \frac{1}{\Gamma(1+q)}. \]

Clearly,
\[ \|\Phi(t)\| \leq M, \quad \|\Psi(t)\| \leq \frac{M}{\Gamma(q)}, \quad t \geq 0. \]

**Lemma 1.** ([10]) \(\Phi(t)\) and \(\Psi(t)\) are strongly continuous on \( X \) for \( t \geq 0 \).

**Lemma 2.** ([10]) \(\Phi(t)\) and \(\Psi(t)\) are norm-continuous on \( X \) for \( t > 0 \).

Based on the work in [8,10–12], the mild solution for the Problem (1) is defined as follows.
Definition 2. A function \( u \in C([-\varpi, T], X) \) satisfying the equation

\[
    u(t) = \begin{cases} 
        \phi(t), & t \in [-\varpi, 0], \\
        \Phi(t)\phi(0) + \int_0^t (t-s)^{\beta-1} \Psi(t-s)f(s,u_s)ds, & t \in [0, T],
    \end{cases}
\]

is called a mild solution of the problem (1.1).

The following lemma is a generalization of Gronwall's inequality.

Lemma 3. Suppose \( b \geq 0, \beta > 0 \) and \( a(t) \) is a nonnegative function locally integrable on \( 0 \leq t \leq T(T < +\infty) \), and suppose \( u(t) \) is nonnegative and locally integrable on \( 0 \leq t \leq T \) with

\[
    u(t) \leq a(t) + b \int_0^t (t-s)^{\beta-1} u(s)ds,
\]

on this interval, then we have that

\[
    u(t) \leq a(t) + \int_0^t \left[ \sum_{n=1}^{\infty} \frac{\Gamma(n\beta)}{\Gamma(n\beta)} (t-s)^{n\beta-1} a(s) \right] ds, 0 \leq t \leq T.
\]

Kuratowski measure of noncompactness:

On each bounded subset \( B \) in the Banach space \( X \), define

\[
    \mu(B) := \inf \{ d > 0 ; \text{ } B \text{ can be covered by a finite number of sets of diameter } < d \}.
\]

Then, \( \mu(.) \) is called the Kuratowski measure of noncompactness on \( B \).

Some basic properties of \( \mu(.) \) are given in the following Lemma.

Lemma 4. Let \( X \) be a Banach space with norm \( \| \cdot \| \) and \( B, C \subseteq X \) be bounded. Then

1. \( \mu(B) = 0 \) if and only if \( B \) is relatively compact;
2. \( \mu(B) = \mu(\overline{B}) = \mu(\text{co} B) \), where \( \overline{B} \) is the closed convex hull of \( B \);
3. \( \mu(B) \leq \mu(C) \) when \( B \subseteq C \);
4. \( \mu(B+C) \leq \mu(B) + \mu(C) \);
5. \( \mu(B \cup C) \leq \max\{ \mu(B), \mu(C) \} \);
6. \( \mu(B(0,r)) = 2r \), where \( B(0,r) = \{ x \in X ; \| x \| \leq r \} \), if \( \text{dim} \ X = +\infty \).

Lemma 5. Let \( X \) be a Banach space, \( Q : X \rightarrow X \) be a completely continuous operator, if the set

\[
    \Lambda = \{ x ; \text{ } x \in X, x = \lambda Qx, \text{ } 0 < \lambda < 1 \}
\]

is bounded. Then \( Q \) has a fixed point.

Lemma 6. Let \( X \) be a Banach space and \( T \) an operator on \( X \). If there exists a positive integer \( n \) such that \( T^n \) is a contractive map, i.e., there exists a constant \( C(0 \leq C < 1) \) such that

\[
    \| T^n x - T^n y \| \leq C \| x - y \|, \text{ } \forall x, y \in X,
\]

then \( T^n \) has a unique fixed point on \( X \) and it is also the unique fixed point of \( T \).

