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Common Fixed Point Theorems for Generalized Geraghty (α, ψ, ϕ)-Quasi Contraction Type Mapping in Partially Ordered Metric-Like Spaces

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Abstract: The aim of this paper is to establish the existence of some common fixed point results for generalized Geraghty (α, ψ, ϕ)-quasi contraction self-mapping in partially ordered metric-like spaces. We display an example and an application to show the superiority of our results. The obtained results progress some well-known fixed (common fixed) point results in the literature. Our main results cannot be specifically attained from the corresponding metric space versions. This paper is scientifically novel because we take Geraghty contraction self-mapping in partially ordered metric-like spaces via $\alpha-$admissible mapping. This opens the door to other possible fixed (common fixed) point results for non-self-mapping and in other generalizing metric spaces.

Keywords: common fixed point; metric-like space; α-Geraghty contraction; triangular α-admissible mapping

1. Introduction

Fixed point theory occupies a central role in the study of solving nonlinear equations of kinds $Sx = x$, where the function S is characterized on abstract space X. It is outstanding that the Banach contraction principle is a standout amongst essential and principal results in the fixed point theorem. It ensures the existence of fixed points for certain self-maps in a complete metric space and provides a helpful technique to find those fixed points. Many authors studied and extended it in many generalizations of metric spaces with new contractive mappings, for example, see References [1–3] and the references therein.

Otherwise, Hitzler and Seda [4] introduce the notation of metric-like (dislocated) metric space as a generalization of a metric space, they introduced variants of the Banach fixed point theorem in such space. Metric like spaces were revealed by Amini-Harandi [5] who proved the existence of fixed point results. This interesting subject has been mediated by certain authors, for example, see References [6–8]. In partial metric spaces and partially ordered metric-like spaces, the usual contractive condition is weakened and many researchers apply their results to problems of existence and uniqueness of solutions for some boundary value problems of differential and Integral equations, for example, see References [9–22] and the references therein.

Additionally, Geraghty [23] characterized a kind of the set of functions $\mathfrak{S}$ to be classified as the functions β:$[0, \infty) \to [0, 1)$ such that if $\{t_n\}$ is a sequence in $[0, +\infty)$ with $\beta(t_n) \to 1$, then $t_n \to 0$.

By using the function $\beta \in \mathfrak{S}$, Geraghty [23] presented the following exceptional theorem

Theorem 1. *Suppose (Y,d) is a complete metric space. Assume that $T:Y \to Y$ and $\beta:[0,\infty) \to [0,1)$ are functions such that for all $u,v \in Y$,*

$$d(Tu,Tv) \leq \beta(d(u,v))d(u,v), \tag{1}$$

where $\beta \in \mathcal{S}$, then T has a fixed point and has to be unique.

The main results of Geraghty have engaged many of authors, see References [24–26] and the references therein.

Recently, Amini-Harandi and Emami [27] reconsidered Theorem 1 as the framework of partially ordered metric spaces and they presented taking into account existence theorem.

Theorem 2. *Let (Y,d) be a partially ordered complete metric space. Assume $S:Y \to Y$ is a mapping such that there exists $u_0 \in Y$ with $u_0 \preceq Su_0$ and $\alpha \in \mathcal{F}$ such that*

$$d(Su,Sv) \leq \alpha(d(u,v))d(u,v), \text{ for any } u,v \in Y \text{ with } u \succeq v. \tag{2}$$

Hence, S has a fixed point supported that either S is continuous or Y is such that if an increasing sequence $\{u_n\} \to u$, then $u_n \leq u$ for all n.

In 2015, Karapinar [28] demonstrated the following specific results:

Theorem 3. *[28] Let (Y,σ) be a complete metric-like space. Assume that $S:Y \to Y$ is a mapping. If there exists $\beta \in \mathfrak{S}$ such that*

$$\sigma(Su,Sv) \leq \beta(\sigma(u,v))\sigma(u,v) \tag{3}$$

for all $u,v \in Y$, then S has a unique fixed point $u^* \in Y$ with $\sigma(u^*,u^*) = 0$.

The notion of quasi-contraction presented by Reference [29], is known as one of the foremost common contractive self-mappings.

