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Quantiles in Abstract Convex Structures

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Abstract: In this short paper, we aim at a qualitative framework for modeling multivariate decision problems where each alternative is characterized by a set of properties. To this extent, we consider convex spaces as underlying universes and make use of lattice operations in convex spaces to formalize the notion of quantiles. We also put in evidence that many important models of decision problems can be viewed as convex spaces-based models. Several properties of aggregation operators are translated into this general setting, and independence and invariance are used to provide axiomatic characterizations of quantiles.

Keywords: convex space; aggregation operator; invariance; independence; quantile

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1. Introduction

The aim of this paper is to propose a general unified framework for defining aggregation operators. Our framework is abstract and algebraic in nature and in this framework we generalize some results of the literature [1–4].

We consider convex structures where the notion considered here (see [5]) is not restricted to the context of vector spaces. The basic idea of our approach is to describe the space of alternatives in terms of a "topological" relation. We can prove that lattices, median spaces and interval spaces are convex spaces and also that to every property spaces (see [1,2,4]) is associated with a convex structure.

We then focus on aggregation operators $f\colon X^A \to X$ where X is convexity space and A is a nonempty set. We study aggregation operators that satisfy properties of monotonicity and independence and we consider aggregation operators that are based on *decisive* subsets of A. Moreover, we consider operators that are componentwise compatible with the structure of convexity space of X.

We propose also a particular version of Arrow's theorem thus considering a link between aggregation theory and social choice theory as in [6]. It appears that there are many connections between the work presented here with the results of [3,4,7–12]. Applications of these types of results can be found in in [4,13,14].

The structure of the paper is as follows. In Section 2 we introduce convex spaces and we provide the necessary definitions. Section 3 is devoted to describe some important examples. Finally in Sections 3 and 4 we study some classes of agggregation operators acting on abstract convex structures.

2. Abstract Convex Structures

The notion of convexity is a basic mathematical structure that is used to analyze many different problems and there are in the literature various kinds of generalized, topological, or axiomatically defined convexities. There are generalizations that are motivated by concrete problems and those that are stated from an axiomatic point of view, where the notion of abstract convexity is based on properties of a family of sets.

In this paper the general notion of abstract convexity structure that is studied in [5] is considered.

Definition 1. *A family $\mathcal{C}$ of subsets of a set X is a convexity on a set X if $\emptyset$ and X belong to $\mathcal{C}$ and $\mathcal{C}$ is closed under arbitrary intersections and closed under unions of chains.*

The elements of $\mathcal{C}$ are called convex sets of X and the pair $(X, \mathcal{C})$ is called a convex space.

Moreover, the convexity notion allows us to define the notion of the convex hull operator, which is similar to that of the closure operator in topology.

Definition 2. *If X is a set with a convexity $\mathcal{C}$ and A is a subset of X, then the convex hull of $A \subseteq X$ is the set*

$$convA = \bigcap\{C \in \mathcal{C} : A \subseteq C\}. \tag{1}$$

This operator enjoys certain properties that are identical to those of usual convexity: for instance $convA$ is the smallest convex set that contains set A. It is also clear that C is convex if and only if $convC = C$.

The convex hull of a set $\{x_1, \ldots, x_n\}$is called an n-polytope and is denoted by $[x_1, \ldots, x_n]$. A 2-polytope $[a, b]$ is called the segment joining a, b.

A convexity C is called N-ary $(N \in \mathbb{N})$ if $A \subseteq \mathcal{C}$ whenever $convF \subseteq A$ for all $F \subseteq A$ where F has at most N elements. A 2-ary convexity is called an interval convexity.

We also consider biconvex spaces, i.e., triples of the form $(X, \mathcal{A}, \mathcal{B})$ where $\mathcal{A}, \mathcal{B}$ are two convexities on a set X, called the lower and the upper convexity. Obviously every convex space $(X, \mathcal{C})$ can be viewed as a biconvex space $(X, \mathcal{C}, \mathcal{C})$.

If X, Y are convex spaces with convexities $\mathcal{C}, \mathcal{D}$, respectively, we consider the following definition of a compatible map between two convex spaces.

Definition 3. *A map $\gamma\colon X \to Y$ is convex if $\gamma^{-1}(C) \in \mathcal{C}$ for every $\in \mathcal{C}$ and such that when $C_i \in \mathcal{C}$ for $i \in I$*

$$\gamma(\bigcap_{i \in I} C_i) = \bigcap_{i \in I} \gamma(C_i). \tag{2}$$

For a general theory of convexity we refer to [5].

3. Some Examples

We present some examples and classes of convex spaces. First of all we note that every real vector space together with the collection of all convex sets in the usual meaning, is a 2-arity convex space.