Before we give the main theorems, we need the following lemma.
Lemma 7. Let $a, b \geq 0, \beta > 0$. Suppose that $u(t)$ is nonnegative continuous function on $0 \leq t \leq T$ with
\[ u(t) \leq a + b \int_0^t (t-s)^{\beta-1} \max_{0 \leq \tau \leq s} u(\tau) ds \]
on this interval. Then
\[ u(t) \leq a + a \sum_{n=1}^{+\infty} \frac{(b \Gamma(\beta))^n}{\Gamma(n \beta)} \left( n \beta - 1 \right) (t-s)^n \]
$0 \leq t \leq T$.

Proof. Write
\[ v(t) := \max_{0 \leq s \leq t} u(s). \]
Then $v(t)$ is a non-decreasing nonnegative continuous function on $[0, T]$.

Given $0 < t \leq T$. Then for any $s, 0 \leq s \leq t$,
\[ u(s) \leq a + b \int_0^s (s-r)^{\beta-1} v(r) dr \]
\[ \leq a + b \int_0^s r^{\beta-1} v(t-r) dr \]
\[ \leq a + b \int_0^t r^{\beta-1} v(t-r) dr \]
\[ = a + b \int_0^t (t-s)^{\beta-1} v(s) ds. \]
Hence,
\[ v(t) \leq a + b \int_0^t (t-s)^{\beta-1} v(s) ds. \]
By Lemma 3, we have
\[ v(t) \leq a + a \int_0^t \left( \sum_{n=1}^{+\infty} \frac{(b \Gamma(\beta))^n}{\Gamma(n \beta)} \right) (t-s)^n v(s) ds, \quad 0 \leq t \leq T, \]
Therefore,
\[ v(t) \leq a + a \sum_{n=1}^{+\infty} \frac{(b \Gamma(\beta))^n}{\Gamma(n \beta)} \left( n \beta - 1 \right) (t-s)^n, \quad \forall t \in [0, T]. \]
The proof ends then. \( \Box \)

First we discuss the case $f$ is not necessarily Lipschitz.

In this case, $A$ needs to not only generate an analytic semigroup, but also needs to generate a compact semigroup.

Our first main result is as follows, where the space $X$ could be an infinite dimensional space.

Theorem 1. Let $A$ be the infinitesimal generator of a compact analytic semigroup of uniformly bounded linear operator, and $f : [0, T] \times C([-\omega, 0], X) \to X$ is continuous. If there are almost everywhere nonnegative measurable functions $l_1(t), l_2(t)$ on $[0, T]$ such that
\[ \|f(t, \varphi)\| \leq l_1(t) + l_2(t) \|\varphi\|_{[-\omega, 0]} \]
for a.e. $t \in [0, T], \varphi \in C([-\omega, 0], X)$ where
\[ \sup_{t \in [0, T]} \int_0^t (t-s)^{n-1} l_1(s) ds < \infty, \quad l_2(t) \in L^\infty([0, T]), \]
then for any $\varphi \in C([-\omega, 0], X)$, the Problem (1) has at least one mild solution on $[-\omega, T]$. 
Proof. For every $\phi \in C([-\omega, 0])$, we define
\[ y(t) := \phi(t) \quad (t \in [-\omega, 0]), \quad y(t) := \Phi(t)\phi(0) \quad (t \geq 0). \]

By Lemma 1, we see that $y \in C([-\omega, T], X)$.