A mapping $S:Y \to Y$ is expressed to be a quasi contraction if there exists $0 \leq \lambda < 1$ such that

$$d(Su,Sv) \leq \lambda \max\{d(u,v).d(u,fv), d(u,fv), d(fu,v), d(u,fv)\}, \tag{4}$$

for any $u,v \in Y$.

In this paper, we show the generalized Geraghty (α,ψ,ϕ)-quasi contraction type mapping in partially ordered metric like space, then we present some fixed and common fixed point theorems for such mappings in an ordered complete metric-like space. We investigate this new contractive mapping as a generalized weakly contractive mapping in our main results, then we display an example and an application to support our obtained results.

2. Preliminaries

In this section, we review a few valuable definitions and assistant results that will be required within the following sections.

Definition 1. *[5] Let Y be a nonempty set. A function $\sigma:Y \times Y \to [0,\infty)$ is expressed to be a metric-like space on X if for any $u,v,z \in Y$, the accompanying stipulations satisfied:*

(σ_1) $\sigma(u,v) = 0 \Rightarrow u = v$,
(σ_2) $\sigma(u,v) = \sigma(v,u)$,
(σ_3) $\sigma(u,z) \leq \sigma(u,v) + \sigma(v,z)$.

The pair (Y,σ) is called a metric-like space.

Obviously, we can consider that every metric space and partial metric space could be a metric-like space. However, this assertion isn't valid.

Example 1. *[5] Let $Y = \{0,1\}$ and*

$$\sigma(u,v) = \begin{cases} 2, & \text{if } u = v = 0; \\ 1, & \text{otherwise.} \end{cases} \tag{5}$$

We note that $\sigma(0,0) \not\leq \sigma(0,1)$. So, (Y,σ) is a metric-like space and at the same time it is not a partial metric space.

Additonally, each metric-like σ on Y create a topology τ_σ on Y whose use as a basis of the group of open σ-balls

$$B_\sigma(Y,\epsilon) = \{u \in Y : \mid \sigma(u,v) - \sigma(u,u) \mid < \epsilon\}, \text{ for all } u,v \in Y \text{ and } \epsilon > 0.$$

Let (Y,σ) be a metric-like space and $f{:}Y \to Y$ be a continuous mapping. Then

$$\lim_{n\to\infty} u_n = u \Rightarrow \lim_{n\to\infty} fu_n = fu.$$

A sequence $\{u_n\}$ of elements of Y is considered σ-Cauchy if the limit $\lim_{n,m\to\infty} \sigma(u_n,u_m)$ exists as a finite number. The metric-like space (Y,σ) is considered complete if for each σ-Cauchy sequence $\{u_n\}$, there is some $u \in Y$ such that

$$\lim_{n\to\infty} \sigma(u_n,u) = \sigma(u,u) = \lim_{n,m\to\infty} \sigma(u_n,u_m).$$

Remark 1. *[30] Let $Y = \{0,1\}$, and $\sigma(u,v) = 1$ for each $u,v \in Y$ and $u_n = 1$ for each $n \in \mathbb{N}$. Then, it is easy to see that $u_n \to 0$ and $u_n \to 1$ and so in metric-like spaces the limit of a convergent sequence is not necessarily unique.*

Lemma 1. *[30] Let (Y,σ) be a metric-like space. Let $\{u_n\}$ be a sequence in Y such that $u_n \to u$ where $u \in Y$ and $\sigma(u,u) = 0$. Then, for all $u,v \in Y$, we have $\lim_{n\to\infty} \sigma(u_n,v) = \sigma(u,v)$.*

Example 2. *[5] Let $Y = \mathbb{R}$ and $\sigma{:}Y \times Y \to [0,+\infty)$ be defined by*

$$\sigma(u,v) = \begin{cases} 2n, & \text{if } u = v = 0; \\ n, & \text{otherwise.} \end{cases}$$

Then, we can consider (Y,σ) to be a metric-like space, but it does not satisfy the conditions of the partial metric space, as $\sigma(0,0) \not\leq \sigma(0,1)$.

