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A New Type of Generalization on W—Asymptotically $\mathcal{J}_\lambda$—Statistical Equivalence with the Number of α

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Abstract: In our paper, by using the concept of $W-$asymptotically $\mathcal{J}-$ statistical equivalence of order α which has been previously defined, we present the definitions of $W-$asymptotically $\mathcal{J}_\lambda-$statistical equivalence of order α, $W-$strongly asymptotically $\mathcal{J}_\lambda-$statistical equivalence of order α, and $W-$strongly Cesáro asymptotically $\mathcal{J}-$statistical equivalence of order α where $0 < \alpha \leq 1$. We also extend these notions with a sequence of positive real numbers, $p = (p_k)$, and we investigate how our results change if p is constant.

Keywords: $\mathcal{J}-$statistical convergence; asymptotic equivalence; set sequences; $\lambda = (\lambda_n)$ sequence

MSC: 40G15; 40A35

1. Introduction and Background

To make it easier to understand, we prefer to give introduction section in five parts. In first part, we give the main definitions related to statistical convergence: $\lambda-$statistical convergence, $\mathcal{J}$—convergence, $\mathcal{J}$—statistical convergence, and $\mathcal{J}_\lambda-$ statistical convergence. In the second part, we mention asymptotic equivalence and S_λ^L-asymptotic equivalence. In the third part, we explain set sequences, and we give some important definitions for set sequences in the Wijsman sense. In the fourth part, we explain how statistical convergence and $\mathcal{J}-$convergence were expanded using the number of α $(0 < \alpha < 1)$. Finally, in last part, we explain the purpose and innovations of our study.

1.1. $\mathcal{J}_\lambda-$Statistical Convergence

Statistical convergence is a concept which was formally introduced by Fast [1] and Steinhaus [2], independently. Later on, Schoenberg reintroduced this concept in his own study [3]. This new type of convergence has been used in different areas by several authors in references [4–8]. Statistical convergence is based on the definition of the natural density of the set $K \subseteq \mathbb{N}$ and we define the natural density of K by $d(K) = \lim_{n\to\infty} \frac{|K_n|}{n}$. In this definition, $K_n = \{k \in K : k \leq n\}$ and $|K_n|$ gives the number of elements in K_n.

Using this information, we say that a sequence (x) of real numbers is statistically convergent to the number L if $d(K(\varepsilon)) = d(\{k \leq n : |x_k - L| \geq \varepsilon\}) = 0$. In this case we write $st - \lim x_k = L$, and usually, S denotes the set of all statistical convergent sequences.

Let $\lambda = (\lambda_n)$ be a positive number sequence which is non-decreasing and tending to ∞. Also, for this sequence $\lambda_{n+1} \leq \lambda_n + 1,\ \lambda_1 = 1$. We denote the set of this kind of sequence by Λ, and we have the interval $I_n = [n - \lambda_n + 1, n]$. Mursaleen [9] defined $\lambda-$statistical convergence such that

$$\lim_{n\to\infty} \frac{1}{\lambda_n} |\{k \in I_n : |x_k - L| \geq \varepsilon\}| = 0$$

for any $\varepsilon > 0$, and he denoted this new method by S_λ. On the other hand, Kostyrko, Šalát and Wilezyński [10] introduced a new type of convergence which is defined in a metric space and is called $\mathcal{J}$—convergence. This type of convergence is based on the definition of an ideal $\mathcal{J}$ in $\mathbb{N}$.

A family of sets, $\mathcal{J} \subseteq 2^{\mathbb{N}}$, is an ideal if the following properties are provided:

(i) $\emptyset \in \mathcal{J}$; (ii) $A, B \in \mathcal{J}$ implies $A \cup B \in \mathcal{J}$; and (iii) for each $A \in \mathcal{J}$ and each $B \subseteq A$ implies $B \in \mathcal{J}$.

We say that $\mathcal{J}$ is non-trivial if $\mathbb{N} \notin \mathcal{J}$ and $\mathcal{J}$ is admissible if $\{n\} \in \mathcal{J}$ for every $n \in \mathbb{N}$.

A family of sets, $\mathcal{F} \subseteq 2^{\mathbb{N}}$, is a filter if the following properties are provided:

(i) $\emptyset \notin \mathcal{F}$; (ii) if $A, B \in \mathcal{F}$ then we have $A \cap B \in \mathcal{F}$; and (iii) for each $A \in \mathcal{F}$ and each $A \subseteq B$, we have $B \in \mathcal{F}$.

If $\mathcal{J}$ is an ideal in $\mathbb{N}$, then we have,

$$F(\mathcal{J}) = \{A \subset \mathbb{N} : \mathbb{N} \backslash A \in \mathcal{J}\}$$

is a filter in $\mathbb{N}$.

Definition 1. *([10]) A sequence of reals $x = (x_k)$ is $\mathcal{J}-$ convergent to $L \in \mathbb{R}$ if and only if the set*

$$A_\varepsilon = \{k \in \mathbb{N} : |x_k - L| \geq \varepsilon\} \in \mathcal{J}$$

for each $\varepsilon > 0$. In this case, we say that L is the $\mathcal{J}-$limit of the sequence (x).

