axioms

MDPI

Article

Some Identities for Euler and Bernoulli Polynomials and Their Zeros

Taekyun Kim [1] and Cheon Seoung Ryoo [2,*]

1 Department of Mathematics, Kwangwoon University, Seoul 139-701, Korea; tkkim@kw.ac.kr
2 Department of Mathematics, Hannam University, Daejeon 306-791, Korea
* Correspondence: ryoocs@hnu.kr; Tel.: +82-42-629-7525

Received: 30 June 2018; Accepted: 11 August 2018; Published: 14 August 2018

Abstract: In this paper, we study some special polynomials which are related to Euler and Bernoulli polynomials. In addition, we give some identities for these polynomials. Finally, we investigate the zeros of these polynomials by using the computer.

Keywords: Appell sequence; Appell numbers and polynomials; Bernoulli and Euler polynomials; cosine–Bernoulli and cosine–Euler polynomials; sine–Bernoulli and sine–Euler polynomials

MSC: 11B68; 11S40; 11S80

---

1. Introduction

Many mathematicians have studied in the area of the Bernoulli numbers and polynomials, Euler numbers and polynomials, Genocchi numbers and polynomials, and tangent numbers and polynomials. The class of Appell polynomial sequences is one of the important classes of polynomial sequences. The Appell polynomial sequences arise in numerous problems of applied mathematics, mathematical physics and several other mathematical branches (see [1–14]). The Appell polynomials can be defined by considering the following generating function:

$$\begin{aligned} A(t)e^{xt} &= A_0(x) + A_1(x)\frac{t}{1!} + A_2(x)\frac{t^2}{2!} + \cdots + A_n(x)\frac{t^n}{n!} + \cdots \\ &= \sum_{n=0}^{\infty} A_n(x)\frac{t^n}{n!}, \text{ (see [5,7,8])}, \end{aligned} \tag{1}$$

where

$$A(t) = A_0 + A_1\frac{t}{1!} + A_2\frac{t^2}{2!} + \cdots + A_n\frac{t^n}{n!} + \cdots, A_0 \neq 0.$$

Alternatively, the sequence $A_n(x)$ is Appell sequence for $(g(t), t)$ if and only if

$$\frac{1}{g(t)}e^{xt} = \sum_{n=0}^{\infty} A_n(x)\frac{t^n}{n!}, \text{ (see [5,7,8])},$$

where

$$g(t) = \sum_{n=0}^{\infty} g_n\frac{t^n}{n!}, g_0 \neq 0.$$

Differentiating generating Equation (1) with respect to x and equating coefficients of $\frac{t^n}{n!}$, we have

$$\frac{d}{dx}A_n(x) = nA_{n-1}(x), n = 0, 1, 2, 3, \cdots.$$

The typical examples of Appell polynomials are the Bernoulli and Euler polynomials (see [1–14]). It is well known that the Bernoulli polynomials are defined by the generating function to be

$$\frac{t}{e^t-1}e^{xt}=\sum_{n=0}^{\infty}B_n(x)\frac{t^n}{n!}. \tag{2}$$

When $x=0, B_n=B_n(0)$ are called the Bernoulli numbers. The Euler polynomials are given by the generating function to be

$$\frac{2}{e^t+1}e^{xt}=\sum_{n=0}^{\infty}E_n(x)\frac{t^n}{n!}. \tag{3}$$

When $x=0, E_n=E_n(0)$ are called the Euler numbers.

The Bernoulli polynomials $\mathbf{B}_n^{(r)}(x)$ of order r are defined by the following generating function

$$\left(\frac{t}{e^t-1}\right)^r e^{xt}=\sum_{n=0}^{\infty}\mathbf{B}_n^{(r)}(x)\frac{t^n}{n!}, \quad (|t|<2\pi). \tag{4}$$

The Frobenius–Euler polynomials of order r, denoted by $\mathbf{H}_n^{(r)}(u,x)$, are defined as

$$\left(\frac{1-u}{e^t-u}\right)^r e^{xt}=\sum_{n=0}^{\infty}\mathbf{H}_n^{(r)}(u,x)\frac{t^n}{n!}. \tag{5}$$

The values at $x=0$ are called Frobenius–Euler numbers of order r; when $r=1$, the polynomials or numbers are called ordinary Frobenius–Euler polynomials or numbers.

In this paper, we study some special polynomials which are related to Euler and Bernoulli polynomials. In addition, we give some identities for these polynomials. Finally, we investigate the zeros of these polynomials by using the computer.

2. Cosine–Bernoulli, Sine–Bernoulli, Cosine–Euler and Sine–Euler Polynomials

In this section, we define the cosine–Bernoulli, sine–Bernoulli, cosine–Euler and sine–Euler polynomials. Now, we consider the Euler polynomials that are given by the generating function to be

$$\frac{2}{e^t+1}e^{(x+iy)t}=\sum_{n=0}^{\infty}E_n(x+iy)\frac{t^n}{n!}. \tag{6}$$

On the other hand, we observe that

$$e^{(x+iy)t}=e^{xt}e^{iyt}=e^{xt}(\cos yt+i\sin yt). \tag{7}$$

From Equations (6) and (7), we have

$$\sum_{n=0}^{\infty}E_n(x+iy)\frac{t^n}{n!}=\frac{2}{e^t+1}e^{(x+iy)t}=\frac{2}{e^t+1}e^{xt}(\cos yt+i\sin yt), \tag{8}$$

and

$$\sum_{n=0}^{\infty}E_n(x-iy)\frac{t^n}{n!}=\frac{2}{e^t+1}e^{(x-iy)t}=\frac{2}{e^t+1}e^{xt}(\cos yt-i\sin yt). \tag{9}$$

Thus, by (8) and (9), we can derive

$$\frac{2}{e^t+1}e^{xt}\cos yt=\sum_{n=0}^{\infty}\left(\frac{E_n(x+iy)+E_n(x-iy)}{2}\right)\frac{t^n}{n!}, \tag{10}$$

and
$$\frac{2}{e^t+1}e^{xt}\sin yt=\sum_{n=0}^{\infty}\left(\frac{E_n(x+iy)-E_n(x-iy)}{2i}\right)\frac{t^n}{n!}. \tag{11}$$

It follows that we define the following cosine–Euler polynomials and sine–Euler polynomials.

Definition 1. *The cosine–Euler polynomials $E_n^{(C)}(x,y)$ and sine–Euler polynomials $E_n^{(S)}(x,y)$ are defined by means of the generating functions*
$$\sum_{n=0}^{\infty}E_n^{(C)}(x,y)\frac{t^n}{n!}=\frac{2}{e^t+1}e^{xt}\cos yt, \tag{12}$$
and
$$\sum_{n=0}^{\infty}E_n^{(S)}(x,y)\frac{t^n}{n!}=\frac{2}{e^t+1}e^{xt}\sin yt, \tag{13}$$
respectively.