Samet et al. [31] displayed the definition of α-admissible mapping as followings:

Definition 2. *[31] Let $S{:}X \to X$ and $\alpha{:}X \times X \to [0,\infty)$ are two functions. Then, S is called α-admissible if $\forall u,v \in X$ with $\alpha(u,v) \geq 1$ implies $\alpha(fu,fv) \geq 1$.*

Definition 3. *[32] Let $S,T{:}X \to X$ be two mappings and $\alpha{:}X \times X \to \mathbb{R}$ be a function. We consider that the pair (S,T) is α-admissible if*

$$u,v \in X,\ \alpha(u,v) \geq 1 \Rightarrow \alpha(Su,Tv) \geq 1 \text{ and } \alpha(Tu,Sv) \geq 1$$

Definition 4. *[33] Let $S{:}X \to X$ and $\alpha{:}X \times X \to [0,\infty)$. Then, S is called a triangular α-admissible mapping if*

(1) *S is α-admissible,*
(2) *$\alpha(u,z) \geq 1$ and $\alpha(z,v) \geq 1$ imply $\alpha(u,v) \geq 1$.*

Definition 5. *[32] Let $S, T : X \to X$ and $\alpha : X \times X \to [0,\infty)$. Then, (S,T) is called a triangular α-admissible mapping if*

(1) *The pair (S,T) is α-admissible,*
(2) *$\alpha(u,z) \geq 1$ and $\alpha(z,v) \geq 1$ imply $\alpha(u,v) \geq 1$.*

Let Ψ indicate the set of functions $\psi : [0,\infty) \to [0,\infty)$ that approve the following stipulations:

(1) ψ is strictly continuous increasing,
(2) $\psi(t) = 0 \Leftrightarrow t = 0$.

and Φ indicates the set of all continuous functions $\phi : [0,\infty) \to [0,\infty)$ with $\phi(t) > \psi(t)$ for all $t > 0$ and $\phi(0) = 0$.

Definition 6. *[12] Let $(X, d, \preceq)$ be a partially ordered metric space. Assume $f, g : X \to X$ are two mappings. Then:*

(1) *For all $x, y \in X$ are said to be comparable if $x \preceq y$ or $y \preceq x$ holds,*
(2) *f is said to be nondecreasing if $x \preceq y$ implies $fx \preceq fy$,*
(3) *f, g are called weakly increasing if $fx \preceq gfx$ and $gx \preceq fgx$ for all $x \in X$,*
(4) *f is called weakly increasing if f and I are weakly increasing, where I is denoted to the identity mapping on X.*

3. Main Results

In this section, we present the notation of generalized Geraghty (α, ψ, ϕ)-quasi contraction self-mappings in partially ordered metric-like space. Then, we present some fixed and common fixed point theorems for such self-mappings. We investigate this new contractive self-mapping as a generalized weakly contractive self-mapping which is a generalization of the results of Reference [34]. Results of this kind are amongst the most useful in fixed point theory and it's applications.

Definition 7. *Let (X, σ) be a partially ordered metric-like space and $S, T : X \to X$ be two mappings. Then, we consider that the pair (S,T) is generalized Geraghty (α, ψ, ϕ)-quasi contraction self-mapping if there exist $\alpha : X \times X \to [0,\infty)$, $\beta \in \mathfrak{S}$, $\psi \in \Psi$ and $\phi : [0,\infty) \to [0,\infty)$ are continuous functions with $\phi(t) \leq \psi(t)$ for all $t > 0$ such that*

$$\alpha(x,y)\psi(\sigma(Sx,Ty)) \leq \lambda\beta(\psi(M_{x,y}))\phi(M_{x,y}), \tag{6}$$

holds for all elements $x, y \in X$ and $0 \leq \lambda < 1$, where

$$M_{x,y} = \max\{\sigma(x,y), \sigma(x,Sx), \sigma(y,Ty), \sigma(Sx,y), \sigma(x,Ty)\}.$$

The following two lemmas will be utilized proficiently within the verification of our fundamental result.

Lemma 2. *If $\psi \in \Psi$ and $\phi : [0,\infty) \to [0,\infty)$ are continuous function that satisfy the condition $\psi(t) > \phi(t)$ for all $t > 0$, then $\phi(0) = 0$.*

Proof. From the assumption $\phi(t) < \psi(t)$, since ψ and ϕ are continuous, we have

$$0 \leq \phi(0) = \lim_{t \to 0} \phi(t) \leq \lim_{t \to 0} \psi(t) = \psi(0) = 0.$$