Ordered spaces The usual convexity on $\mathbb{R}$ can be defined in terms of ordering as follows: a set C is convex if and only if when $a, b \in C$ and $a \leq x \leq b$ implies $x \in C$. We can define in the same way a convexity on a partially ordered set (see [5], p. 6). Such a convexity is called the order convexity.

Lattices If $\langle L, \wedge, \vee \rangle$ is a lattice we denote by $\mathcal{L}$ and $\mathcal{U}$ the collections of all ideals and all filters respectively (the empty set and the whole lattice are treated as (non-proper) ideals and filters). Since the union of a chain of filters (ideals) is a filter (ideal), these are two convexities on L that will be called the lower and the upper lattice convexity respectively. Moreover there exists a convexity $\mathcal{C}$ generated by $\mathcal{L} \bigcup \mathcal{U}$ the least convexity containing all ideals and filters. This convexity will be called the lattice convexity on L.

Please note that if L is linearly ordered then G equals the order convexity. The convexity of the dual lattice is the same as the original one.

It is possible to consider lattices as convex spaces (with the lattice convexity) as well as bi-convex spaces (with the lower and upper lattice convexities). It is easy to check that a proper halfspace is either a prime filter or a prime ideal. It can be proved also that the lattice convexity is an interval convexity and that

$$[a, b] = \{x \in L : a \wedge b \leq x \leq a \vee b\}. \tag{3}$$

Median spaces A median space is a convexity space X with an interval convexity such that for each $a, b, c \in X$ there exists a unique point in $[a, b] \cap [a, c] \cap [b, c]$. We call it the median of a, b, c and denote by $m(a, b, c)$. This defines a map $m\colon X^3 \to X$, called the median operator on X. In any convexity space, every point in $[a, b] \cap [a, c] \cap [b, c]$ is called a median of a, b, c. There is a natural way to define the structure of a median space by means of the median operator (see [5]).

Property-based domains A property-based domain (as defined in [1]) is a pair $(X, \mathcal{H})$ where X is a non-empty set and $\mathcal{H}$ is a collection of non-empty subsets of X and if $x, y \in X$ and $x \neq y$ there exists $H \in \mathcal{H}$ such that $x \in H$ and $y \notin H$. The elements of $\mathcal{H}$ are referred to as properties and if $x \in H$ we say that x has property represented by the subset H. This definition is slightly more general than that of [3] and of [4], in fact it is not assumed that the set X is finite and we do not consider that the set H^c is a property if H is a property.

The "property space" model provides a very general framework for representing preferences and then aggregation of preferences. In every property-based domain we can define a convexity defined as follows. A subset $S \subseteq X$ is said to be convex if it is intersection of properties.

Arrowian framework The problem of preference aggregation can be viewed as a property-based domain and then as a convex space. We consider a set of alternatives A and a set $\mathcal{R}$ of binary relations in A. We can consider different requirements on the set $\mathcal{R}$ and so $\mathcal{R}$ can be the set of preorders or the set of linear orders in A.

If we define for each pair $a, b \in A$ the set

$$H_{a,b} = \{R \in \mathcal{R} : aRb\} \tag{4}$$

the family $\mathcal{H} = \{H_{a,b} : a, b \in A\}$ defines a property-based domain structure on the set $\mathcal{R}$. See [4] for more details on Arrowian framework.

4. Aggregation Functional over Convex Spaces

Aggregation operators are mathematical functions that are used to combine several inputs into a single representative outcome; see [15] for a comprehensive overview on aggregation theory. Aggregation operators play an important role in several fields such as decision sciences, computer and information sciences, economics and social sciences and there are a large number of different aggregation operators that differ on the assumptions on the inputs and about the information that we want to consider in the model.

Definition 4. *If N is an arbitrary nonempty set and X is a convex space, then an aggregation functional is a map $F\colon X^A \to \mathcal{P}(X)$.*

Our framework is very general, we do not assume that the sets X and A are finite or that the map $F\colon X^A \to \mathcal{P}(X)$ is surjective. Moreover, we consider the case in which there are more than one equivalent solutions and also the case in which there are no solutions. For each $c \in X$, we denote by $\mathbf{c}$ the constant c map in L^A.

The following properties of an aggregation functional are key to our analysis.

Monotonicity If $C \in \mathcal{C}$, $F(f) \subseteq C$ and $y \in C$ then $F(g) \subseteq C$ where $g(i) = y$ and $f(j) = g(j)$ if $j \neq i$.

Idempotence $c \in F(\mathbf{c})$ for every $c \in X$.

Independence If $C, D \in \mathcal{C}$ $F(f) \subseteq C$ and for all $i \in A$, $f(i) \in C$ if and only if $g(i) \in D$ we have that $F(g) \subseteq D$.