Note that $E_n^{(C)}(x,0)=E_n(x), E_n^{(S)}(x,0)=0, (n\geq 0)$. The cosine–Euler and sine–Euler polynomials can be determined explicitly. A few of them are
$$\begin{aligned}
&E_0^{(C)}(x,y)=1,\quad E_1^{(C)}(x,y)=-\frac{1}{2}+x,\\
&E_2^{(C)}(x,y)=-x+x^2-y^2,\\
&E_3^{(C)}(x,y)=\frac{1}{4}-\frac{3x^2}{2}+x^3+\frac{3y^2}{2}-3xy^2,\\
&E_4^{(C)}(x,y)=x-2x^3+x^4+6xy^2-6x^2y^2+y^4,
\end{aligned}$$
and
$$\begin{aligned}
&E_0^{(S)}(x,y)=0,\quad E_1^{(S)}(x,y)=y,\\
&E_2^{(S)}(x,y)=-y+2xy,\\
&E_3^{(S)}(x,y)=-3xy+3x^2y-y^3,\\
&E_4^{(S)}(x,y)=y-6x^2y+4x^3y+2y^3-4xy^3.
\end{aligned}$$
By (10)–(13), we have
$$\begin{aligned}
E_n^{(C)}(x,y)&=\frac{E_n(x+iy)+E_n(x-iy)}{2},\\
E_n^{(S)}(x,y)&=\frac{E_n(x+iy)-E_n(x-iy)}{2i}.
\end{aligned}$$
Clearly, we can get the following explicit representations of $E_n(x+iy)$
$$\begin{aligned}
E_n(x+iy)&=\sum_{k=0}^{n}\binom{n}{k}(x+iy)^{n-k}E_k,\\
E_n(x+iy)&=\sum_{k=0}^{n}\binom{n}{k}(iy)^{n-k}E_k(x).
\end{aligned}$$

Let
$$e^{xt}\cos yt=\sum_{k=0}^{\infty}C_k(x,y)\frac{t^k}{k!},\qquad e^{xt}\sin yt=\sum_{k=0}^{\infty}S_k(x,y)\frac{t^k}{k!}. \tag{14}$$

Then, by Taylor expansions of $e^{xt}\cos yt$ and $e^{xt}\sin yt$, we get

$$e^{xt}\cos yt = \sum_{k=0}^{\infty}\left(\sum_{m=0}^{[\frac{k}{2}]}\binom{k}{2m}(-1)^m x^{k-2m}y^{2m}\right)\frac{t^k}{k!} \tag{15}$$

and

$$e^{xt}\sin yt = \sum_{k=0}^{\infty}\left(\sum_{m=0}^{[\frac{k-1}{2}]}\binom{k}{2m+1}(-1)^m x^{k-2m-1}y^{2m+1}\right)\frac{t^k}{k!}, \tag{16}$$

where [ ] denotes taking the integer part. By (14)–(16), we get

$$C_k(x,y) = \sum_{m=0}^{[\frac{k}{2}]}\binom{k}{2m}(-1)^m x^{k-2m}y^{2m},$$

and

$$S_k(x,y) = \sum_{m=0}^{[\frac{k-1}{2}]}\binom{k}{2m+1}(-1)^m x^{k-2m-1}y^{2m+1}, (k \geq 0).$$

The two polynomials can be determined explicitly. A few of them are

$$\begin{aligned}
&C_0(x,y) = 1, \quad C_1(x,y) = x, \quad C_2(x,y) = x^2 - y^2, \\
&C_3(x,y) = x^3 - 3xy^2, \quad C_4(x,y) = x^4 - 6x^2y^2 + y^4, \\
&C_5(x,y) = x^5 - 10x^3y^2 + 5xy^4, \quad C_6(x,y) = x^6 - 15x^4y^2 + 15x^2y^4 - y^6,
\end{aligned}$$

and

$$\begin{aligned}
&S_0(x,y) = 0, \quad S_1(x,y) = y, \quad S_2(x,y) = 2xy, \\
&S_3(x,y) = 3x^2y - y^3, \quad S_4(x,y) = 4x^3y - 4xy^3, \\
&S_5(x,y) = 5x^4y - 10x^2y^3 + y^5, \quad S_6(x,y) = 6x^5y - 20x^3y^3 + 6xy^5.
\end{aligned}$$

Now, we observe that

$$\begin{aligned}
\frac{2}{e^t+1}e^{xt}\cos yt &= \left(\sum_{l=0}^{\infty}E_l\frac{t^l}{l!}\right)\left(\sum_{m=0}^{\infty}C_m(x,y)\frac{t^m}{m!}\right) \\
&= \sum_{n=0}^{\infty}\left(\sum_{l=0}^{n}\binom{n}{l}E_lC_{n-l}(x,y)\right)\frac{t^n}{n!}.
\end{aligned} \tag{17}$$

Therefore, we obtain the following theorem:

Theorem 1. *For $n \geq 0$, we have*

$$E_n^{(C)}(x,y) = \sum_{l=0}^{n}\binom{n}{l}E_lC_{n-l}(x,y)$$

and

$$E_n^{(S)}(x,y) = \sum_{l=0}^{n}\binom{n}{l}E_lS_{n-l}(x,y).$$

From (12), we have

$$2e^{xt}\cos yt = \left(\sum_{l=0}^{\infty} E_l^{(C)}(x,y)\frac{t^l}{l!}\right)\left(e^t+1\right)$$
$$= \sum_{n=0}^{\infty}\left(\sum_{l=0}^{n}\binom{n}{l}E_l^{(C)}(x,y)+E_n^{(C)}(x,y)\right)\frac{t^n}{n!}. \tag{18}$$

By (14) and (18), we get

$$C_n(x,y) = \frac{1}{2}\left(\sum_{l=0}^{n}\binom{n}{l}E_l^{(C)}(x,y)+E_n^{(C)}(x,y)\right). \tag{19}$$

Therefore, we obtain the following theorem:

Theorem 2. *For $n \geq 0$, we have*

$$C_n(x,y) = \frac{1}{2}\left(\sum_{l=0}^{n}\binom{n}{l}E_l^{(C)}(x,y)+E_n^{(C)}(x,y)\right),$$

and

$$S_n(x,y) = \frac{1}{2}\left(\sum_{l=0}^{n}\binom{n}{l}E_l^{(S)}(x,y)+E_n^{(S)}(x,y)\right).$$

From (12), we note that

$$\sum_{n=0}^{\infty} E_n^{(C)}(1-x,y)\frac{t^n}{n!} = \frac{2}{e^t+1}e^{(1-x)t}\cos yt$$
$$= \frac{2}{e^{-t}+1}e^{-xt}\cos(-yt)$$
$$= \left(\sum_{l=0}^{\infty}(-1)^l E_l\frac{t^l}{l!}\right)\left(\sum_{m=0}^{\infty}(-1)^m C_{m,}(x,y)\frac{t^m}{m!}\right)$$
$$= \sum_{n=0}^{\infty}\left(\sum_{l=0}^{n}\binom{n}{l}E_l C_{n-l}(x,y)\right)\frac{(-1)^n}{n!}t^n. \tag{20}$$

Therefore, we obtain the following theorem:

Theorem 3. *For $n \geq 0$, we have*

$$E_n^{(C)}(1-x,y) = (-1)^n\sum_{l=0}^{n}\binom{n}{l}E_l C_{n-l}(x,y)$$
$$= (-1)^n E_n^{(C)}(x,y),$$

and

$$E_n^{(S)}(1-x,y) = (-1)^{n+1}E_n^{(S)}(x,y)$$
$$= (-1)^{n+1}\sum_{l=0}^{n}\binom{n}{l}E_l S_{n-l}(x,y).$$

Now, we observe that

$$\begin{aligned}\sum_{n=0}^{\infty} E_n^{(C)}(x+1,y)\frac{t^n}{n!} &= \frac{2}{e^t+1}e^{(x+1)t}\cos yt \\ &= \frac{2}{e^t+1}e^{xt}(e^t-1+1)\cos yt \\ &= 2e^{xt}\cos yt - \frac{2}{e^t+1}e^{xt}\cos yt \\ &= \sum_{n=0}^{\infty}\left(2C_n(x,y) - E_n^{(C)}(x,y)\right)\frac{t^n}{n!}.\end{aligned} \tag{21}$$

By comparing the coefficients on the both sides, we get

$$E_n^{(C)}(x+1,q) + E_n^{(C)}(x,y) = 2C_n(x,y),\ (n \geq 0). \tag{22}$$

Therefore, we obtain the following theorem:

Theorem 4. *For $n \geq 0$, we have*

$$E_n^{(C)}(x+1,y) + E_n^{(C)}(x,y) = 2C_n(x,y),$$

and

$$E_n^{(S)}(x+1,y) + E_n^{(S)}(x,y) = 2S_n(x,y).$$

From (14) and (15), we have

$$\sum_{k=0}^{\infty} C_k(0,y)\frac{t^k}{k!} = \sum_{m=0}^{\infty}(-1)^m y^{2m}\frac{t^{2m}}{(2m)!}. \tag{23}$$