□

Lemma 3. *Let $S, T: X \to X$ be two mappings and $\alpha: X \times X \to [0, \infty)$ be a function such that S, T are triangular $\alpha-$admissible. Suppose that there exists $x_0 \in X$ such that $\alpha(x_0, Sx_0) \geq 1$. Define a sequence $\{x_n\}$ in X by $Sx_{2n} = x_{2n+1}$ and $Tx_{2n+1} = x_{2n+2}$. Then $\alpha(x_n, x_m) \geq 1$ for all $m, n \in \mathbb{N}$ with $n < m$.*

Proof. Since $\alpha(x_0, Sx_0) \geq 1$ and S, T are $\alpha-$admissible, we get

$$\alpha(x_0, x_1) = \alpha(x_0, Sx_0) \geq 1.$$

By triangular $\alpha-$admissibility, we get

$$\alpha(Sx_0, Tx_1) = \alpha(x_1, x_2) \geq 1$$

and

$$\alpha(TSx_0, STx_1) = \alpha(x_2, x_3) \geq 1.$$

Again, since $\alpha(x_2, x_3) \geq 1$, then

$$\alpha(Sx_2, Tx_3) = \alpha(x_3, x_4) \geq 1$$

and

$$\alpha(TSx_2, STx_3) = \alpha(x_4, Sx_5) \geq 1.$$

By proceeding the above process, we conclude that $\alpha(x_n, x_{n+1}) \geq 1$ for all $n \in \mathbb{N} \cup \{0\}$.

Now, we prove that $\alpha(x_n, x_m) \geq 1$, for all$m, n \in \mathbb{N}$ with $n < m$. Since

$$\begin{cases} \alpha(x_n, x_{n+1}) \geq 1, \\ \alpha(x_{n+1}, x_{n+2}) \geq 1, \end{cases}$$

then, we have

$$\alpha(x_n, x_{n+2}) \geq 1.$$

Again, since

$$\begin{cases} \alpha(x_n, x_{n+2}) \geq 1 \\ \alpha(x_{n+2}, x_{n+3}) \geq 1, \end{cases}$$

we deduce that

$$\alpha(x_n, x_{n+3}) \geq 1.$$

By continuing this process, we have

$$\alpha(x_n, x_m) \geq 1$$

for all $n \in \mathbb{N}$ with $m > n$. □

Lemma 4. *Let $(X, \preceq, \sigma)$ be a partially ordered metric-like space. Assume S, T are two self-mappings of X which the pair (S, T) is generalized (α, ψ, ϕ)-quasi contraction self-mappings. Fix $x_1 \in X$ and define a sequence $\{x_n\}$ by $x_{2n+1} = Sx_{2n}$ and $x_{2n+2} = Tx_{2n+1}$ for all $n \in \mathbb{N}$. If $\lim_{n \to \infty} \sigma(x_n, x_{n+1}) = 0$ and the sequence $\{x_n\}$ is nondecreasing, then $\{x_n\}$ is a Cauchy sequence.*

Proof. Since S, T are a generalized (α, ψ, ϕ)-quasi contraction non-self mapping, then there exist $\psi \in \Psi, \phi \in \Phi$ such that
$$\alpha(x,y)\psi(\sigma(Sx,Ty)) \leq \lambda\beta(\psi(M_{x,y}))\phi(M_{x,y}), \tag{7}$$
holds for all elements $x, y \in X$ and $0 \leq \lambda < 1$, where

$$M_{x,y} = \max\{\sigma(x,y), \sigma(x,Sx), \sigma(y,Ty), \sigma(Sx,y), \sigma(x,Ty)\}.$$

Now, we show that the sequence $\{x_n\}$ is Cauchy sequence. Assume, for contradiction's sake, that $\{x_n\}$ isn't Cauchy sequence. Therefore, there exist $\epsilon > 0$ and two subsequences $\{n_k\}$ and $\{m_k\}$ of the sequence $\{x_n\}$ such that $\sigma(x_{2n_k}, x_{2m_k}), \sigma(x_{2n_k-1}, x_{2m_k})$ and $\sigma(x_{2n_k}, x_{2m_k+1})$ converge to ϵ^+ when $k \to \infty$.
$$n_k > m_k > k,\ \sigma(x_{2n_k}, x_{2m_k-2}) < \epsilon,\ \sigma(x_{2n_k}, x_{2m_k}) \geq \epsilon. \tag{8}$$