Set
\[ M_1 := \sup_{t \in [0,T]} \int_0^t (t-s)^{q-1} l_1(s) \, ds, \quad M_2 := \|l_2\|_{\infty}, \quad M_3 := \max_{s \in [-\omega, T]} \|y(s)\|. \]

Let
\[ u(t) := x(t) + y(t), \quad t \in [-\omega, T]. \]

Then, it is obvious that $u$ satisfies Equation (2) if and only if $x_0 = 0$ and for $t \in [0, T]$,
\[ x(t) = \int_0^t (t-s)^{q-1} \Psi(t-s) f(s, x_s + y_s). \]

We consider the operator $P : C_0(X) \to C_0(X)$ as follows:
\[ (Px)(t) = \begin{cases} 0, & t \in [-\omega, 0], \\ \int_0^t (t-s)^{q-1} \Psi(t-s) f(s, x_s + y_s) \, ds, & t \in [0, T]. \end{cases} \]  \hspace{1cm} (3)

Because $f$ is continuous, by using the Lebesgue dominated convergence theorem, it is easy to prove that $P : C_0(X) \to C_0(X)$ is continuous. Set $B_r = \{x \in C_0(X), \|x\|_{C_0(X)} \leq r\}$, $r > 0$. Next, we will show that $P$ is a compact operator on $B_r$.

Clearly, $\{(Px)(0) : x \in B_r\}$ is compact.

For $t \in (0, T]$, let $0 < \epsilon_1 < t$, $\epsilon_2 > 0$, $x \in B_r$.

Then, we obtain
\[
(Px)(t) = \int_0^{t-\epsilon_1} (t-s)^{q-1} \int_{\epsilon_2}^{\infty} q \vartheta \eta_{t} \beta((t-s)^{\vartheta} \theta) f(s, x_s + y_s) \, d\vartheta ds \\
+ \int_0^{t-\epsilon_1} (t-s)^{q-1} \int_0^{\epsilon_2} q \vartheta \eta_{t} \beta((t-s)^{\vartheta} \theta) f(s, x_s + y_s) \, d\vartheta ds \\
+ \int_{t-\epsilon_1}^{t} (t-s)^{q-1} \int_{\epsilon_2}^{\infty} q \vartheta \eta_{t} \beta((t-s)^{\vartheta} \theta) f(s, x_s + y_s) \, d\vartheta ds \\
+ \int_{t-\epsilon_1}^{t} (t-s)^{q-1} \int_0^{\epsilon_2} q \vartheta \eta_{t} \beta((t-s)^{\vartheta} \theta) f(s, x_s + y_s) \, d\vartheta ds.
\]