Therefore, by Theorem 4 and (23), we obtain the following corollary:

Corollary 1. *For $n \geq 0$, we have*

$$E_{2n}^{(C)}(1,y) + E_{2n}^{(C)}(0,y) = 2(-1)^n y^{2n},$$

and

$$E_{2n+1}^{(S)}(1,y) + E_{2n+1}^{(S)}(0,y) = 2(-1)^n y^{2n+1}.$$

By (12), we get

$$\begin{aligned}\sum_{n=0}^{\infty} E_n^{(C)}(x+r,y)\frac{t^n}{n!} &= \left(\frac{2e^{xt}}{e^t+1}\cos yt\right)e^{rt} \\ &= \left(\sum_{l=0}^{\infty} E_l^{(C)}(x,y)\frac{t^l}{l!}\right)\left(\sum_{k=0}^{\infty} r^k\frac{t^k}{k!}\right) \\ &= \sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}\binom{n}{k}E_k^{(C)}(x,y)r^{n-k}\right)\frac{t^n}{n!}.\end{aligned} \tag{24}$$

Therefore, by comparing the coefficients on the both sides, we obtain the following theorem:

Theorem 5. *For $n \geq 0, r \in \mathbb{N}$, we have*

$$E_n^{(C)}(x+r,y) = \sum_{k=0}^{n} \binom{n}{k} E_k^{(C)}(x,y) r^{n-k},$$

and

$$E_n^{(S)}(x+r,y) = \sum_{k=0}^{n} \binom{n}{k} E_k^{(S)}(x,y) r^{n-k}.$$

Taking $r = 1$ in Theorem 5, we obtain the following corollary:

Corollary 2. *For $n \geq 0$, we have*

$$2C_n(x,y) = E_n^{(C)}(x,y) + \sum_{k=0}^{n} \binom{n}{k} E_k^{(C)}(x,y),$$

and

$$2S_n(x,y) = E_n^{(S)}(x,y) + \sum_{k=0}^{n} \binom{n}{k} E_k^{(S)}(x,y).$$

From Corollary 2, we note that

$$E_n^{(C)}(0,y) + \sum_{k=0}^{n} \binom{n}{k} E_k^{(C)}(0,y) = \begin{cases} 0, & \text{if } n = 2m+1, \\ 2(-1)^m y^{2m}, & \text{if } n = 2m, \end{cases} \tag{25}$$

and

$$E_n^{(S)}(0,y) + \sum_{k=0}^{n} \binom{n}{k} E_k^{(S)}(0,y) = \begin{cases} 2(-1)^m y^{2m+1}, & \text{if } n = 2m+1, \\ 0, & \text{if } n = 2m. \end{cases} \tag{26}$$

By (12), we get

$$\begin{aligned} \sum_{n=1}^{\infty} \frac{\partial}{\partial x} E_n^{(C)}(x,y) \frac{t^n}{n!} &= \frac{\partial}{\partial x} \left(\frac{2}{e^t+1} e^{xt} \cos yt \right) \\ &= \frac{2}{e^t+1} t e^{xt} \cos yt \\ &= \sum_{n=1}^{\infty} \left(n E_{n-1}^{(C)}(x,y) \right) \frac{t^n}{n!}. \end{aligned} \tag{27}$$

Comparing the coefficients on the both sides of (27), we have

$$\frac{\partial}{\partial x} E_n^{(C)}(x,y) = n E_{n-1}^{(C)}(x,y).$$

Similarly, for $n \geq 1$, we have

$$\begin{aligned} \frac{\partial}{\partial x} E_n^{(S)}(x,y) &= n E_{n-1}^{(S)}(x,y), \\ \frac{\partial}{\partial y} E_n^{(C)}(x,y) &= -n E_{n-1}^{(S)}(x,y), \\ \frac{\partial}{\partial y} E_n^{(S)}(x,y) &= n E_{n-1}^{(C)}(x,y). \end{aligned}$$

Now, we consider the Bernoulli polynomials that are given by the generating function to be

$$\frac{t}{e^t-1}e^{(x+iy)t} = \sum_{n=0}^{\infty} B_n(x+iy)\frac{t^n}{n!}.$$

We also have

$$\sum_{n=0}^{\infty} B_n(x+iy)\frac{t^n}{n!} = \frac{t}{e^t-1}e^{(x+iy)t} = \frac{t}{e^t-1}e^{xt}(\cos yt + i\sin yt), \tag{28}$$

and

$$\sum_{n=0}^{\infty} B_n(x-iy)\frac{t^n}{n!} = \frac{t}{e^t-1}e^{(x-iy)t} = \frac{t}{e^t-1}e^{xt}(\cos yt - i\sin yt). \tag{29}$$

Thus, by (28) and (29), we can derive

$$\frac{t}{e^t-1}e^{xt}\cos yt = \sum_{n=0}^{\infty}\left(\frac{B_n(x+iy)+B_n(x-iy)}{2}\right)\frac{t^n}{n!}, \tag{30}$$

and

$$\frac{t}{e^t-1}e^{xt}\sin yt = \sum_{n=0}^{\infty}\left(\frac{B_n(x+iy)-B_n(x-iy)}{2i}\right)\frac{t^n}{n!}. \tag{31}$$

It follows that we define the following cosine–Bernoulli and sine–Bernoulli polynomials.

Definition 2. *The cosine–Bernoulli polynomials* $B_n^{(C)}(x,y)$ *and sine–Bernoulli polynomials* $B_n^{(S)}(x,y)$ *are defined by means of the generating functions*

$$\sum_{n=0}^{\infty} B_n^{(C)}(x,y)\frac{t^n}{n!} = \frac{t}{e^t-1}e^{xt}\cos yt, \tag{32}$$

and

$$\sum_{n=0}^{\infty} B_n^{(S)}(x,y)\frac{t^n}{n!} = \frac{t}{e^t-1}e^{xt}\sin yt, \tag{33}$$

respectively.

By (30), (31), (32), and (33), we have

$$B_n^{(C)}(x,y) = \frac{B_n(x+iy)+B_n(x-iy)}{2},$$
$$B_n^{(S)}(x,y) = \frac{B_n(x+iy)-B_n(x-iy)}{2i}.$$

Note that $B_n^{(C)}(x,0) = B_n(x)$ are the Bernoulli polynomials. The cosine–Bernoulli and sine–Bernoulli polynomials can be determined explicitly. A few of them are

$$B_0^{(C)}(x,y) = 1, \quad B_1^{(C)}(x,y) = -\frac{1}{2}+x,$$
$$B_2^{(C)}(x,y) = \frac{1}{6} - x + x^2 - y^2,$$
$$B_3^{(C)}(x,y) = \frac{x}{2} - \frac{3x^2}{2} + x^3 + \frac{3y^2}{2} - 3xy^2,$$
$$B_4^{(C)}(x,y) = -\frac{1}{30} + x^2 - 2x^3 + x^4 - y^2 + 6xy^2 - 6x^2y^2 + y^4,$$

and

$$\begin{aligned}
&B_0^{(S)}(x,y) = 0, \quad B_1^{(S)}(x,y) = y, \quad B_2^{(S)}(x,y) = -y + 2xy, \\
&B_3^{(S)}(x,y) = \frac{y}{2} - 3xy + 3x^2y - y^3, \\
&B_4^{(S)}(x,y) = 2xy - 6x^2y + 4x^3y + 2y^3 - 4xy^3.
\end{aligned}$$

From (32), we have

$$\begin{aligned}
\sum_{n=0}^{\infty} B_n^{(C)}(x,y)\frac{t^n}{n!} &= \frac{t}{e^t-1}e^{xt}\cos yt, \\
&= \left(\sum_{l=0}^{\infty} B_n \frac{t^l}{l!}\right)\left(\sum_{m=0}^{\infty} C_m(x,y)\frac{t^m}{m!}\right) \\
&= \sum_{n=0}^{\infty}\left(\sum_{l=0}^{n}\binom{n}{l}B_l C_{n-l}(x,y)\right)\frac{t^n}{n!}.
\end{aligned} \tag{34}$$