By the above inequalities and triangle inequality property, we imply that

$$\begin{aligned} \epsilon &\leq \sigma(x_{2n_k}, x_{2m_k}) \leq \sigma(x_{2n_k}, x_{2m_k-2}) + \sigma(x_{2m_k-2}, x_{2m_k-1}) + \sigma(x_{2m_k-1}, x_{2m_k}) \\ &< \epsilon + \sigma(x_{2m_k-2}, x_{2m_k-1}) + \sigma(x_{2m_k-1}, x_{2m_k}). \end{aligned}$$

In view of $\lim_{n\to\infty} \sigma(x_n, x_{n+1}) = 0$ and letting $k \to \infty$ in the above inequalities, we obtain

$$\lim_{k\to\infty} \sigma(x_{2n_k}, x_{2m_k}) = \epsilon. \tag{9}$$

By the triangle inequality, we have

$$\begin{aligned} \sigma(x_{2n_k}, x_{2m_k}) &\leq \sigma(x_{2n_k}, x_{2n_k+1}) + \sigma(x_{2n_k+1}, x_{2m_k}) \\ &\leq \sigma(x_{2n_k}, x_{2n_k+1}) + \sigma(x_{2n_k+1}, x_{2m_k+1}) + \sigma(x_{2m_k+1}, x_{2m_k}) \\ &\leq \sigma(x_{2n_k}, x_{2n_k+1}) + \sigma(x_{2n_k+1}, x_{2n_k+2}) + \sigma(x_{2n_k+2}, x_{2m_k}) + 2\sigma(x_{2m_k}, x_{2m_k+1}) \\ &\leq 2\sigma(x_{2n_k}, x_{2n_k+1}) + 2\sigma(x_{2m_k+2}, x_{2m_k+1}) + \sigma(x_{2n_k}, x_{2m_k}) + 2\sigma(x_{2m_k}, x_{2m_k+1}). \end{aligned}$$

Taking the limit as $k \to \infty$ in the above inequalities and using Equation (9), we get

$$\lim_{k\to\infty} \sigma(x_{2n_k}, x_{2m_k}) = \lim_{k\to\infty} \sigma(x_{2n_k+1}, x_{2m_k}) = \lim_{k\to\infty} \sigma(x_{2n_k+1}, x_{2m_k+1}) = \epsilon. \tag{10}$$

Since $x_{n_k+1} \preceq x_{m_k}$ and $\alpha(x_{n_k+1}, x_{m_k}) \geq 1$ for all $k \in \mathbb{N}$, so by substituting x with x_{n_k+1} and y with x_{m_k} in Equation (7), it follows that

$$\psi(\sigma(x_{n_k+1}, x_{m_k})) \leq \alpha(x_{n_k+1}, x_{m_k})\psi(\sigma(Sx_{n_k}, Tx_{m_k-1})) \leq \lambda\beta(\psi(M_{x,y}))\phi(M_{x,y}), \tag{11}$$

holds for all elements $x, y \in X$ and $0 \leq \lambda < 1$, where

$$\begin{aligned} M_{x_{n_k}, x_{m_k-1}} &= \max\{\sigma(x_{n_k}, x_{m_k-1}), \sigma(x_{n_k}, Sx_{n_k}), \sigma(x_{m_k-1}, Tx_{m_k-1}), \\ &\quad \sigma(Sx_{n_k}, x_{m_k-1}), \sigma(x_{n_k}, Tx_{m_k-1})\} \\ &= \max\{\sigma(x_{n_k}, x_{m_k-1}), \sigma(x_{n_k}, x_{n_k+1}), \sigma(x_{m_k-1}, x_{m_k}), \\ &\quad \sigma(x_{n_k+1}, x_{m_k-1}), \sigma(x_{n_k}, x_{m_k})\}. \end{aligned}$$

Taking the limit as $k \to \infty$ of the above inequality and applying Equations (9), (10), we get

$$\lim_{k\to\infty} M_{x_{2n_k}, x_{2m_k}} = \epsilon. \tag{12}$$

Letting $k \to \infty$ in Equation (11) and using $\phi \in \Phi$, $\beta \in \mathcal{S}$ and Equation (12), we deduce that

$$\begin{aligned} \psi(\epsilon) &\leq \lambda\beta(\psi(\epsilon))\phi(\epsilon) \\ &< \lambda\phi(\epsilon) \\ &< \lambda\psi(\epsilon). \end{aligned}$$