Invariance For every convex map $\gamma\colon X \to X$, $F(\gamma \circ f) = \gamma(F(f))$

5. Quantiles in Convex Spaces

We briefly consider aggregation functionals based on a complete lattices. As it is well known the quantile is a generalization of the concept of median and it plays an important role in statistical and economic literature. We study quantile in an ordinal framework and we consider an axiomatic representation of quantiles as in [1,7,8]. Here we provide a definition and characterization of quantiles for lattice-valued operators.

If A is a nonempty set and L a bounded lattice a non-additive measure on A with values in L is a function $m\colon 2^A \to L$ such that $m(\emptyset) = 0$, $m(A) = 1$ and $m(C) \leq m(D)$ whenever $C \subseteq D$.

Definition 5. *If α is an element of L, then the lattice-valued quantile of level α is the functional $Q_\alpha\colon L^A \to L$ defined by*

$$Q_\alpha(f, m) = \bigvee\{x : m(\{f \geq x\}) \geq \alpha\}.$$

It can be proved that this definition extends the well known definition of quantile for real-valued functions (see [8]).

We recall the definition of completely distributive lattice. A complete lattice L is said to be a completely distributive is the following distributive law holds

$$\bigwedge_{i\in I}\Big(\bigvee_{j\in J} x_{ij}\Big) = \bigvee_{f\in J^I}\Big(\bigwedge_{i\in I} x_{if(i)}\Big),$$

for every doubly indexed subset $\{x_{ij} : i \in I, j \in J\}$ of L. Please note that every complete chain (in particular, the extended real line and each product of complete chains) is completely distributive. Moreover, complete distributivity reduces to distributivity in the case of finite lattices.

A collection of sets $\mathcal{U} \subseteq 2^A$ is said to be an *upper set* in A if $X \in \mathcal{U}$ and $X \subset Y$ implies that $Y \in \mathcal{U}$. Then we can prove the following results.

Proposition 1. *Let L be a completely distributive lattice. An aggregation functional $F\colon L^A \to L$ is a lattice-valued quantile with respect to a non-additive measure $m\colon 2^A \to L$ if and only if there exists a upper set $\mathcal{U}$ such that*

$$F(f) = \bigvee\{x \in L : \textit{there exists} \quad U \in \mathcal{U} \quad \textit{such that} \quad f(i) \geq x \quad \textit{for every} \quad i \in U\} \tag{5}$$

or if and only if there exists a upper set $\mathcal{U}$ such that

$$F(f) = \bigwedge\{x \in L : \textit{there exists} \quad U \in \mathcal{U} \quad \textit{such that} \quad f(i) \leq x \quad \textit{for every} \quad i \in U\} \tag{6}$$

Proof of Proposition 1. By Proposition 1 in [7] if L is a completely distributive lattice an aggregation functional $F\colon L^A \to L$ is a lattice-valued quantile with respect to a non-additive measure $m\colon \mathcal{A} \to L$ if and only if there exists a upper set $\mathcal{U}$ such that

$$F(f) = \bigvee_{U\in\mathcal{U}} \bigwedge_{i\in U} f(i)$$

or if and only if there exists a upper set $\mathcal{U}$ such that

$$F(f) = \bigwedge_{U\in\mathcal{U}} \bigvee_{i\in U} f(i).$$

Then we can prove that F is a lattice-valued quantile when $F(f) \geq x$ if and only if there exists $U \in \mathcal{U}$ such that $f(i) \geq x$ for every $i \in U$. Then we get

$$F(f) = \bigvee\{x \in L : \text{there exists} \quad U \in \mathcal{U} \quad \text{such that} \quad f(i) \geq x \quad \text{for every} \quad i \in U\}.$$

The second statement follows similarly. □

Since we know that the lattice convexity is an interval convexity we can prove the following characterization of lattice-valued quantiles.

Proposition 2. *If L is a completely distributive lattice, then an aggregation functional $F\colon L^A \to L$ is a lattice-valued quantile with respect to a non-additive measure $m\colon \mathcal{A} \to L$ if and only if there exists an upper set $\mathcal{U}$ such that*

$$F(f) = \bigcap\{C \in \mathcal{C} : \{i : f(i) \in C\} \in \mathcal{U}\} \tag{7}$$

Then the elements in $F(f)$ belong to a convex set C if and only if $f(i)$ belongs to C for a "decisive " or a "large enough"set.

Let us define quantiles in an abstract convex structures.