Comparing the coefficients on the both sides of (34), we obtain the following theorem:

Theorem 6. *For $n \geq 0$, we have*

$$B_n^{(C)}(x,y) = \sum_{l=0}^{n}\binom{n}{l}B_l C_{n-l}(x,y),$$

and

$$B_n^{(S)}(x,y) = \sum_{l=0}^{n}\binom{n}{l}B_l S_{n-l}(x,y).$$

By replacing x by $1-x$ in (32), we get

$$\begin{aligned}
\sum_{n=0}^{\infty} B_n^{(C)}(1-x,y)\frac{t^n}{n!} &= \frac{t}{e^t-1}e^{(1-x)t}\cos yt \\
&= \frac{t}{1-e^{-t}}e^{-xt}\cos yt \\
&= \sum_{n=0}^{\infty}(-1)^n B_n(x,y)\frac{t^n}{n!}.
\end{aligned} \tag{35}$$

Therefore, we obtain the following theorem:

Theorem 7. *For $n \geq 0$, we have*

$$B_n^{(C)}(1-x,y) = (-1)^n B_n^{(C)}(x,y),$$

and

$$B_n^{(S)}(1-x,y) = (-1)^{n+1} B_n^{(S)}(x,y).$$

Now, we observe that

$$\begin{aligned}\sum_{n=0}^{\infty} B_n^{(C)}(x+1,q)\frac{t^n}{n!} &= \frac{t}{e^t-1}e^{(x+1)t}\cos yt \\ &= te^{xt}\cos yt + \frac{t}{e^t-1}e^{xt}\cos yt \\ &= \sum_{n=1}^{\infty} nC_{n-1}(x,y)\frac{t^n}{n!} + \sum_{n=0}^{\infty} B_n^{(C)}(x,y)\frac{t^n}{n!} \\ &= \sum_{n=0}^{\infty}\left(nC_{n-1}(x,y) + B_n^{(C)}(x,y)\right)\frac{t^n}{n!}.\end{aligned} \tag{36}$$

Thus, by (36), we get

$$B_n^{(C)}(x+1,y) = nC_{n-1}(x,y) + B_n^{(C)}(x,y), (n \geq 1). \tag{37}$$

Therefore, by (37), we obtain the following theorem:

Theorem 8. *For $n \geq 1$, we have*

$$B_n^{(C)}(x+1,y) - B_n^{(C)}(x,y) = nC_{n-1}(x,y),$$

and

$$B_n^{(S)}(x+1,y) - B_n^{(S)}(x,y) = nS_{n-1}(x,y).$$

Now, we define the new type polynomials that are given by the generating functions to be

$$\frac{2}{e^t+1}\cos yt = \sum_{n=0}^{\infty} E_n^{(C)}(y)\frac{t^n}{n!}, \tag{38}$$

and

$$\frac{2}{e^t+1}\sin yt = \sum_{n=0}^{\infty} E_n^{(S)}(y)\frac{t^n}{n!}, \tag{39}$$

respectively.

Note that $E_n^{(C)}(0) = E_n$, $E_n^{(S)}(0) = 0$, $E_n^{(C)}(0,y) = E_n^{(C)}(y)$, $E_n^{(S)}(0,y) = E_n^{(S)}(y), (n \geq 0)$. The new type polynomials can be determined explicitly. A few of them are

$$\begin{aligned}&E_0^{(C)}(y) = 1, \quad E_1^{(C)}(x,y) = -\frac{1}{2}, \quad E_2^{(C)}(x,y) = -y^2, \quad E_3^{(C)}(y) = \frac{1}{4} + \frac{3y^2}{2}, \\ &E_4^{(C)}(y) = y^4, \quad E_5^{(C)}(y) = -\frac{1}{2} - \frac{5y^2}{2} - \frac{5y^4}{2}, \quad E_6^{(C)}(y) = -y^6,\end{aligned}$$

and

$$\begin{aligned}&E_0^{(S)}(x,y) = 0, \quad E_1^{(S)}(x,y) = y, \quad E_2^{(S)}(x,y) = -y, \quad E_3^{(S)}(x,y) = -y^3, \\ &E_4^{(S)}(x,y) = y + 2y^3 \quad E_5^{(S)}(x,y) = y^5, \quad E_6^{(S)}(x,y) = -3y - 5y^3 - 3y^5.\end{aligned}$$

From (38) and (39), we derive the following equations:

$$\frac{2}{e^t+1}\cos yt = \sum_{k=0}^{\infty}\left(\sum_{m=0}^{[\frac{k}{2}]}\binom{k}{2m}(-1)^m E_{k-2m}y^{2m}\right)\frac{t^k}{k!} \tag{40}$$

and

$$\frac{2}{e^t+1}\sin yt=\sum_{k=0}^{\infty}\left(\sum_{m=0}^{[\frac{k-1}{2}]}\binom{k}{2m+1}(-1)^m E_{k-2m-1}y^{2m+1}\right)\frac{t^k}{k!}. \tag{41}$$

By (38)–(41), we get

$$E_n^{(C)}(y)=\sum_{m=0}^{[\frac{n}{2}]}\binom{n}{2m}(-1)^m y^{2m}E_{n-2m}, \tag{42}$$

and

$$E_n^{(S)}(y)=\sum_{m=0}^{[\frac{n-1}{2}]}\binom{n}{2m+1}(-1)^m y^{2m+1}E_{n-2m-1}, (k\geq 0). \tag{43}$$

From (12), (13), (38) and (39), we derive the following theorem:

Theorem 9. *For $n\geq 0$, we have*

$$E_n^{(C)}(x,y)=\sum_{k=0}^{n}\binom{n}{k}x^{n-k}E_k^{(C)}(y),$$

and

$$E_n^{(S)}(x,y)=\sum_{k=0}^{n}\binom{n}{k}x^{n-k}E_k^{(S)}(y).$$

Now, we define the new type polynomials that are given by the generating functions to be

$$\frac{t}{e^t-1}\cos yt=\sum_{n=0}^{\infty}B_n^{(C)}(y)\frac{t^n}{n!}, \tag{44}$$

and

$$\frac{t}{e^t-1}\sin yt=\sum_{n=0}^{\infty}B_n^{(S)}(y)\frac{t^n}{n!}, \tag{45}$$

respectively.

Note that $B_n^{(C)}(0)=B_n$, $B_n^{(S)}(0)=0$, $B_n^{(C)}(0,y)=B_n^{(C)}(y)$, $B_n^{(S)}(0,y)=B_n^{(S)}(y)$, $(n\geq 0)$. The new type polynomials can be determined explicitly. A few of them are

$$\begin{aligned}&B_0^{(C)}(x,y)=1,\quad B_1^{(C)}(x,y)=-\frac{1}{2},\quad B_2^{(C)}(x,y)=\frac{1}{6}-y^2,\\&B_3^{(C)}(x,y)=\frac{3y^2}{2},\quad B_4^{(C)}(x,y)=-\frac{1}{30}-y^2+y^4,\quad B_5^{(C)}(x,y)=\frac{5y^4}{2},\end{aligned}$$

and

$$\begin{aligned}&B_0^{(S)}(x,y)=0,\quad B_1^{(S)}(x,y)=y,\quad B_2^{(S)}(x,y)=-y,\\&B_3^{(S)}(x,y)=\frac{y}{2}-y^3,\quad B_4^{(S)}(x,y)=2y^3,\quad B_5^{(S)}(x,y)=-\frac{y}{6}-\frac{5y^3}{3}+y^5.\end{aligned}$$