This is possible only if $\epsilon = 0$. Which contradicts the positivity of ϵ. Therefore, we get the desired result. □

Theorem 4. *Let (X, σ) be a partially ordered metric like space. Assume that $S, T{:}X \to X$ are two self-mappings fulfilling the following conditions:*

(1) *(S, T) is triangular α-admissible and there exists an $x_0 \in X$ such that $\alpha(x_0, Sx_0) \geq 1$,*
(2) *the pair (S, T) is weakly increasing,*
(3) *the pair (S, T) is a generalized Geraghty (α, ψ, ϕ)-quasi contraction non-self mapping,*
(4) *S and T are σ-continuous mappings.*

Then, the pair (S, T) has a common fixed point $z \in X$ with $\sigma(z, z) = 0$. Moreover, assume that if $x_1, x_2 \in X$ such $\sigma(x_1, x_1) = \sigma(x_2, x_2) = 0$ implies that x_1 and x_2 are comparable elements. Then the common fixed point of the pair (S, T) is unique.

Proof. Let $x_0 \in X$ such that $\alpha(x_0, Sx_0) \geq 1$. Define the sequence $\{x_n\}$in X as follows:

$$x_{2n+1} = Sx_{2n} \;\; x_{2n+2} = Tx_{2n+1} \;\; for\ all \;\; n \geq 0. \tag{13}$$

Suppose that $x_{2n} \neq x_{2n+1}$ for all $n \in \mathbb{N}_0$. Then, $\sigma(x_{2n}, x_{2n+1}) > 0$ for all $n \in \mathbb{N}_0$. Indeed, if $x_{2n} \neq x_{2n+1}$, which is a contradiction. By using the assumption of Equations (1), (2), and Lemma 3, we have

$$\alpha(x_n, x_{n+1}) \geq 1 \tag{14}$$

for all $n \in \mathbb{N} \cup \{0\}$.

Since the pair (S, T) is weakly increasing, we have

$$x_1 = Sx_0 \preceq TSx_0 = x_2 = Sx_1 = \preceq \ldots x_{2n} \preceq TSx_{2n} = x_{2n+2} \preceq \ldots.$$

Thus, $x_n \preceq x_{n+1}$, for all $n \in \mathbb{N}$. Since $\alpha(x_{2n}, x_{2n+1}) \geq 1$, by applying Equation (6), we obtain

$$\begin{aligned} \psi(\sigma(x_{2n+1}, x_{2n+2})) &= \psi(\sigma(Sx_{2n}, Tx_{2n+1})) \\ &\leq \alpha(x_{2n}, x_{2n+1})\psi(\sigma(Sx_{2n}, Tx_{2n+1})) \\ &\leq \lambda\beta(\psi(M_{x_{2n}, x_{2n+1}}))\phi(M_{x_{2n}, x_{2n+1}}). \end{aligned} \tag{15}$$

Set $\sigma_n = \sigma(x_{2n+1}, x_{2n+2})$. We have

$$\begin{aligned} \psi(\sigma_n) &= \psi(\sigma(x_{2n+1}, x_{2n+2})) && (16) \\ &\leq \lambda\beta(\psi(M_{x_{2n}, x_{2n+1}}))\phi(M_{x_{2n}, x_{2n+1}}). && (17) \end{aligned}$$

For the rest, for each n assume that $(\sigma_n \neq 0)$.

$$\begin{aligned} M_{x_{2n}, x_{2n+1}} &= \max\{\sigma(x_{2n}, x_{2n+1}), \sigma(x_{2n}, Sx_{2n}), \sigma(x_{2n+1}, Tx_{2n+1}), \sigma(Sx_{2n}, x_{2n+1}), \sigma(x_{2n}, Tx_{2n+1})\} \\ &= \max\{\sigma(x_{2n}, x_{2n+1}), \sigma(x_{2n}, x_{2n+1}), \sigma(x_{2n+1}, x_{2n+2}), \sigma(x_{2n+1}, x_{2n+1}), \sigma(x_{2n}, x_{2n+2})\} \\ &= \max\{\sigma(x_{2n}, x_{2n+1}), \sigma(x_{2n+1}, x_{2n+2}), \sigma(x_{2n}, x_{2n+2})\} \\ &= \max\{\sigma_{n-1}, \sigma_n, \sigma_{n-1} + \sigma_n\} \end{aligned}$$