Since $(\epsilon_1^\vartheta \epsilon_2)$ is compact, and the set
\[
\{ \int_0^{t-\epsilon_1} (t-s)^{q-1} \int_{\epsilon_2}^{\infty} q \vartheta \eta_{t} \beta((t-s)^{\vartheta} \theta - \epsilon_1^\vartheta \epsilon_2) f(s, x_s + y_s) \, d\vartheta ds : x \in B_r \}
\]
is bounded, we see that the set
\[
\{ \beta(\epsilon_1^\vartheta \epsilon_2) \int_0^{t-\epsilon_1} (t-s)^{q-1} \int_{\epsilon_2}^{\infty} q \vartheta \eta_{t} \beta((t-s)^{\vartheta} \theta - \epsilon_1^\vartheta \epsilon_2) f(s, x_s + y_s) \, d\vartheta ds : x \in B_r \}
\]
is relatively compact in $X$. Lemma 4(1) tells us
\[
\mu(\{ \beta(\epsilon_1^\vartheta \epsilon_2) \int_0^{t-\epsilon_1} (t-s)^{q-1} \int_{\epsilon_2}^{\infty} q \vartheta \eta_{t} \beta((t-s)^{\vartheta} \theta - \epsilon_1^\vartheta \epsilon_2) f(s, x_s + y_s) \, d\vartheta ds : x \in B_r \}) = 0.
\]
Moreover, it is clear that
\[
\int_{0}^{t-\epsilon} (t-s)^{\theta-1} \int_{\varepsilon}^{\infty} q \eta_{q}(\theta) \mathbb{B}((t-s)^{\theta}) f(s, x_{s} + y_{s}) d\theta ds \\
= \mathbb{E}(\varepsilon_{1}^{\varepsilon_{2}}) \int_{0}^{t-\epsilon} (t-s)^{\theta-1} \int_{\varepsilon}^{\infty} q \eta_{q}(\theta) \mathbb{B}((t-s)^{\theta} - \varepsilon_{1}^{\varepsilon_{2}}) f(s, x_{s} + y_{s}) d\theta ds.
\]
Thus, we get
\[
\mu(\{ \int_{0}^{t-\epsilon} (t-s)^{\theta-1} \int_{\varepsilon}^{\infty} q \eta_{q}(\theta) \mathbb{B}((t-s)^{\theta}) f(s, x_{s} + y_{s}) d\theta ds : x \in B_{r} \}) = 0.
\]
On the other hand, it is easy to see that there exists a positive constant $C$ such that
\[
\| \int_{0}^{t-\epsilon} (t-s)^{\theta-1} \int_{\varepsilon}^{\infty} q \eta_{q}(\theta) \mathbb{B}((t-s)^{\theta}) f(s, x_{s} + y_{s}) d\theta ds \|
\leq C \int_{0}^{\varepsilon} q \eta_{q}(\theta) d\theta, \quad \forall x \in B_{r}.
\]
By Lemma 4(6), we have
\[
\mu(\{ \int_{0}^{t-\epsilon} (t-s)^{\theta-1} \int_{\varepsilon}^{\infty} q \eta_{q}(\theta) \mathbb{B}((t-s)^{\theta}) f(s, x_{s} + y_{s}) d\theta ds : x \in B_{r} \})
\leq 2C \int_{0}^{\varepsilon} q \eta_{q}(\theta) d\theta.
\]
This means that,
\[
\lim_{\varepsilon_{1}, \varepsilon_{2} \to 0+} \mu(\{ \int_{0}^{t-\epsilon} (t-s)^{\theta-1} \int_{\varepsilon}^{\infty} q \eta_{q}(\theta) \mathbb{B}((t-s)^{\theta}) f(s, x_{s} + y_{s}) d\theta ds : x \in B_{r} \}) = 0.
\]
Similarly, we can prove that
\[
\lim_{\varepsilon_{1}, \varepsilon_{2} \to 0+} \mu(\{ \int_{t-\epsilon}^{t} (t-s)^{\theta-1} \int_{\varepsilon}^{\infty} q \eta_{q}(\theta) \mathbb{B}((t-s)^{\theta}) f(s, x_{s} + y_{s}) d\theta ds : x \in B_{r} \}) = 0,
\]
\[
\lim_{\varepsilon_{1}, \varepsilon_{2} \to 0+} \mu(\{ \int_{t-\epsilon}^{t} (t-s)^{\theta-1} \int_{\varepsilon}^{\infty} q \eta_{q}(\theta) \mathbb{B}((t-s)^{\theta}) f(s, x_{s} + y_{s}) d\theta ds : x \in B_{r} \}) = 0.
\]
By Lemma 4(4), we obtain
\[
\mu(\{ \int_{0}^{t} (t-s)^{\theta-1} \int_{0}^{\infty} q \eta_{q}(\theta) \mathbb{B}((t-s)^{\theta}) f(s, x_{s} + y_{s}) d\theta ds : x \in B_{r} \})
\leq \mu(\{ \int_{0}^{t-\epsilon} (t-s)^{\theta-1} \int_{\varepsilon}^{\infty} q \eta_{q}(\theta) \mathbb{B}((t-s)^{\theta}) f(s, x_{s} + y_{s}) d\theta ds : x \in B_{r} \})
+ \mu(\{ \int_{t-\epsilon}^{t} (t-s)^{\theta-1} \int_{\varepsilon}^{\infty} q \eta_{q}(\theta) \mathbb{B}((t-s)^{\theta}) f(s, x_{s} + y_{s}) d\theta ds : x \in B_{r} \})
+ \mu(\{ \int_{t-\epsilon}^{t} (t-s)^{\theta-1} \int_{0}^{\varepsilon} q \eta_{q}(\theta) \mathbb{B}((t-s)^{\theta}) f(s, x_{s} + y_{s}) d\theta ds : x \in B_{r} \}).
\]
Letting $\varepsilon_{1}, \varepsilon_{2} \to 0+$, we get
\[
\mu(\{ \int_{0}^{t} (t-s)^{\theta-1} \int_{0}^{\infty} q \eta_{q}(\theta) \mathbb{B}((t-s)^{\theta}) f(s, x_{s} + y_{s}) d\theta ds : x \in B_{r} \}) = 0.
\]
Consequently, we see that $\{(Px)(t) : x \in B_{r} \}$ is relatively compact in $X$ for all $t \in [0, T]$. 
Clearly, for $t \in [0, T)$,
\[
\|(Px)(t) - (Px)(0)\| \leq \frac{M}{\Gamma(q)} \int_0^t (t - s)^{q-1} \|f(s, x_s + y_s)\|ds.
\]
Thus, for $0 < t_1 < t_2 \leq T$, we obtain
\[
\|(Px)(t_2) - (Px)(t_1)\| \leq \int_0^{t_1} (t_2 - s)^{q-1} \|\Psi(t_2 - s) - \Psi(t_1 - s)\| f(s, x_s + y_s)\|ds
\]
\[
+ \int_0^{t_1} [(t_2 - s)^{q-1} - (t_1 - s)^{q-1}] \|\Psi(t_1 - s)\| f(s, x_s + y_s)\|ds
\]
\[
+ \int_{t_1}^{t_2} (t_2 - s)^{q-1} \|\Psi(t_2 - s)\| f(s, x_s + y_s)\|ds.
\]
This, together with Lemma 2, implies that $P(B_r)$ is equicontinuous on $[0, T]$. Obviously $P(B_r)$ is bounded in $C_0(X)$. By the Arzela-Ascoli theorem, we know that $P$ is a compact operator. Hence, $P$ is completely continuous in $C_0(X)$.