From (44) and (45), we derive the following equations:

$$\frac{t}{e^t-1}\cos yt=\sum_{k=0}^{\infty}\left(\sum_{m=0}^{[\frac{k}{2}]}\binom{k}{2m}(-1)^m B_{k-2m}y^{2m}\right)\frac{t^k}{k!} \tag{46}$$

and

$$\frac{t}{e^t-1}\sin yt=\sum_{k=0}^{\infty}\left(\sum_{m=0}^{[\frac{k-1}{2}]}\binom{k}{2m+1}(-1)^m B_{k-2m-1}y^{2m+1}\right)\frac{t^k}{k!}. \tag{47}$$

By (44)–(47), we get

$$B_n^{(C)}(y) = \sum_{m=0}^{[\frac{n}{2}]} \binom{n}{2m} (-1)^m y^{2m} B_{n-2m}, \tag{48}$$

and

$$B_n^{(S)}(y) = \sum_{m=0}^{[\frac{n-1}{2}]} \binom{n}{2m+1} (-1)^m y^{2m+1} B_{n-2m-1}, (k \geq 0). \tag{49}$$

From (32), (33), (44) and (45), we derive the following theorem:

Theorem 10. *For $n \geq 0$, we have*

$$B_n^{(C)}(x,y) = \sum_{k=0}^{n} \binom{n}{k} x^{n-k} B_k^{(C)}(y),$$

and

$$B_n^{(S)}(x,y) = \sum_{k=0}^{n} \binom{n}{k} x^{n-k} B_k^{(S)}(y).$$

We remember that the classical Stirling numbers of the first kind $S_1(n,k)$ and $S_2(n,k)$ are defined by the relations (see [12])

$$(x)_n = \sum_{k=0}^{n} S_1(n,k) x^k \text{ and } x^n = \sum_{k=0}^{n} S_2(n,k)(x)_k, \tag{50}$$

respectively. Here, $(x)_n = x(x-1)\cdots(x-n+1)$ denotes the falling factorial polynomial of order n. The numbers $S_2(n,m)$ also admit a representation in terms of a generating function

$$(e^t - 1)^m = m! \sum_{n=m}^{\infty} S_2(n,m) \frac{t^n}{n!}. \tag{51}$$

By (12), (51) and by using Cauchy product, we get

$$\begin{aligned}
\sum_{n=0}^{\infty} E_n^{(C)}(x,y) \frac{t^n}{n!} &= \left(\frac{2}{e^t+1}\right) (1-(1-e^{-t}))^{-x} \cos yt \\
&= \left(\frac{2}{e^t+1}\right) \cos yt \sum_{l=0}^{\infty} \binom{x+l-1}{l} (1-e^{-t})^l \\
&= \sum_{l=0}^{\infty} < x >_l \frac{(e^t-1)^l}{l!} \left(\frac{2}{e^t+1}\right) e^{-lt} \cos yt \\
&= \sum_{l=0}^{\infty} < x >_l \sum_{n=0}^{\infty} S_2(n,l) \frac{t^n}{n!} \sum_{n=0}^{\infty} E_n^{(C)}(-l,y) \frac{t^n}{n!} \\
&= \sum_{n=0}^{\infty} \left(\sum_{l=0}^{\infty} \sum_{i=l}^{n} \binom{n}{i} S_2(i,l) E_{n-i}^{(C)}(-l,y) < x >_l \right) \frac{t^n}{n!},
\end{aligned} \tag{52}$$

where $< x >_l = x(x+1)\cdots(x+l-1) (l \geq 1)$ with $< x >_0 = 1$.

By comparing the coefficients on both sides of (52), we have the following theorem:

Theorem 11. *For $n \in \mathbb{Z}_+$, we have*

$$E_n^{(C)}(x,y) = \sum_{l=0}^{\infty}\sum_{i=l}^{n}\binom{n}{i}S_2(i,l)E_{n-i}^{(C)}(-l,y) < x >_l,$$
$$E_n^{(S)}(x,y) = \sum_{l=0}^{\infty}\sum_{i=l}^{n}\binom{n}{i}S_2(i,l)E_{n-i}^{(S)}(-l,y) < x >_l .$$

By (12), (38), (50), (51) and by using Cauchy product, we have

$$\begin{aligned}\sum_{n=0}^{\infty} E_n^{(C)}(x,y)\frac{t^n}{n!} &= \left(\frac{2}{e^t+1}\right)((e^t-1)+1)^x\cos(yt) \\ &= \frac{2}{e^t+1}\cos(yt)\sum_{l=0}^{\infty}\binom{x}{l}(e^t-1)^l \\ &= \sum_{l=0}^{\infty}(x)_l\frac{(e^t-1)^l}{l!}\left(\frac{2}{e^t+1}\cos(yt)\right) \\ &= \sum_{l=0}^{\infty}(x)_l\sum_{n=0}^{\infty}S_2(n,l)\frac{t^n}{n!}\sum_{n=0}^{\infty}E_n^{(C)}(y)\frac{t^n}{n!} \\ &= \sum_{n=0}^{\infty}\left(\sum_{l=0}^{\infty}\sum_{i=l}^{n}\binom{n}{i}(x)_lS_2(i,l)E_{n-i}^{(C)}(y)\right)\frac{t^n}{n!}.\end{aligned} \tag{53}$$

By comparing the coefficients on both sides of (53), we have the following theorem:

Theorem 12. *For $n \in \mathbb{Z}_+$, we have*

$$E_n^{(C)}(x,y) = \sum_{l=0}^{\infty}\sum_{i=l}^{n}\binom{n}{i}(x)_lS_2(i,l)E_{n-i}^{(C)}(y),$$
$$E_n^{(S)}(x,y) = \sum_{l=0}^{\infty}\sum_{i=l}^{n}\binom{n}{i}(x)_lS_2(i,l)E_{n-i}^{(S)}(y).$$

By (4), (12), (38), (50), (51) and by using Cauchy product, we have

$$\begin{aligned}&\sum_{n=0}^{\infty} E_n^{(C)}(x,y)\frac{t^n}{n!} \\ &= \left(\frac{2}{e^t+1}\right)e^{xt}\cos(yt) \\ &= \frac{(e^t-1)^r}{r!}\frac{r!}{t^r}\left(\frac{t}{e^t-1}\right)^r e^{xt}\sum_{n=0}^{\infty}E_n^{(C)}(y)\frac{t^n}{n!} \\ &= \frac{(e^t-1)^r}{r!}\left(\sum_{n=0}^{\infty}\mathbf{B}_n^{(r)}(x)\frac{t^n}{n!}\right)\left(\sum_{n=0}^{\infty}E_n^{(C)}(y)\frac{t^n}{n!}\right)\frac{r!}{t^r} \\ &= \sum_{n=0}^{\infty}\left(\sum_{l=0}^{n}\frac{\binom{n}{l}}{\binom{l+r}{r}}S_2(l+r,r)\sum_{i=0}^{n-l}\binom{n-l}{i}\mathbf{B}_i^{(r)}(x)E_{n-l-i}^{(C)}(y)\right)\frac{t^n}{n!}.\end{aligned}$$

By comparing the coefficients on both sides, we have the following theorem:

Theorem 13. *For $n \in \mathbb{Z}_+$ and $r \in \mathbb{N}$, we have*

$$E_n^{(C)}(x,y) = \sum_{l=0}^{n} \frac{\binom{n}{l}}{\binom{l+r}{r}} S_2(l+r,r) \sum_{i=0}^{n-l} \binom{n-l}{i} E_{n-l-i}^{(C)}(y) \mathbf{B}_i^{(r)}(x),$$
$$E_n^{(S)}(x,y) = \sum_{l=0}^{n} \frac{\binom{n}{l}}{\binom{l+r}{r}} S_2(l+r,r) \sum_{i=0}^{n-l} \binom{n-l}{i} E_{n-l-i}^{(S)}(y) \mathbf{B}_i^{(r)}(x).$$