$\mathcal{J}$—convergence generalizes many types of convergence such as usual convergence and statistical convergence. If we choose the ideals $\mathcal{J}_f = \{A \subset \mathbb{N} : A \text{ is finite}\}$ and $\mathcal{J}_d = \{A \subset \mathbb{N} : d(A) = 0\}$, then we obtain usual convergence and statistical convergence, respectively.

Based on the statistical convergence and $\mathcal{J}-$convergence, an important role was located in this area, $\mathcal{J}$—statistical convergence, which was introduced by Das, Savaş and Ghosal [11] as follows:

Definition 2. *([11]) A sequence $x = (x_k)$ is $\mathcal{J}-$statistically convergent to L if*

$$\left\{n \in \mathbb{N} : \frac{1}{n} \left|\{k \leq n : |x_k - L| \geq \varepsilon\}\right| \geq \delta\right\} \in \mathcal{J}$$

for every $\varepsilon > 0$ and $\delta > 0$.

1.2. Asymptotic Equivalence

Asymptotic equivalence was first introduced by Pobyvanets [12] and some main definitions and asymptotic reguler matrices were given by Marouf [13]. Bilgin [14] defined $f-$Asymptotically equivalent sequences, and on the other hand, asymptotically statistically equivalent sequences were presented by Patterson [15]. Gümüş and Savaş [16] gave the definition of $\mathcal{J}$—asymptotically $\lambda-$statistically equivalent sequences by using the $\lambda = (\lambda_n)$ sequence, and they were also interested in some inclusion relations between other related spaces.

According to Marouf, if $x = (x_k)$ and $y = (y_k)$ are two non-negative sequences, we say that they are asymptotically equivalent if

$$\lim_k \frac{x_k}{y_k} = 1.$$

This is denoted by $x \sim y$.

Definition 3. *([16]) Let $\mathcal{J}$ be an admissible ideal and $\lambda \in \Lambda$. Two number sequences $x = (x_k)$ and $y = (y_k)$ are $S^L(\mathcal{J})-$asymptotically equivalent of multiple L (or $\mathcal{J}-$ asymptotically $\lambda-$statistically equivalent) if every $\delta, \varepsilon > 0$,*

$$\left\{ n \in \mathbb{N} : \frac{1}{\lambda_n} \left| \left\{ k \in I_n : \left| \frac{x_k}{y_k} - L \right| \geq \varepsilon \right\} \right| \geq \delta \right\} \in \mathcal{J}.$$

1.3. Set Sequences

In recent years, studies on set sequences has become popular. Firstly, usual convergence has been extended to convergence of sequences of sets. The first definitions of this subject were based on Baronti and Papini's [17] work in 1986. Now, we revisit the definitions of convergence, boundedness, and the Cesáro summability of set sequences. Throughout the paper, (X, ρ) is a metric space, and $A, A_k, B_k \subseteq X$ represents non-empty closed subsets of X for all $k \in \mathbb{N}$.

(X, ρ) is a metric, $x \in X$ is a point in X, and A is any non-empty subset of X. The distance from x to A is defined by

$$\rho(x, A) = \inf_{a \in A} \rho(x, A).$$

Definition 4. *([17]) In any metric space, the set sequence $\{A_k\}$ is Wijsman convergent to A if*

$$\lim_{k \to \infty} \rho(x, A_k) = \rho(x, A)$$

for each $x \in X$. We write for this case $W - \lim_{k \to \infty} A_k = A$.

We would like to give a well known example of this subject.

Example 1. *In the (x, y)-plane, consider the $A_k = \{(x, y) : x^2 + y^2 + 2kx = 0\}$ sequence of circles. We can easily see that for $k \to \infty$, this sequence is Wijsman convergent to the y-axis, i.e., $A = \{(x, y) : x = 0\}$.*

Definition 5. *([17]) In any metric space, the set sequence $\{A_k\}$ is bounded if*

$$\sup_k \rho(x, A_k) < \infty$$

for each $x \in X$. This is shown as $\{A_k\} \in L_\infty$.

Definition 6. *([17]) In any metric space, the set sequence $\{A_k\}$ is Wijsman Cesáro summable to A if*

$$\lim_{n \to \infty} \frac{1}{n} \sum_{k=1}^{n} \rho(x, A_k) = d(x, A)$$

for each $x \in X$, and $\{A_k\}$ is Wijsman strongly Cesáto summable to A if

$$\lim_{n \to \infty} \frac{1}{n} \sum_{k=1}^{n} |\rho(x, A_k) - \rho(x, A)| = 0$$

for each $x \in X$.

Nuray and Rhodes [18] introduced Wijsman statistical convergence for set sequences by combining statistical convergence with this new concept. Similarly, Kisi and Nuray [19] defined Wijsman $\mathcal{J}-$convergence for set sequences with an ideal $\mathcal{J}$.

Definition 7. *([18]) Let (X,ρ) be a metric space. For any non-empty closed subsets, $A, A_k \subseteq X$, we say that the sequence $\{A_k\}$ is Wijsman statistically convergent to A if $\{d(x,A_k)\}$ is statistically convergent to $d(x,A)$, i.e., $\varepsilon > 0$ and $x \in X$,*

$$\lim_{n\to\infty} \frac{1}{n}|\{k \le n : |\rho(x,A_k) - \rho(x,A)| \ge \varepsilon\}| = 0.$$

In this case, we write $st - \lim_W A_k = A$ or $A_k \to A(WS)$. We denote the set of all Wijsman statistically convergent sequences by WS.