Set $\Lambda := \{x \in C_0(X) : x = \lambda Px, 0 < \lambda < 1\}$. Take $x \in \Lambda$. Then for each $t \in [0, T]$,
\[
x(t) = \lambda \int_0^t (t - s)^{q-1} \Psi(t - s) f(s, x_s + y_s)ds.
\]
Thus
\[
\|x(t)\| \leq \frac{M}{\Gamma(q)} M_1 + \frac{M}{\Gamma(q)} \int_0^t (t - s)^{q-1} M_2 \|x_s\|_{[-\infty, 0]} + \|y_s\|_{[-\infty, 0]}ds
\]
\[
\leq \frac{MM_1}{\Gamma(q)} + \frac{M}{\Gamma(q)} M_2 M_3 \int_0^t (t - s)^{q-1}ds + \frac{MM_2}{\Gamma(q)} \int_0^t (t - s)^{q-1} \max_{0 \leq r \leq s} \|x(r)\|ds,
\]
Write
\[
C_1 = \frac{MM_1}{\Gamma(q)} + \frac{MM_2 M_3 T^q}{q}, \quad C_2 = \frac{MM_2}{\Gamma(q)}.
\]
Then
\[
\|x(t)\| \leq C_1 + C_2 \int_0^t (t - s)^{q-1} \max_{0 \leq r \leq s} \|x(r)\|ds.
\]
By Lemma 7, we have
\[
\|x(t)\| \leq C_1 + C_2 \sum_{n=1}^{+\infty} \left( \frac{C_2 \Gamma(n\beta)}{\Gamma(n\beta)} T^{n\beta} \right) \frac{T^{n\beta}}{n\beta} < \infty, \quad 0 \leq t \leq T.
\]
Therefore, the set $\Lambda$ is bounded. By virtue of Lemma 5, we see that $P$ has a fixed point $x(t)$. Thus, $u(t) = x(t) + y(t)$ is a mild solution of the Problem (1). •

**Remark 1.** If the semigroup $B(t)$ (generated by $A$) satisfies that there exists a $\omega > 0$ such that $B(t)$ is compact for all $t \in (0, \omega)$, then we can see from the proof above that the theorem still holds.