By (5), (12), (38), (50), (51) and by using the Cauchy product, we get

$$\begin{aligned}
&\sum_{n=0}^{\infty} E_n^{(C)}(x,y)\frac{t^n}{n!} = \left(\frac{2}{e^t+1}\right) e^{xt}\cos(yt) \\
&= \frac{(e^t-u)^r}{(1-u)^r}\left(\frac{1-u}{e^t-u}\right)^r e^{xt}\left(\frac{2}{e^t+1}\right)\cos(yt) \\
&= \sum_{n=0}^{\infty} \mathbf{H}_n^{(r)}(u,x)\frac{t^n}{n!}\sum_{i=0}^{r}\binom{r}{i} e^{it}(-u)^{r-i}\frac{1}{(1-u)^r}\left(\frac{2}{e^t+1}\right)\cos(yt) \\
&= \frac{1}{(1-u)^r}\sum_{i=0}^{r}\binom{r}{i}(-u)^{r-i}\sum_{n=0}^{\infty}\mathbf{H}_n^{(r)}(u,x)\frac{t^n}{n!}\sum_{n=0}^{\infty}E_n^{(C)}(i,y)\frac{t^n}{n!} \\
&= \sum_{n=0}^{\infty}\left(\frac{1}{(1-u)^r}\sum_{i=0}^{r}\binom{r}{i}(-u)^{r-i}\sum_{l=0}^{n}\binom{n}{l}\mathbf{H}_l^{(r)}(u,x)E_{n-l}^{(C)}(i,y)\right)\frac{t^n}{n!}.
\end{aligned}$$

By comparing the coefficients on both sides, we have the following theorem:

Theorem 14. *For $n \in \mathbb{Z}_+$ and $r \in \mathbb{N}$, we have*

$$E_n^{(C)}(x,y) = \frac{1}{(1-u)^r}\sum_{i=0}^{r}\sum_{l=0}^{n}\binom{r}{i}\binom{n}{l}(-u)^{r-i}\mathbf{H}_l^{(r)}(u,x)E_{n-l}^{(C)}(i,y),$$
$$E_n^{(S)}(x,y) = \frac{1}{(1-u)^r}\sum_{i=0}^{r}\sum_{l=0}^{n}\binom{r}{i}\binom{n}{l}(-u)^{r-i}\mathbf{H}_l^{(r)}(u,x)E_{n-l}^{(S)}(i,y).$$

By Theorems 12–14, we have the following corollary.

Corollary 3. *For $n \in \mathbb{Z}_+$ and $r \in \mathbb{N}$, we have*

$$\begin{aligned}
&\sum_{l=0}^{\infty}\sum_{i=l}^{n}\binom{n}{i}(x)_l S_2(i,l)E_{n-i}^{(C)}(y) \\
&= \frac{1}{(1-u)^r}\sum_{i=0}^{r}\sum_{l=0}^{n}\binom{r}{i}\binom{n}{l}(-u)^{r-i}\mathbf{H}_l^{(r)}(u,x)E_{n-l}^{(C)}(i,y) \\
&= \sum_{l=0}^{n}\frac{\binom{n}{l}}{\binom{l+r}{r}}S_2(l+r,r)\sum_{i=0}^{n-l}\binom{n-l}{i}E_{n-l-i}^{(C)}(y)\mathbf{B}_i^{(r)}(x).
\end{aligned}$$

3. Distribution of Zeros of the Cosine–Euler and Sine–Euler Polynomials

This section aims to demonstrate the benefit of using numerical investigation to support theoretical prediction and to discover a new interesting pattern of the zeros of the cosine–Euler and sine–Euler polynomials. Using a computer, a realistic study for the cosine–Euler polynomials $E_n^{(C)}(x,y)$ and sine–Euler polynomials $E_n^{(S)}(x,y)$ is very interesting. It is the aim of this paper to observe an interesting phenomenon of "scattering" of the zeros of the the cosine–Euler polynomials $E_n^{(C)}(x,y)$ and sine–Euler polynomials $E_n^{(S)}(x,y)$ in a complex plane. We investigate the beautiful zeros of the cosine–Euler and sine–Euler polynomials by using a computer. We plot the zeros of the cosine–Euler polynomials $E_n^{(C)}(x,y)$ (Figure 1).

In Figure 1 (top-left), we choose $n = 30$ and $y = -3$. In Figure 1 (top-right), we choose $n = 30$ and $y = 0$. In Figure 1 (bottom-left), we choose $n = 30$ and $y = 1/2$. In Figure 1 (bottom-right), we choose $n = 30$ and $y = 3$.

We plot the zeros of the sine–Euler polynomials $E_n^{(S)}(x,y)$ (Figure 2).

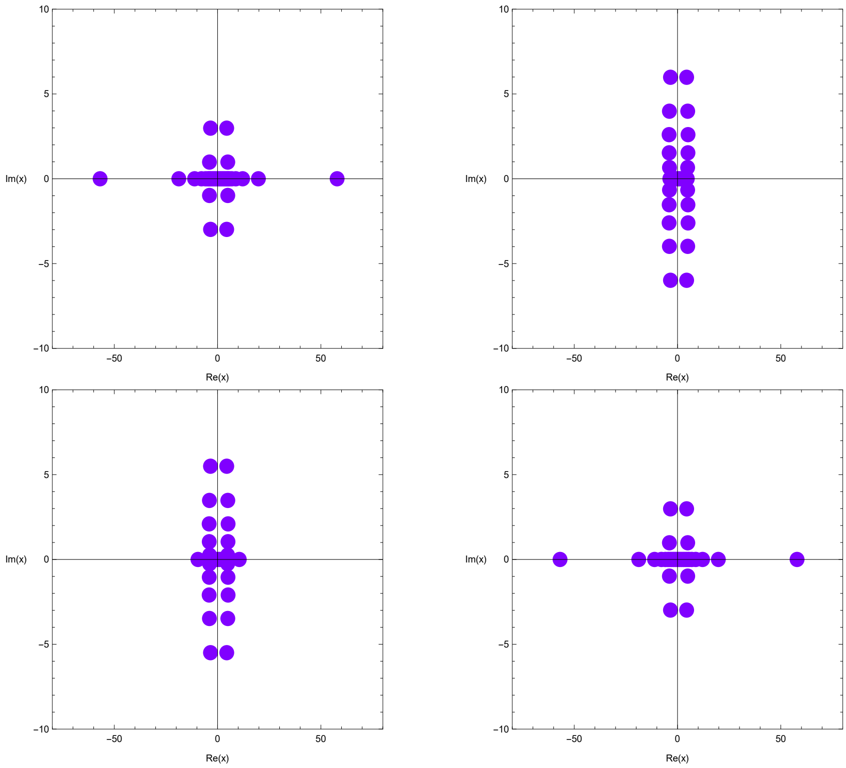

Figure 1. Zeros of $E_n^{(C)}(x,y)$.

In Figure 2 (top-left), we choose $n = 30$ and $x = -3$. In Figure 2 (top-right), we choose $n = 30$ and $x = -1$. In Figure 2 (bottom-left), we choose $n = 30$ and $x = 1$. In Figure 2 (bottom-right), we choose $n = 30$ and $x = 3$.

We observe that $E_n^{(C)}(x,a), x \in \mathbb{C}$ has $Re(x) = \frac{1}{2}$ reflection symmetry in addition to the usual $Im(x) = 0$ reflection symmetry analytic complex functions, where $a \in \mathbb{R}$(Figures 1 and 2).