If for some $n \in \mathbb{N}$, $\max\{\sigma_{n-1}, \sigma_n, \sigma_{n-1} + \sigma_n\} = \sigma_n$ then from Equation (16), we find that $\psi(\sigma_n) < \lambda\psi(\sigma_n)$ which is a contradiction with respect to $0 \le \lambda < 1$. We deduce $\max\{\sigma_{n-1}, \sigma_n, \sigma_{n-1} + \sigma_n\} = \max\{\sigma_{n-1}, \sigma_{n-1} + \sigma_n\}$. Therefore Equation (16) becomes

$$\psi(\sigma_n) < \lambda\psi(\max\{\sigma_{n-1}, \sigma_{n-1} + \sigma_n\}).$$

Put

$$\gamma = \max\{\lambda, \frac{\lambda}{1-\lambda}\}.$$

Thus,

$$\psi(\sigma_n) \le \gamma\beta(\psi(\sigma_{n-1}))\phi(\sigma_{n-1}), for\ all\ n \in \mathbb{N}_0. \tag{18}$$

It is clear that $\gamma < 1$. Therefore, the sequence $\{\sigma(x_n, x_{n+1})\}$ is a decreasing sequence. Thus, there exists $r \ge 0$ such that

$$\lim_{n\to\infty} \sigma(x_n, x_{n+1}) = r.$$

Now, we show that $r = 0$. Presume to the contrary, that is $r > 0$. Since $\beta \in \mathcal{S}$ and by using the condition of Theorem 4 and taking the limit as $k \to \infty$ in Equation (18), we conclude

$$\psi(r) \le \lambda\beta(\psi(r))\phi(r) < \lambda\phi(r) < \lambda\psi(r),$$

which could be a contradiction. So $r = 0$. Then,

$$\lim_{n\to\infty} \sigma(x_n, x_{n+1}) = 0.$$

Lemma 4 implies that $\{x_n\}$ is a Cauchy sequence and from the completeness of (X, σ), then there exists a $x^* \in X$ in order that

$$\lim_{n\to\infty} \sigma(x_n, x^*) = \sigma(x^*, x^*) = \lim_{n,m\to\infty} \sigma(x_n, x_m). \tag{19}$$

Whereas, S and T are continuous, we conclude

$$\lim_{n\to\infty} \sigma(x_{n+1}, Tx^*) = \lim_{n\to\infty} \sigma(Sx_n, Tx^*) = \sigma(Sx^*, Tx^*), \tag{20}$$

$$\lim_{n\to\infty} \sigma(Sx^*, x_{n+1}) = \lim_{n\to\infty} \sigma(Sx^*, Tx_n) = \sigma(Sx^*, Tx^*). \tag{21}$$

By Lemma 1 and Equation (19), we obtain that

$$\lim_{n\to\infty} \sigma(x_{n+1}, Tx^*) = \sigma(x^*, Tx^*) \tag{22}$$

and

$$\lim_{n\to\infty} \sigma(Sx^*, x_{n+1}) = \sigma(Sx^*, x^*). \tag{23}$$

By merging Equations (20) and (22), we deduce that $\sigma(x^*, Tx^*) = \sigma(Sx^*, x^*)$. In addition, by Equations (21) and (23), we deduce that $\sigma(Sx^*, x^*) = \sigma(Sx^*, Tx^*)$. So

$$\sigma(x^*, Tx^*) = \sigma(Sx^*, x^*) = \sigma(Sx^*, Tx^*). \tag{24}$$

Presently, we display that $\sigma(x^*, Tx^*) = 0$. Assume the opposite, that is, $\sigma(x^*, Tx^*) > 0$, we get

$$\begin{aligned} \psi(\sigma(x^*, Tx^*)) &= \psi(\sigma(Sx^*, Tx^*)) \\ &\le \lambda\beta(\psi(M_{x^*,x^*}))\phi(M_{x^*,x^*}), \end{aligned} \tag{25}$$

where

$$\begin{aligned} M_{x^*,x^*} &= \max\{\sigma(x^*,x^*),\sigma(x^*,Sx^*),\sigma(x^*,Tx^*),\sigma(Sx^*,x^*),\sigma(x^*,Tx^*),\} \\ &= \max\{\sigma(x^*,Tx^*),\sigma(x^*,Sx^*)\} \\ &= \max\{\sigma(x^*,Tx^*),\sigma(x^*,Tx^*)\}. \end{aligned}$$