**Remark 2.** The mild solution in this case is usually not unique.

**Remark 3.** Suppose that $g : X \rightarrow X$ is not Lipschitz continuous, i.e., there does not exist a positive constant $C$ such that
\[
\|g(x) - g(y)\| \leq C\|x - y\|, \forall x, y \in X,
\]
but there exists a positive constant $M$ such that $\|g(x)\| \leq M\|x\|, \forall x \in X$ (therefore $g$ is bounded on $X$). Set
\[ f(t, \varphi) = c_1(t)x_0 + c_2(t)\varphi(\int_{-\omega}^{0} \varphi(s)ds). \]

Let \( x_0 \in X \) be a fixed element, and \( c_i(t) (i = 1, 2) \) be continuous functions on \([0, T]\), and \( \varphi \in C([-\omega, 0], X) \).

Then \( f \) satisfies the condition of this theorem, but \( f \) is usually not Lipschitz continuous.

Next we discuss the case when \( f \) is Lipschitz continuous.

In this case, \( A \) needs only to generate an analytic semigroup.

Our second main result is as follows.

**Theorem 2.** Let \( A \) be the infinitesimal generator of an analytic semigroup of uniformly bounded linear operator, and \( f : [0, T] \times C([-\omega, 0], X) \to X \) be continuous. If \( f \) satisfies the Lipschitz condition, i.e., there exists a constant \( L > 0 \) such that

\[ \| f(t, \varphi_1) - f(t, \varphi_2) \| \leq L \| \varphi_1 - \varphi_2 \|([-\omega, 0]), \quad \forall t \in [0, T], \varphi_i \in C([-\omega, 0], X), i = 1, 2, \]

then for any \( \varphi \in C([-\omega, 0], X) \), the problem (1) has a unique mild solution on \([-\omega, T]\).

**Proof.** As in the proof of last theorem, for every \( \varphi \in C([-\omega, 0]) \), we define \( y(t), u(t) \) and the operator \( P : C_0(X) \to C_0(X) \). Then we know that \( y \in C([-\omega, T], X) \), \( u \) satisfies Equation (2) if and only if \( x_0 = 0 \) and for \( t \in [0, T] \),

\[ x(t) = \int_0^t (t-s)^{\gamma-1}\Psi(t-s) f(s, x_s + y_s), \]

and \( P : C_0(X) \to C_0(X) \) is continuous.

For any \( t \in [0, T] \), \( x, \dot{x} \in C_0(X) \),

\[ \| (P\dot{x})(t) - (P\dot{x})(t) \| \leq \int_0^t (t-s)^{\gamma-1} \frac{M}{\Gamma(q)} L \| x_s - \dot{x}_s \|([-\omega, 0]) ds \]

\[ \leq \frac{ML}{\Gamma(q)} \| x - \dot{x} \|_{C_0(X)} \]

\[ \| (P^2x)(t) - (P^2\dot{x})(t) \| \leq \int_0^t (t-s)^{\gamma-1} \| \Psi(t-s) \|_{L^1([-\omega, 0])} \| (P\dot{x})_s - (P\dot{x})_s \|([-\omega, 0]) ds \]

\[ \leq \int_0^t (t-s)^{\gamma-1} \frac{M}{\Gamma(q)} L \max_{0 \leq \tau \leq s} \| (P\dot{x})(\tau) - (P\dot{x})(\tau) \| ds \]

\[ \leq \left( \frac{ML}{\Gamma(q)} \right)^2 B(q, 1) \int_0^t (t-s)^{\gamma-1}s^q ds \| x - \dot{x} \|_{C_0(X)}. \]