Since

$$\begin{aligned}\sum_{n=0}^{\infty} E_n^{(C)}(1-x,-y)\frac{(-1)^n t^n}{n!} &= \frac{2}{e^{-t}+1}e^{(1-x)(-t)}\cos yt \\ &= \frac{2}{e^t+1}e^{xt}\cos yt = \sum_{n=0}^{\infty} E_n^{(C)}(x,y)\frac{t^n}{n!},\end{aligned}$$

we obtain

$$\begin{aligned}&E_n^{(C)}(x,y) = (-1)^n E_n^{(C)}(1-x,-y), \quad E_n^{(C)}(x,y) = (-1)^n E_n^{(C)}(1-x,y), \\ &E_n^{(S)}(x,y) = (-1)^n E_n^{(S)}(1-x,-y), \quad E_n^{(S)}(x,y) = (-1)^{n+1} E_n^{(S)}(1-x,y).\end{aligned}$$

Hence, we have the following theorem:

Theorem 15. *If* $n \equiv 1 \pmod 2$, *then*

$$E_n^{(C)}(1/2, y) = 0, \quad B_n^{(C)}(1/2, y) = 0, \text{ for } n \in \mathbb{N}.$$

If $n \equiv 0 \pmod 2$, *then*

$$E_n^{(S)}(1/2, y) = 0, \quad B_n^{(S)}(1/2, y) = 0, \text{ for } n \in \mathbb{N}.$$

Our numerical results for numbers of real and complex zeros of the cosine–Euler polynomials $E_n^{(C)}(x, y) = 0$ are displayed (Table 1).

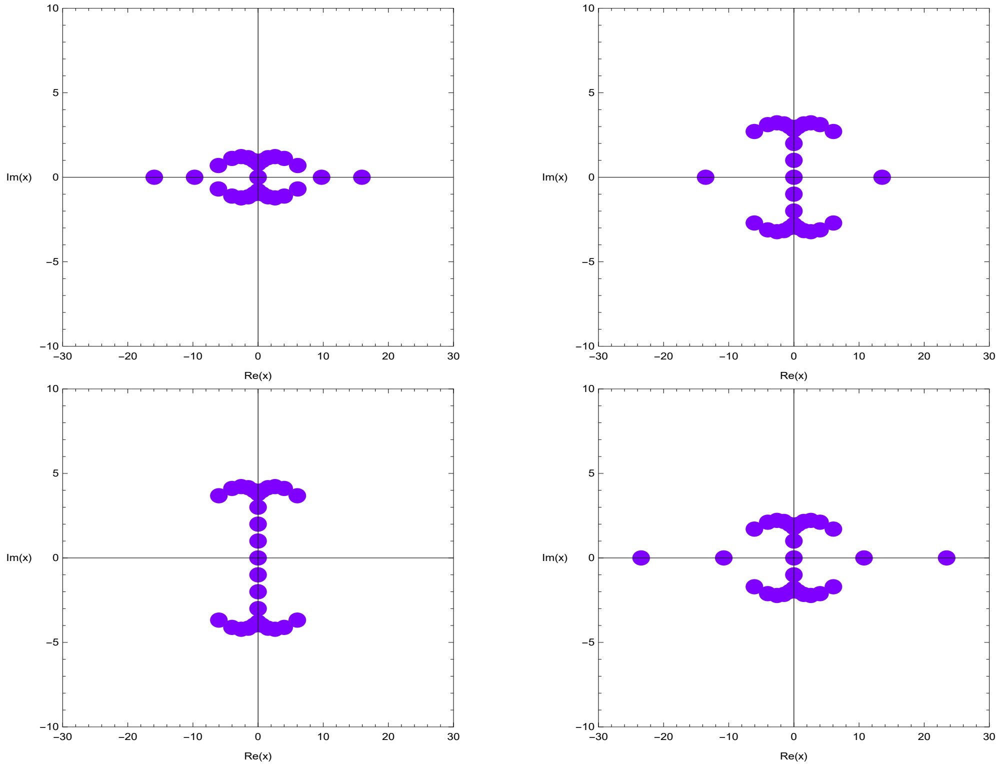

Figure 2. Zeros of $E_n^{(S)}(x, y)$.

Table 1. Numbers of real and complex zeros of $E_n^{(C)}(x, y)$.

Degree n	$y = -3$		$y = 2$	
	Real Zeros	Complex Zeros	Real Zeros	Complex Zeros
1	1	0	1	0
2	2	0	2	0
3	3	0	3	0
4	4	0	4	0
5	5	0	5	0
6	6	0	6	0
7	7	0	7	0
8	8	0	8	0
9	9	0	9	0
10	10	0	10	0

Our numerical results for numbers of real and complex zeros of the sine–Euler polynomials $E_n^{(S)}(x, y) = 0$ are displayed (Table 2).

Stacks of zeros of the cosine–Euler polynomials $E_n^{(C)}(x,y)$ for $1 \le n \le 40$ from a 3D structure are presented (Figure 3).

In Figure 3 (left), we choose $y = -3$. In Figure 3 (right), we choose $y = 1/2$. The plot of real zeros of the cosine–Euler polynomials $E_n^{(C)}(x,y)$ for $1 \le n \le 40$ structure are presented (Figure 4).

In Figure 4 (left), we choose $y = -3$. In Figure 4 (right), we choose $y = 1/2$. Stacks of zeros of the sine–Euler polynomials $E_n^{(S)}(x,y)$ for $1 \le n \le 40$ from a 3D structure are presented (Figure 5).

Table 2. Numbers of real and complex zeros of $E_n^{(S)}(x,y)$.

Degree n	$x = -3$		$x = 1$	
	Real Zeros	Complex Zeros	Real Zeros	Complex Zeros
1	1	0	1	0
2	1	0	1	0
3	3	0	3	0
4	3	0	3	0
5	5	0	5	0
6	5	0	1	4
7	7	0	7	0
8	7	0	1	6
9	9	0	9	0
10	9	0	1	8

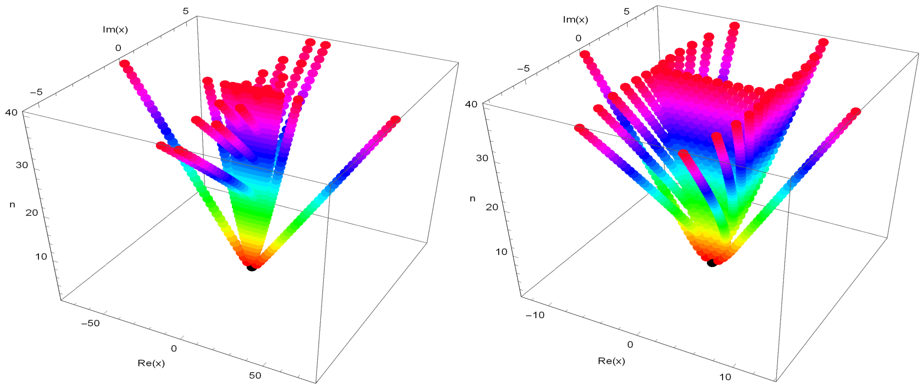

Figure 3. Stacks of zeros of $E_n^{(C)}(x,y), 1 \le n \le 40$.

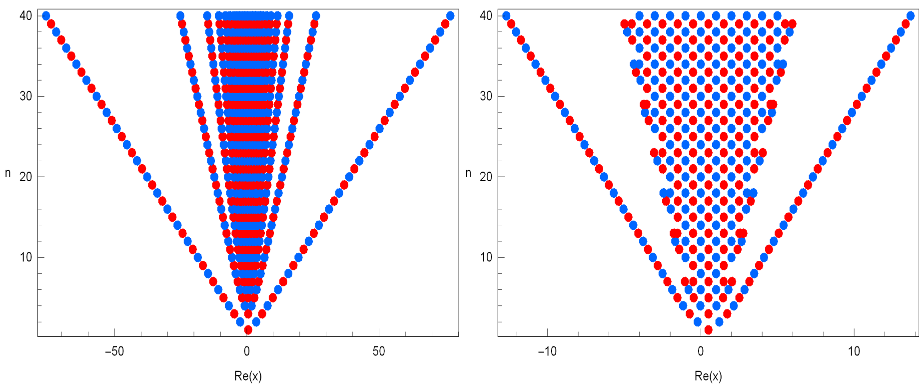

Figure 4. Real zeros of $E_n^{(C)}(x,y)$.