Definition 8. *([19]) Let (X,ρ) be a metric space and $\mathcal{J} \subseteq 2^{\mathbb{N}}$ be a proper ideal in $\mathbb{N}$. For any non-empty closed subsets, $A, A_k \subset X$, we say that the sequence $\{A_k\}$ is Wijsman $\mathcal{J}-$convergent to A, if for each $\varepsilon > 0$, and $x \in X$,, the set is*

$$A\,(x,\varepsilon) = \{k \in \mathbb{N} : |\rho\,(x,A_k) - \rho\,(x,A)| \ge \varepsilon\} \in \mathcal{J}.$$

In this case, we write $\mathcal{I}_W - \lim A_k = A$ or $A_k \to A(\mathcal{I}_W)$, and we denote the set of all Wijsman $\mathcal{J}-$convergent sequences by $\mathcal{J}_W$.

Example 2. *Let $X = \mathbb{R}^2$ and $\{A_k\}$ be a sequence as follows:*

$$A_k = \begin{cases} \{(x,y) \in \mathbb{R}^2 : x^2 + y^2 - 2ky = 0\} & \text{if, } k \ne n^2 \\ \{(x,y) \in \mathbb{R}^2 : y = -1\} & \text{if, } k = n^2 \end{cases}$$

and

$$A = \left\{(x,y) \in \mathbb{R}^2 : y = 0\right\}.$$

The sequence $\{A_k\}$ is not Wijsman convergent to the set A. Howeverm if we choose the ideal $\mathcal{J} = \mathcal{J}_d$, then $\{A_k\}$ is Wijsman $\mathcal{J}-$convergent to set A,, where $\mathcal{J}_d = \{T \subseteq \mathbb{N} : d(T) = 0\}$, and where d is the natural density.

Definition 9. *([19]) In any metric space, let $\mathcal{J} \subseteq 2^{\mathbb{N}}$ be a non-trivial ideal and $A, A_k \subset X$. The sequence $\{A_k\}$ is said to be Wijsman $\mathcal{J}-$statistically convergent to A or $S\,(\mathcal{J}_W)$-convergent to A if*

$$\left\{n \in \mathbb{N} : \frac{1}{n}|\{k \le n : |\rho(x,A_k) - \rho(x,A)| \ge \varepsilon\}| \ge \delta\right\} \in \mathcal{J}$$

for each $\varepsilon > 0$ and each $x \in X$ and $\delta > 0$, and we write $A_k \to A\,(S\,(\mathcal{J}_W))$. The class of all Wijsman $\mathcal{J}-$ statistically convergent sequences is denoted by $S\,(\mathcal{J}_W)$.

Recently, Hazarika and Esi [20] and Savas [21] obtained some results about asymptotically $\mathcal{J}-$statistically equivalent set sequences.

1.4. The Number α

In recent years, many concepts that are considered essential in this area has been reworked using the alpha number. In references [22,23], by using the natural density of order α, the statistical convergence of order α $(0 < \alpha < 1)$ was introduced. The new definition is not exactly parallel to that of statistical convergence. Some other applications of this concept are the $\lambda-$statistical convergence of order α by Çolak and Bektaş [24], the lacunary statistical convergence of order α by Şengül and Et [25], the weighted statistical convergence of order α and its applications by Ghosal [26], and the almost statistical convergence of order α by Et, Altın and Çolak [27]. $\mathcal{J}-$statistical convergence and $\mathcal{J}-$lacunary statistical convergence of order α were introduced by Das and Savaş in 2014 [28]. In all of these studies, n was replaced by n^{α} in the denominator in the definition of natural density, and a different direction was given.

In 2017, Savas [21] gave a new definition about Wijsman asymptotically $\mathcal{J}-$statistical equivalence of order α $(0 < \alpha \leq 1)$ as follows:

Definition 10. *([21]) In any metric space, let $A_k, B_k \subseteq X$ be any non-empty closed subsets such that $d(x, A_k) > 0$ and $d(x, B_k) > 0$ for all $x \in X$. We say that the sequences $\{A_k\}$ and $\{B_k\}$ are Wijsman asymptotically $\mathcal{J}-$ statistically equivalent of order α $(0 < \alpha \leq 1)$ to multiple L if for each $\varepsilon > 0$, $\delta > 0$, and $x \in X$,*

$$\left\{ n \in \mathbb{N} : \frac{1}{n^\alpha} \left| \left\{ k \leq n : \left| \frac{\rho(x, A_k)}{\rho(x, B_k)} - L \right| \geq \varepsilon \right\} \right| \geq \delta \right\} \in \mathcal{J}.$$

In this case, we write $\{A_k\} \overset{[S^L(\mathcal{J}_W)]^\alpha}{\sim} \{B_k\}$. It is obvious that $[S^L(\mathcal{J}_W)]^\alpha$ denotes the the set of all sequences such that $\{A_k\} \overset{[S^L(\mathcal{J}_W)]^\alpha}{\sim} \{B_k\}$.