Definition 6. *If N is an arbitrary nonempty set and X is a convex space, then a quantile is an aggregation functional $F\colon X^A \to \mathcal{P}(X)$ defined by*

$$F(f) = \bigcap\{C \in \mathcal{C} : \{i : f(i) \in C\} \in \mathcal{U}\} \tag{8}$$

where $\mathcal{U}$ is an upper set in A.

Furthermore, we can characterize from an axiomatic point of view quantiles in an abstract convex structure.

Proposition 3. *If N is an arbitrary nonempty set and X is a convex space, then a quantile is a monotone, idempotent and independent aggregation functional. Conversely an aggregation functional $F\colon X^A \to \mathcal{P}(X)$ that is monotone and independent is a quantile.*

Proof of Proposition 3. If C is a convex set in $\mathcal{C}$ and f is an element of X^A we define the set $N(f, C) = \{i \in N : f(i) \in C\}$. Let F be a quantile, $C \in \mathcal{C}$, f an element of X^A such that $F(f) \subseteq C$ and $y \in C$. If we define an element g of X^A by $g(i) = y$ and $f(j) = g(j)$ if $j \neq i$ then $N(f, C) \subseteq N(g, C)$ and so $F(g) \subseteq C$. Moreover, if $c \in C$ we have that $c \in F(\mathbf{c})$ if F is a quantile since $N(\mathbf{c}, C) = A$.

By the definition of quantile if F is a quantile, $C, D \in \mathcal{C}$, $F(f) \subseteq C$ and for all $i \in A$, $f(i) \in C$ if and only if $g(i) \in D$ we can easily prove that $N(f, C) = N(g, D)$ and we get $F(g) \subseteq D$. So we have proved that quantiles are monotone , idempotent and independent functionals.

We note that functional F is monotone and independent if and only if $C, D \in \mathcal{C}$ $F(f) \subseteq C \in \mathcal{C}$ and for all $i \in N$, if $f(i) \in C$ then $g(i) \in D$ we have that $F(g) \in D$.

We say that a set $U \subseteq N$ is decisive with respect to an element $C \in \mathcal{C}$ if there exists $f \in X^A$ such that $N(f, C) = U$ and $F(x) \subseteq C$. Being F monotone and independent a set U is decisive with respect to C if and only if for every $f \in X^A$ such that $N(f, C) = U$, $F(f) \subseteq C$.

Since the functional F is monotone and independent then the set of decisive subset of N does not depend on the convex set C. If $\mathcal{U}$ is the family of decisive subsets of N for every $f \in X^A$, $F(f) \subseteq C$ if and only if $N(F, C) \in \mathcal{U}$. So we have proved that

$$F(f) = \bigcap\{C : N(f, C) \in \mathcal{U}\} = \bigcap\{C \in \mathcal{C} : \{i : f(i) \in C\} \in \mathcal{U}\}.$$

□

The following proposition presents another property of quantiles in convex spaces.

Proposition 4. *f N is an arbitrary nonempty set and X is a convex space, then a quantile is an invariant aggregation functional.*

Proof of Proposition 4. Let $F\colon X^A \to \mathcal{P}(X)$ be a quantile and $\gamma\colon X \to X$ a convex map.

Then $F(\gamma \circ f) = \bigcap\{C \in \mathcal{C} : \{i : (\gamma \circ f)(i) \in C\} \in \mathcal{U}\} = \bigcap\{\gamma(C), C \in \mathcal{C} : \{i : f(i) \in \gamma^{-1}C\} \in \mathcal{U}\} = \bigcap\{\gamma(D), D \in \mathcal{C} : \{i : f(i) \in D\} \in \mathcal{U}\} = \bigcap\{\gamma(C), C \in \mathcal{C} : \{i : f(i) \in C\} \in \mathcal{U}\}$.

If $\mathcal{D} = C \in \mathcal{C} : \{i : f(i) \in C\} \in \mathcal{U}\}$ being γ continuous

$$\bigcap_{C \in \mathcal{D}} \gamma(C) = \gamma(\bigcap_{C \in \mathcal{D}} C)$$

and we get that $F(\gamma \circ f) = \gamma(F(f))$. □

6. Concluding Remarks

We introduced a unified qualitative framework for studying aggregation operators. The approach presented in this paper has taken its inspiration from social choice theory and we generalize some results in social choice in certain respects. This setting has several appealing aspects, for it provides sufficiently rich structures studied in the literature , which allow the definition of quantiles from an ordinal point of view, and which do not depend on the usual arithmetical structure of the reals.

There are however many opportunities for much more detailed research in this area in particular from the point of view of aggregation theory. An obvious topic for future research is to analyze other aggregation functionals defined in convex spaces. There are several extensions avaiable within this framework, for instance, one could consider Sugeno type integral defined by a class of decisive sets.

Conflicts of Interest: The author declares no conflict of interest.

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