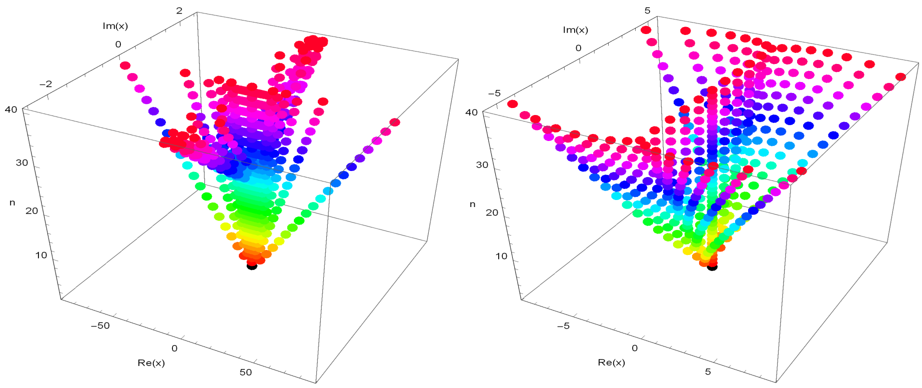

Figure 5. Stacks of zeros of $E_n^{(S)}(x,y), 1 \leq n \leq 40$.

In Figure 5 (left), we choose $x = -3$. In Figure 3 (right), we choose $x = 1$. The plot of real zeros of the sine–Euler polynomials $E_n^{(S)}(x,y)$ for $1 \leq n \leq 40$ structure are presented (Figure 6).

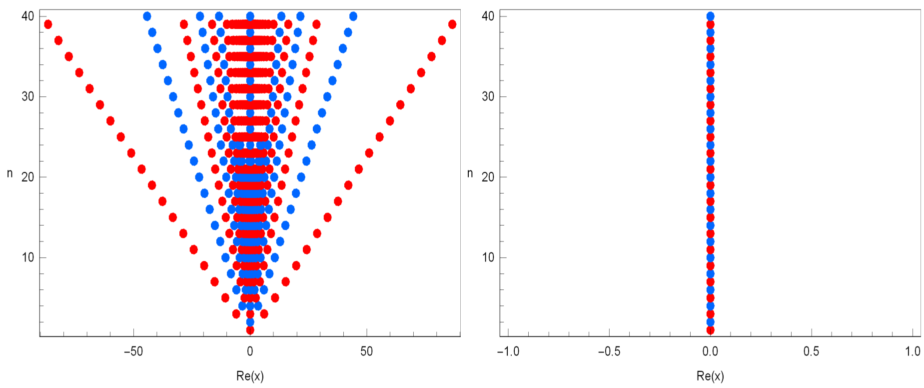

Figure 6. Real zeros of $E_n^{(S)}(x,y)$.

In Figure 6 (left), we choose $x = -3$. In Figure 6 (right), we choose $x = 1$.

We observe a remarkable regular structure of the complex roots of the cosine–Euler polynomials $E_n^{(C)}(x,y)$. We also hope to verify a remarkable regular structure of the complex roots of the cosine–Euler polynomials $E_n^{(C)}(x,y)$. Next, we calculated an approximate solution satisfying $E_n^{(C)}(x,y) = 0, x \in \mathbb{R}$. The results are given in Table 3.

Table 3. Approximate solutions of $E_n^{(C)}(x,-3) = 0, x \in \mathbb{R}$.

Degree n				x			
1				0.50000			
2			−2.5414,	3.5414			
3			−4.7678,	0.50000,	5.7678		
4		−6.8305,	−0.82832,	1.8283,	7.8305		
5		−8.8303,	−1.8336,	0.50000,	2.8336,	9.8303	
6		−10.799,	−2.7017,	−0.40666,	1.4067,	3.7017,	11.799
7	−12.751,	−3.4960,	−1.1389,	0.50000,	2.1389,	4.4960,	13.751

Next, we calculated an approximate solution satisfying $E_n^{(S)}(x,y) = 0, y \in \mathbb{R}$. The results are given in Table 4.

Table 4. Approximate solutionsof $E_n^{(S)}(-3, y) = 0, y \in \mathbb{R}$.

Degree n				y			
1				0.00000			
2				0.00000			
3			−6.0000,	0,	6.0000		
4			−3.3912,	0,	3.3912		
5		−10.687,	−2.4038,	0,	2.4038,	10.687	
6		−5.9045,	−1.8630,	0,	1.8630,	5.9045	
7	−15.241,	−4.1727,	−1.5184,	0,	1.5184,	4.1727,	15.241

Author Contributions: T.K. and C.S.R. wrote and checked the results of the paper; C.S.R. conducted numerical experiments of this paper; T.K. completed the revision of the article.

Funding: This work was supported by the National Research Foundation of Korea (NRF) grant funded by the Korea government (MEST) (No. 2017R1A2B4006092).

Acknowledgments: The authors would like to thank the referees for their valuable comments.

Conflicts of Interest: The authors declare no conflict of interest.

References

1. Ağyüz, E.; Acikgoz, M.; Araci, S. A symmetric identity on the q-Genocchi polynomials of higher-order under third dihedral group D3. *Proc. Jangjeon Math. Soc.* **2015**, *18*, 177–187
2. Bayad, A.; Chikhi, J. Non linear recurrences for Apostol-Bernoulli-Euler numbers of higher order. *Adv. Stud. Contemp. Math. (Kyungshang)* **2012** , *22*, 1–6.
3. Carlitz, L. Recurrences for the Bernoulli and Euler numbers. II. *Math. Nachr.* **1965**, *29*, 151–160. [CrossRef]
4. He, Y.; Kim, T. A higher-order convolution for Bernoulli polynomials of the second kind. *Appl. Math. Comput.* **2018**, *324*, 51–58. [CrossRef]
5. Kim, D.S.; Kim, T. Some identities of Bernoulli and Euler polynomials arising from umbral calculus. *Adv. Stud. Contemp. Math. (Kyungshang)* **2013**, *23*, 159–171. [CrossRef]
6. Kim, D.S.; Kim, T.; Kim, Y.H.; Lee, S.H. Some arithmetic properties of Bernoulli and Euler numbers. *Adv. Stud. Contemp. Math. (Kyungshang)* **2012**, *22*, 467–480.
7. Kim, D.S.; Kim, T.; Kim, Y.-H.; Dolgy, D.V. A note on Eulerian polynomials associated with Bernoulli and Euler numbers and polynomials. *Adv. Stud. Contemp. Math. (Kyungshang)* **2012**, *22*, 379–389.
8. Roman, S. *The Umbral Calculu, Pure and Applied Mathematics, 111*; Academic Press, Inc.: New York, NY, USA; Harcourt Brace Jovanovich Publishers: San Diego, CA, USA, 1984; ISBN 0-12-594380-6.
9. Ryoo, C.S.; Kim, Y.H. A numerical investigation on the structure of the roots of the twisted q-Euler polynomials. *Adv. Stud. Contemp. Math. (Kyungshang)* **2009**, *19*, 131–141.
10. Ryoo, C.S.; Agarwal, R.P. Some identities involving q-poly-tangent numbers and polynomials and distribution of their zeros. *Adv. Differ. Equ.* **2017**, *213*. [CrossRef]
11. Simsek, Y. Identities on the Changhee numbers and Apostol-type Daehee polynomials. *Adv. Stud. Contemp. Math. (Kyungshang)* **2017**, *27*, 199–212.
12. Srivastava, H.M.; Pintér, Á. Remarks on some relationships between the Bernoulli and Euler polynomials. *Appl. Math. Lett*. **2004**, *17*, 375–380. [CrossRef]
13. Srivastava, H.M.; Pintér, Á. Addition theorems for the Appell polynomials and the associated classes of polynomial expansions. *Aequ. Math.* **2013**, *85*, 483–495.
14. Young, P.T. Degenerate Bernoulli polynomials, generalized factorial sums, and their applications. *J. Number Theor.* **2008**, *128*, 738–758 [CrossRef]