1.5. Present Study

It should be mentioned that the generalization of the concept of $W-$asymptotically $\mathcal{J}-$statistical equivalence of order α for $\lambda = (\lambda_n)$ sequences has not been studied until now. So, this brings to mind the question of how our new results will be if we use λ and p sequences. This makes the study interesting. In this study, we searched for the answer to this question. We generalized $W-$asymptotically $\mathcal{J}_\lambda-$statistical equivalence of order α and compared the properties of this new concept with the other type of convergences without λ.

2. Main Results

Following this information, we now consider our main definitions and results. Throughout the paper, (X, ρ) is a metric space, $\mathcal{J} \subseteq 2^\mathbb{N}$ is an admissible ideal, $(\lambda_n) \in \Lambda$ and λ_n^α is the α^{th} power of $(\lambda_n)^\alpha$ of λ_n, that is $\lambda^\alpha = (\lambda_n)^\alpha = (\lambda_1^\alpha, \lambda_2^\alpha, ..., \lambda_n^\alpha, ...)$, and $p = (p_k)$ is a positive real number sequence. We use the W $(Wijsman)$ symbol since our expressions are defined for set sequences.

Definition 11. *Let $A_k, B_k \subseteq X$ be non-empty closed subsets such that $d(x, A_k) > 0$ and $d(x, B_k) > 0$ for all $x \in X$. Then, the sequences $\{A_k\}$ and $\{B_k\}$ are $W-$strongly Cesáro asymptotically $\mathcal{J}-$statistically equivalent of order α $(0 < \alpha \leq 1)$ to multiple L if for each $\delta > 0$ and $x \in X$,*

$$\left\{ n \in \mathbb{N} : \frac{1}{n^\alpha} \sum_{k=1}^{n} \left| \frac{\rho(x, A_k)}{\rho(x, B_k)} - L \right|^{p_k} \geq \delta \right\} \in \mathcal{J}.$$

For this situation, we write $\{A_k\} \overset{[\sigma^{L(p)}(\mathcal{J}_W)]^\alpha}{\sim} \{B_k\}$, and $\left[\sigma^{L(p)}(\mathcal{J}_W)\right]^\alpha$ denotes the set of all sequences $\{A_k\}$ and $\{B_k\}$ such that $\{A_k\} \overset{[\sigma^{L(p)}(\mathcal{J}_W)]^\alpha}{\sim} \{B_k\}$.

Now let us give our definitions with the λ sequence.

Definition 12. *Let $A_k, B_k \subseteq X$ be non-empty closed subsets such that $d(x, A_k) > 0$ and $d(x, B_k) > 0$ for all $x \in X$. Then, the sequences $\{A_k\}$ and $\{B_k\}$ are $W-$ asymptotically $\mathcal{J}_\lambda-$statistical equivalent of order α $(0 < \alpha \leq 1)$ to multiple L and denoted by $\{A_k\} \overset{[S_\lambda^L(\mathcal{J}_W)]^\alpha}{\sim} \{B_k\}$ if for each $\varepsilon > 0$, $\delta > 0$, and $x \in X$,*

$$\left\{ n \in \mathbb{N} : \frac{1}{\lambda_n^\alpha} \left| \left\{ k \in I_n : \left| \frac{\rho(x, A_k)}{\rho(x, B_k)} - L \right| \geq \varepsilon \right\} \right| \geq \delta \right\} \in \mathcal{J}.$$

We denote the set of all sequences of $\{A_k\}$ and $\{B_k\}$ such that $\{A_k\} \overset{[S_\lambda^L(\mathcal{J}_W)]^\alpha}{\sim} \{B_k\}$ by $[S_\lambda^L(\mathcal{J}_W)]^\alpha$.

Definition 13. *Let $A_k, B_k \subseteq X$ be non-empty closed subsets such that $d(x, A_k) > 0$ and $d(x, B_k) > 0$ for all $x \in X$. Then, the sequences $\{A_k\}$ and $\{B_k\}$ are $W-$ strongly asymptotically $\mathcal{J}_\lambda-$statistically equivalent of order α $(0 < \alpha \leq 1)$ to multiple L if for each $\delta > 0$ and each $x \in X$,*

$$\left\{ n \in \mathbb{N} : \frac{1}{\lambda_n^\alpha} \sum_{k \in I_n} \left| \frac{\rho(x, A_k)}{\rho(x, B_k)} - L \right|^{p_k} \geq \delta \right\} \in \mathcal{J}.$$

We denote the set of this kind of sequence by $\left[V_\lambda^{L(p)}(\mathcal{J}_W) \right]^\alpha$.

The next theorem examines the relation between Savas' definition and our second definition.

Theorem 1. (i) *If* $\liminf\limits_{n\to\infty} \frac{\lambda_n^\alpha}{n^\alpha} > 0$ *then* $\left[S^L(\mathcal{J}_W) \right]^\alpha \subset \left[S_\lambda^L(\mathcal{J}_W) \right]^\alpha$.
(ii) *If* $\liminf\limits_{n\to\infty} \frac{\lambda_n^\alpha}{n^\alpha} = 1$ *then* $\left[S_\lambda^L(\mathcal{J}_W) \right]^\alpha \subset \left[S^L(\mathcal{J}_W) \right]^\alpha$.