Write \( s = t\tau \). Then we have

\[ \int_0^t (t-s)^{\gamma-1}s^q ds = \int_0^1 (t-t\tau)^{\gamma-1}(t\tau)^q d\tau \]

\[ = t^q \int_0^1 (1-s)^{\gamma-1}s^q ds \]

\[ = t^q B(q, q + 1). \]

Hence

\[ \| (P^2x)(t) - (P^2\dot{x})(t) \| \leq \left( \frac{ML}{\Gamma(q)} \right)^2 t^q B(q, 1) B(q, q + 1) \| x - \dot{x} \|_{C_0(X)}. \]
We can deduce by induction that
\[ \| (P^nx)(t) - (P^n\tilde{x})(t) \| \leq \left( \frac{ML}{\Gamma(q)} \right)^n \prod_{k=0}^{n-1} B(q, kq + 1) \| x - \tilde{x} \|_{C_0(X)}, \quad n = 1, 2, 3, \ldots \]

In fact, suppose that this inequality holds for \( n = m \), that is, for any \( t \in [0, T] \),
\[ \| (P^mx)(t) - (P^m\tilde{x})(t) \| \leq \left( \frac{ML}{\Gamma(q)} \right)^m \prod_{k=0}^{m-1} B(q, kq + 1) \| x - \tilde{x} \|_{C_0(X)}. \]

Then, by the similar argument as above, we obtain
\[ \| (P^{m+1}x)(t) - (P^{m+1}\tilde{x})(t) \| \leq \left( \frac{ML}{\Gamma(q)} \right)^{m+1} \prod_{k=0}^{m-1} B(q, kq + 1) \| x - \tilde{x} \|_{C_0(X)}. \]

Thus we have proved that
\[ \| (P^nx)(t) - (P^n\tilde{x})(t) \| \leq \left( \frac{ML}{\Gamma(q)} \right)^n \prod_{k=0}^{n-1} B(q, kq + 1) \| x - \tilde{x} \|_{C_0(X)}, \quad n = 1, 2, 3, \ldots \]

Therefore
\[ \| P^n x - P^n\tilde{x} \|_{C_0(X)} \leq \left( \frac{ML^q}{\Gamma(q)} \right)^n \prod_{k=0}^{n-1} B(q, kq + 1) \| x - \tilde{x} \|_{C_0(X)}. \]

So \( P^{n_0} \) is a contractive map on \( C_0(X) \) for a positive integer \( n_0 \). Thus by Lemma 6, we know that \( P \) has a unique fixed point \( x(t) \) on \( C_0(X) \), that is, \( u(t) = x(t) + y(t) \) is the unique mild solution of the Problem (1).

\[ \square \]

**Remark 4.** A similar result holds for the following first-order differential equation in the case \( f \) is Lipschitz continuous
\begin{equation}
\begin{cases}
\frac{du(t)}{dt} + Au(t) = f(t, u(t)), & t > t_0 \\
u(t_0) = u_0,
\end{cases}
\end{equation}

For details, please refer to [20], p. 183–185.

**Remark 5.** If we want to get the unique mild solution, we can do as follows. Set \( Q := P^{n_0} (P^{n_0} \text{ as in the proof of Theorem 2}) \),
\[ x_0 = 0, \quad x_{i+1} = Q x_i \quad (i = 0, 1, 2, 3, \ldots). \]

Then \( u_i(t) = x_i(t) + y(t) \) converges uniformly to the unique mild solution of the equation.

3. Examples

It is known that there are many concrete fractional differential equations from anomalous diffusion on fractals (e.g., some amorphous semiconductors or strongly porous materials), which are concrete models of the abstract Cauchy Problem (1). We refer the reader to [2,16] and references therein.
Moreover, from [2,16] and references therein, we see that the following Example 1 with the delay effect models some type of anomalous dynamical behaviors of anomalous transport processes.