Proof. (i) Assume that $\frac{\lambda_n^\alpha}{n^\alpha} > 0$ and $\{A_k\} \overset{S^L(\mathcal{J}_W)}{\sim} \{B_k\}$. Then, there exists a $\eta > 0$ such that $\frac{\lambda_n^\alpha}{n^\alpha} \geq \eta$ for sufficiently large n. For every $\varepsilon > 0$, we have,

$$\frac{1}{n^\alpha} \left\{ k \leq n : \left| \frac{\rho(x,A_k)}{\rho(x,B_k)} - L \right| \geq \varepsilon \right\} \quad \supseteq \quad \frac{1}{n^\alpha} \left\{ k \in I_n : \left| \frac{\rho(x,A_k)}{\rho(x,B_k)} - L \right| \geq \varepsilon \right\}.$$

If we think about the number of elements of the sets that provide this relation,

$$\begin{aligned} \frac{1}{n^\alpha} \left| \left\{ k \leq n : \left| \frac{\rho(x,A_k)}{\rho(x,B_k)} - L \right| \geq \varepsilon \right\} \right| &\geq \frac{1}{n^\alpha} \left| \left\{ k \in I_n : \left| \frac{\rho(x,A_k)}{\rho(x,B_k)} - L \right| \geq \varepsilon \right\} \right| \\ &\geq \frac{\lambda_n^\alpha}{n^\alpha} \frac{1}{\lambda_n^\alpha} \left| \left\{ k \in I_n : \left| \frac{\rho(x,A_k)}{\rho(x,B_k)} - L \right| \geq \varepsilon \right\} \right| \\ &\geq \eta \frac{1}{\lambda_n^\alpha} \left| \left\{ k \in I_n : \left| \frac{\rho(x,A_k)}{\rho(x,B_k)} - L \right| \geq \varepsilon \right\} \right|, \end{aligned}$$

we get, for any $\delta > 0$,

$$\begin{aligned} &\left\{ n \in \mathbb{N} : \frac{1}{\lambda_n^\alpha} \left| \left\{ k \in I_n : \left| \frac{\rho(x,A_k)}{\rho(x,B_k)} - L \right| \geq \varepsilon \right\} \right| \geq \delta \right\} \\ &\subseteq \left\{ n \in \mathbb{N} : \frac{1}{n^\alpha} \left| \left\{ k \leq n : \left| \frac{\rho(x,A_k)}{\rho(x,B_k)} - L \right| \geq \varepsilon \right\} \right| \geq \delta\eta \right\} \in \mathcal{J}. \end{aligned}$$

Then, we have the proof.
(ii) Let $\delta > 0$. Since $\liminf\limits_{n\to\infty} \frac{\lambda_n^\alpha}{n^\alpha} = 1$, we have $m \in N$ such that $\left| \frac{\lambda_n^\alpha}{n^\alpha} - 1 \right| < \frac{\delta}{2}$ for all $n \geq m$. For $\varepsilon > 0$,

$$\begin{aligned} \frac{1}{n^\alpha} \left| \left\{ k \leq n : \left| \frac{\rho(x, A_k)}{\rho(x, B_k)} - L \right| \geq \varepsilon \right\} \right| &\leq \frac{n^\alpha - \lambda_n^\alpha}{n^\alpha} + \frac{1}{\lambda_n^\alpha} \left| \left\{ k \in I_n : \left| \frac{\rho(x, A_k)}{\rho(x, B_k)} - L \right| \geq \varepsilon \right\} \right| \\ &\leq 1 - (1 - \frac{1}{\delta}) + \frac{1}{\lambda_n^\alpha} \left| \left\{ k \in I_n : \left| \frac{\rho(x, A_k)}{\rho(x, B_k)} - L \right| \geq \varepsilon \right\} \right| \end{aligned}$$

for all $n \geq m$. Hence,

$$\begin{aligned} &\left\{ n \in \mathbb{N} : \frac{1}{n^\alpha} \left| \left\{ k \leq n : \left| \frac{\rho(x,A_k)}{\rho(x,B_k)} - L \right| \geq \varepsilon \right\} \right| \geq \delta \right\} \\ &\subseteq \left\{ n \in \mathbb{N} : \frac{1}{\lambda_n^\alpha} \left| \left\{ k \in I_n : \left| \frac{\rho(x,A_k)}{\rho(x,B_k)} - L \right| \geq \varepsilon \right\} \right| \geq \frac{\delta}{2} \right\} \cup \{1, 2, ...m\}. \end{aligned}$$

We know that the right side belongs to the ideal because of the theorem expression. So we have the proof. □

Now let us investigate how the $p = (p_k)$ sequence affects the previous definitions. Initially, we use the constant $p = (p) = (p, p, p, p, ...)$ sequence of positive real numbers in the following two theorems.

Theorem 2. *(i) If* $\{A_k\} \overset{\left[V_\lambda^{Lp}(\mathcal{J}_W)\right]^\alpha}{\sim} \{B_k\}$ *then* $\{A_k\} \overset{\left[S_\lambda^{L}(\mathcal{J}_W)\right]^\alpha}{\sim} \{B_k\}$.

ii) If $\{A_k\}, \{B_k\} \in L_\infty$ and $\{A_k\} \overset{\left[S_\lambda^{L}(\mathcal{J}_W)\right]^\alpha}{\sim} \{B_k\}$ then $\{A_k\} \overset{\left[V_\lambda^{Lp}(\mathcal{J}_W)\right]^\alpha}{\sim} \{B_k\}$.

Proof. (i) Let $\{A_k\} \overset{\left[V_\lambda^{Lp}(\mathcal{J}_W)\right]^\alpha}{\sim} \{B_k\}$ and $\varepsilon > 0$. For each $x \in X$,

$$\begin{aligned}\sum_{k \in I_n} \left|\frac{\rho(x,A_k)}{\rho(x,B_k)} - L\right|^p &\geq \sum_{\substack{k \in I_n \\ \left|\frac{\rho(x,A_k)}{\rho(x,B_k)} - L\right| \geq \varepsilon}} \left|\frac{\rho(x,A_k)}{\rho(x,B_k)} - L\right|^p \\ &\geq \varepsilon^p \left|\left\{k \in I_n : \left|\frac{\rho(x,A_k)}{\rho(x,B_k)} - L\right| \geq \varepsilon\right\}\right|\end{aligned}$$

and so,

$$\frac{1}{\varepsilon^p}\frac{1}{\lambda_n^\alpha}\sum_{k \in I_n}\left|\frac{\rho(x,A_k)}{\rho(x,B_k)} - L\right| \geq \frac{1}{\lambda_n^\alpha}\left|\left\{k \in I_n : \left|\frac{\rho(x,A_k)}{\rho(x,B_k)} - L\right| \geq \varepsilon\right\}\right|.$$

Then, for any $\delta > 0$ we have,

$$\begin{aligned}&\left\{n \in \mathbb{N} : \frac{1}{\lambda_n^\alpha}\left|\left\{k \in I_n : \left|\frac{\rho(x,A_k)}{\rho(x,B_k)} - L\right| \geq \varepsilon\right\}\right| \geq \delta\right\} \\ &\subseteq \left\{n \in \mathbb{N} : \frac{1}{\lambda_n^\alpha}\sum_{k \in I_n}\left|\frac{\rho(x,A_k)}{\rho(x,B_k)} - L\right|^p \geq \varepsilon^p\delta\right\}.\end{aligned}$$

Therefore, $\{A_k\} \overset{\left[S_\lambda^{L}(\mathcal{J}_W)\right]^\alpha}{\sim} \{B_k\}$.

(ii) Assume that $\{A_k\}, \{B_k\} \in L_\infty$ and $\{A_k\} \overset{\left[S_\lambda^{L}(\mathcal{J}_W)\right]^\alpha}{\sim} \{B_k\}$. There is an M such that $\left|\frac{\rho(x,A_k)}{\rho(x,B_k)} - L\right| \leq M$ for each $x \in X$ and all k. For each $\varepsilon > 0$,

$$\begin{aligned}\frac{1}{\lambda_n^\alpha}\sum_{k \in I_n}\left|\frac{\rho(x,A_k)}{\rho(x,B_k)} - L\right|^p &= \frac{1}{\lambda_n^\alpha}\sum_{\substack{k \in I_n \\ \left|\frac{\rho(x,A_k)}{\rho(x,B_k)} - L\right| \geq \varepsilon}}\left|\frac{\rho(x,A_k)}{\rho(x,B_k)} - L\right|^p \\ &+ \frac{1}{\lambda_n^\alpha}\sum_{\substack{k \in I_n \\ \left|\frac{\rho(x,A_k)}{\rho(x,B_k)} - L\right| < \varepsilon}}\left|\frac{\rho(x,A_k)}{\rho(x,B_k)} - L\right|^p \\ &\leq \frac{1}{\lambda_n^\alpha}M^p\left|\left\{k \in I_n : \left|\frac{\rho(x,A_k)}{\rho(x,B_k)} - L\right| \geq \varepsilon\right\}\right| \\ &+ \frac{1}{\lambda_n^\alpha}\varepsilon^p\left|\left\{k \in I_n : \left|\frac{\rho(x,A_k)}{\rho(x,B_k)} - L\right| < \varepsilon\right\}\right| \\ &\leq \frac{M^p}{\lambda_n^\alpha}\left|\left\{k \in I_n : \left|\frac{\rho(x,A_k)}{\rho(x,B_k)} - L\right| \geq \varepsilon\right\}\right| + \varepsilon^p\end{aligned}$$

and then for any $\delta > 0$,

$$\left\{ n \in \mathbb{N} : \frac{1}{\lambda_n^\alpha} \sum_{k \in I_n} \left| \frac{\rho(x, A_k)}{\rho(x, B_k)} - L \right|^p \geq \varepsilon \right\}$$

$$\subseteq \left\{ n \in \mathbb{N} : \frac{1}{\lambda_n^\alpha} \left| \left\{ k \in I_n : \left| \frac{\rho(x, A_k)}{\rho(x, B_k)} - L \right| \geq \varepsilon \right\} \right| \geq \frac{\varepsilon^p}{M^p} \right\} \in \mathcal{J}.$$

□

Now let us examine the above theorems for a non-constant $p = (p_k)$ sequence of positive real numbers.