**Example 1.** Let
\[ X = \{ u(x); \ u(x) \in L^2[0, \pi], u(x) \text{ is a real function} \} \]
and define its natural norm and inner product respectively, for \( u, v \in X \), by
\[
\| u \|_X = \left( \int_0^\pi u(x)^2 dx \right)^{\frac{1}{2}}, \quad < u, v > = \int_0^\pi u(x)v(x) dx.
\]

Consider the following Cauchy problem for fractional partial differential equations with finite delay:
\[
\left\{ \begin{array}{l}
cD^q_t u(t, x) = Au(t, x) + f(t, u_t), t \in [0, T], x \in [0, \pi] \\
u(t, x) = \phi(t, x), t \in [-\omega, 0],
\end{array} \right. \quad (5)
\]
where \( q \in (0, 1), T, \omega > 0 \) are constants.

Let the operator \( A : D(A) \subset X \rightarrow X \) be define by
\[
D(A) := \{ v \in X : v'' \in X, v(0) = v(\pi) = 0 \}, \quad Au = \frac{\partial^2 u}{\partial x^2}.
\]
It is well known (cf., e.g., [18]) that—\( A \) has a discrete spectrum with eigenvalues of the form \( n^2, n \in \mathbb{N} \), and corresponding normalized eigenfunctions given by
\[
z_n = \sqrt{\frac{2}{\pi}} \sin(nx), \quad n = 1, 2, \ldots
\]
Moreover, \( A \) generates a compact analytic semigroup \( B(t) (t \geq 0) \) on \( X \), and
\[
B(t)u = \sum_{n=1}^{+\infty} e^{-n^2 t} < u, z_n > z_n.
\]
It is not difficult to verify that
\[
\| B(t) \| \leq e^{-t} \text{ for all } t \geq 0.
\]
Hence, we take \( M = 1 \). Thus, when \( f \) satisfies the conditions in Remark 3 and \( \phi \) is a continuous function, we see by Theorem 1, the Problem (5) has at least one mild solution.

**Remark 6.** For the special case \( A=0 \),
\[
\left\{ \begin{array}{l}
cD^q_t u(t) = f(t, u_t), t \in [0, T], \\
u(t) = \phi(t), t \in [-\omega, 0],
\end{array} \right. \quad (6)
\]
where \( q \in (0, 1), T, \omega > 0 \) are constants, \( f \) satisfies the condition in Remark 3, and \( \phi \) is a continuous function. Then the Problem (6) has at least one mild solution.

**Example 2.** Consider the following problem
\[
\left\{ \begin{array}{l}
cD^q_t u(t) = Au(t) + f(t, u_t), t \in [0, T], \\
u(t) = \phi(t), t \in [-\omega, 0],
\end{array} \right. \quad (7)
\]
where $X$ is a Banach space, $q \in (0, 1)$, $T, \omega > 0$ are constants, $A$ is the infinitesimal generator of an analytic semigroup of uniformly bounded linear operator on a Banach space $X$,

$$f(t, \varphi) = c_1(t)x_0 + c_2(t) \int_{-\omega}^{0} \varphi(s)ds,$$

$x_0 \in X$ is a fixed element, $c_i(t) (i = 1, 2)$ are continuous functions on $[0, T]$, and $\varphi \in C([-\omega, 0], X)$.

It is easy to verify that $f$ satisfies the condition of Theorem 2. So the Problem (3) has a unique mild solution.

Remark 7. For the special case $A=0$,

$$\left\{ \begin{array}{ll} ^{c}D_q^\alpha u(t) = f(t, u_t), & t \in [0, T], \\ u(t) = \phi(t), & t \in [-\omega, 0], \end{array} \right. \tag{8}$$

where $q \in (0, 1)$, $T, \omega > 0$ are constants, $f(t, \varphi) = c_1(t)x_0 + c_2(t) \int_{-\omega}^{0} \varphi(s)ds$, $x_0 \in X$ a Banach space is a fixed element, $c_i(t) (i = 1, 2)$ are continuous functions on $[0, T]$, $\varphi \in C([-\omega, 0], X)$. So the Problem (8) has a unique mild solution.

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