Theorem 3. *(i) Let $p = (p_k)$ be a positive real number sequence, $\inf p_k = h$ and $\sup_k p = H$. $\{A_k\} \overset{\left[V_\lambda^{L(p)}(\mathcal{J}_W)\right]^\alpha}{\sim} \{B_k\}$ implies $\{A_k\} \overset{\left[S_\lambda^{L}(\mathcal{J}_W)\right]^\alpha}{\sim} \{B_k\}$.*

(ii) Let $\{A_k\}$ and $\{B_k\}$ be bounded sequences, $\inf p_k = h$, $\sup_k p = H$ and $\varepsilon > 0$. Then, $\{A_k\} \overset{\left[S_\lambda^{L}(\mathcal{J}_W)\right]^\alpha}{\sim} \{B_k\}$ implies $\{A_k\} \overset{\left[V_\lambda^{L(p)}(\mathcal{J}_W)\right]^\alpha}{\sim} \{B_k\}$.

Proof. (i) Assume that $\{A_k\} \overset{\left[V_\lambda^{L(p)}(\mathcal{J}_W)\right]^\alpha}{\sim} \{B_k\}$ and $\varepsilon > 0$. Then, we can write

$$\begin{aligned}
\frac{1}{\lambda_n^\alpha} \sum_{k \in I_n} \left| \frac{\rho(x, A_k)}{\rho(x, B_k)} - L \right|^{p_k} &= \frac{1}{\lambda_n^\alpha} \sum_{\substack{k \in I_n \\ \left| \frac{\rho(x, A_k)}{\rho(x, B_k)} - L \right| \geq \varepsilon}} \left| \frac{\rho(x, A_k)}{\rho(x, B_k)} - L \right|^{p_k} + \frac{1}{\lambda_n^\alpha} \sum_{\substack{k \in I_n \\ \left| \frac{\rho(x, A_k)}{\rho(x, B_k)} - L \right| < \varepsilon}} \left| \frac{\rho(x, A_k)}{\rho(x, B_k)} - L \right|^{p_k} \\
&\geq \frac{1}{\lambda_n^\alpha} \sum_{\substack{k \in I_n \\ \left| \frac{\rho(x, A_k)}{\rho(x, B_k)} - L \right| \geq \varepsilon}} \left| \frac{\rho(x, A_k)}{\rho(x, B_k)} - L \right|^{p_k} \\
&\geq \frac{1}{\lambda_n^\alpha} \sum_{\substack{k \in I_n \\ \left| \frac{\rho(x, A_k)}{\rho(x, B_k)} - L \right| \geq \varepsilon}} (\varepsilon)^{p_k} \\
&\geq \frac{1}{\lambda_n^\alpha} \sum_{\substack{k \in I_n \\ \left| \frac{\rho(x, A_k)}{\rho(x, B_k)} - L \right| \geq \varepsilon}} \min\left\{ (\varepsilon)^h, (\varepsilon)^H \right\} \\
&\geq \frac{1}{\lambda_n^\alpha} \min\left\{ (\varepsilon)^h, (\varepsilon)^H \right\} \left\{ k \in I_n : \left| \frac{\rho(x, A_k)}{\rho(x, B_k)} - L \right| \geq \varepsilon \right\}
\end{aligned}$$

and so for $\delta > 0$, we have

$$\left\{ n \in \mathbb{N} : \left| \left\{ k \in I_n : \left| \frac{\rho(x, A_k)}{\rho(x, B_k)} - L \right| \geq \varepsilon \right\} \right| \geq \delta \right\}$$

$$\subseteq \left\{ n \in \mathbb{N} : \frac{1}{\lambda_n^\alpha} \sum_{k \in I_n} \left| \frac{\rho(x, A_k)}{\rho(x, B_k)} - L \right|^{p_k} \geq \delta \min\left\{ (\varepsilon)^h, (\varepsilon)^H \right\} \right\} \in \mathcal{J}.$$

(ii) From the theorem's statement there is an integer (M) such that $\left|\frac{\rho(x,A_k)}{\rho(x,B_k)} - L\right| \leq M$ for each $x \in X$ and all k. For each $\varepsilon > 0$,

$$
\begin{aligned}
\frac{1}{\lambda_n^{\alpha}} \sum_{k \in I_n} \left|\frac{\rho(x,A_k)}{\rho(x,B_k)} - L\right|^{p_k} &= \frac{1}{\lambda_n^{\alpha}} \sum_{\substack{k \in I_n \\ \left|\frac{\rho(x,A_k)}{\rho(x,B_k)} - L\right| \geq \varepsilon}} \left|\frac{\rho(x,A_k)}{\rho(x,B_k)} - L\right|^{p_k} \\
&+ \frac{1}{\lambda_n^{\alpha}} \sum_{\substack{k \in I_n \\ \left|\frac{\rho(x,A_k)}{\rho(x,B_k)} - L\right| < \varepsilon}} \left|\frac{\rho(x,A_k)}{\rho(x,B_k)} - L\right|^{p_k} \\
&\leq \frac{1}{\lambda_n^{\alpha}} \left|\left\{k \in I_n : \left|\frac{\rho(x,A_k)}{\rho(x,B_k)} - L\right| \geq \frac{\varepsilon}{2}\right\}\right| \max\left\{M^h, M^H\right\} \\
&+ \frac{1}{\lambda_n^{\alpha}} \left|\left\{k \in I_n : \left|\frac{\rho(x,A_k)}{\rho(x,B_k)} - L\right| < \frac{\varepsilon}{2}\right\}\right| \frac{\max(\varepsilon)^{p_k}}{2} \\
&\leq \max\left\{M^h, M^H\right\} \frac{1}{\lambda_n^{\alpha}} \left|\left\{k \in I_n : \left|\frac{\rho(x,A_k)}{\rho(x,B_k)} - L\right| \geq \frac{\varepsilon}{2}\right\}\right| + \frac{\max\left\{(\varepsilon)^h, (\varepsilon)^H\right\}}{2}
\end{aligned}
$$

and

$$
\left\{n \in \mathbb{N} : \frac{1}{\lambda_n^{\alpha}} \sum_{k \in I_n} \left|\frac{\rho(x,A_k)}{\rho(x,B_k)} - L\right|^{p_k} \geq \varepsilon\right\}
$$

$$
\subseteq \left\{n \in \mathbb{N} : \frac{1}{\lambda_n^{\alpha}} \left|\left\{k \in I_n : \left|\frac{\rho(x,A_k)}{\rho(x,B_k)} - L\right| \geq \frac{\varepsilon}{2}\right\}\right| \geq \frac{2\varepsilon - \max\left\{(\varepsilon)^h, (\varepsilon)^H\right\}}{2\max\left\{M^h, M^H\right\}}\right\} \in \mathcal{J}.
$$

□

Finally, in the last theorem we investigate the relationship between $\mathcal{W}-$ strongly asymptotically $\mathcal{J}_\lambda-$statistical equivalence of order α and $\mathcal{W}-$strongly Cesáro asymptotically $\mathcal{J}-$statistical equivalence of order α.

Theorem 4. *If* $\{A_k\} \overset{\left[V_\lambda^{L(p)}(\mathcal{J}_W)\right]^{\alpha}}{\sim} \{B_k\}$, *then* $\{A_k\} \overset{\left[\sigma^{L(p)}(\mathcal{J}_W)\right]^{\alpha}}{\sim} \{B_k\}$.

Proof. Now, assume that $\{A_k\} \overset{\left[V_\lambda^{L(p)}(\mathcal{J}_W)\right]^{\alpha}}{\sim} \{B_k\}$ and $\varepsilon > 0$.

$$
\begin{aligned}
\frac{1}{n^{\alpha}} \sum_{k=1}^{n} \left|\frac{\rho(x,A_k)}{\rho(x,B_k)} - L\right|^{p_k} &= \frac{1}{n^{\alpha}} \sum_{k \in I_n} \left|\frac{\rho(x,A_k)}{\rho(x,B_k)} - L\right|^{p_k} + \frac{1}{n^{\alpha}} \sum_{k=1}^{n-\lambda n} \left|\frac{\rho(x,A_k)}{\rho(x,B_k)} - L\right|^{p_k} \\
&\leq \frac{1}{\lambda_n^{\alpha}} \sum_{k \in I_n} \left|\frac{\rho(x,A_k)}{\rho(x,B_k)} - L\right|^{p_k} + \frac{1}{\lambda_n^{\alpha}} \sum_{k=1}^{n-\lambda n} \left|\frac{\rho(x,A_k)}{\rho(x,B_k)} - L\right|^{p_k} \\
&\leq \frac{2}{\lambda_n^{\alpha}} \sum_{k \in I_n} \left|\frac{\rho(x,A_k)}{\rho(x,B_k)} - L\right|^{p_k}.
\end{aligned}
$$

□

According to these operations,

$$
\left\{n \in \mathbb{N} : \frac{1}{n^{\alpha}} \sum_{k=1}^{n} \left|\frac{\rho(x,A_k)}{\rho(x,B_k)} - L\right|^{p_k} \geq \varepsilon\right\} \subseteq \left\{n \in \mathbb{N} : \frac{1}{\lambda_n^{\alpha}} \sum_{k \in I_n} \left|\frac{\rho(x,A_k)}{\rho(x,B_k)} - L\right|^{p_k} \geq \frac{\varepsilon}{2}\right\} \in \mathcal{J}.
$$

3. Conclusions and Future Developments

In our paper, we obtained some different results by defining the $W-$asymptotically $\mathcal{J}-$statistical equivalence of order α for $\lambda = (\lambda_n)$ sequences. Later on, we generalized our results by using a positive real number sequence $p = (p_k)$. Firstly, we compared the $W-$asymptotically $\mathcal{J}-$statistical equivalence of order α and the $W-$asymptotically $\mathcal{J}_\lambda-$statistical equivalence of order α for set sequences. These results are important to understand the role of λ. In other theorems, we investigated the relations between $W-$asymptotically $\mathcal{J}_\lambda-$statistical equivalence and $W-$strongly asymptotically $\mathcal{J}_\lambda-$statistically equivalence of order α according to whether p is constant or not. Then, we searched for the relation between $W-$strongly Cesáro asymptotically $\mathcal{J}-$statistically equivalent sequences of order α and $W-$ asymptotically $\mathcal{J}_\lambda-$statistical equivalent sequences of order α.

We know that the p sequence mentioned in this article is a sequence of positive integers. It is a matter of curiosity as to how the results will be obtained if the p sequence does not provide these conditions. On the other hand, it would be interesting to compare the results obtained using a different sequence to λ with the results in this article.

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