Therefore, from Equation (25), we get

$$\begin{aligned} \psi(\sigma(x^*,Tx^*)) &\leq \beta(\psi(\sigma(x^*,Tx^*)))\phi(\sigma(x^*,Tx^*)) \\ &< \lambda\phi(\sigma(x^*,Tx^*) \\ &< \lambda\psi(\sigma(x^*,Tx^*)) \end{aligned} \tag{26}$$

Since $\psi \in \Psi$, we have $\sigma(x^*,Tx^*) < \lambda\sigma(x^*,Tx^*)$ which is a discrepancy. Thus, we have $\sigma(x^*,Tx^*) = 0$. Hence, $Tx^* = x^*$. From Equation (24), we deduce that $\sigma(x^*,Sx^*) = 0$. Therefore, $Sx^* = x^*$. Hence, x^* is a common fixed point of S and T. To demonstrate the uniqueness of the common fixed point, we suppose that $\bar{x}$ is another fixed point of S and T. Directly, we prove that $\sigma(\bar{x},\bar{x}) = 0$. Assume the antithesis, that is, $\sigma(\bar{x},\bar{x}) > 0$. Since $\bar{x} \preceq \bar{x}$, we get

$$\begin{aligned} \psi(\sigma(\bar{x},\bar{x})) &= \psi(\sigma(S\bar{x},T\bar{x})) \\ &\leq \lambda\beta(\psi(\sigma(\bar{x},\bar{x})))\phi(\sigma(\bar{x},\bar{x})) \\ &< \lambda\phi(\sigma(\bar{x},\bar{x})) \\ &< \lambda\psi(\sigma(\bar{x},\bar{x})) \end{aligned}$$

which is a discrepancy. Thus, $\sigma(\bar{x},\bar{x}) = 0$. Therefore, by the further conditions on X, we deduce that x^* and $\bar{x}$ are comparable. Presently, suppose that $\sigma(x^*,\bar{x}) \neq 0$. Then

$$\begin{aligned} \psi(\sigma(x^*,\bar{x})) &= \psi(\sigma(Sx^*,T\bar{x})) \\ &\leq \lambda(\psi(\sigma(x^*,\bar{x})))\phi(\sigma(x^*,\bar{x})) \\ &< \lambda\phi(\sigma(x^*,\bar{x})) \end{aligned}$$

which is a discrepancy with the condition of Theorem 4. Therefore, $\sigma(x^*,\bar{x}) = 0$. Hence, $x^* = \bar{x}$. Thus, S and T have a unique common fixed point. □

It is additionally conceivable to expel the continuity of S and T by exchanging a weaker condition. $(\mathcal{C})$ If $\{x_n\}$ is a nondecreasing sequence in X such that $\alpha(x_n,x_{n+1}) \geq 1$ for all $n \in \mathbb{N}\cup\{0\}$ and $x_n \to u \in X$ as $n \to \infty$, then there exists a subsequence $\{x_{n_l}\}$ of $\{x_n\}$ such that $x_{n_l} \preceq u$ for all l.

Theorem 5. *Let (X,σ) be a partially ordered metric-like space. Assume that $S,T{:}X \to X$ are two self-mappings fulfilling the following conditions:*

(1) *the pair (S,T) is triangular α-admissible,*
(2) *there exists an $x_0 \in X$ such that $\alpha(x_0,Sx_0) \geq 1$,*
(3) *the pair (S,T) is a generalized Geraghty (α,ψ,ϕ)-quasi contraction non-self mapping,*
(4) *the pair (S,T) is weakly increasing,*
(5) *$(\mathcal{C})$ holds.*

Then, the pair (S,T) has a common fixed point $v \in X$ with $\sigma(v,v) = 0$. Moreover, suppose that if $x_1,x_2 \in X$ such $\sigma(x_1,x_1) = \sigma(x_2,x_2) = 0$ implies that x_1 and x_2 are comparable. Then, the common fixed point of the pair (S,T) is unique.

Proof. Here, we define $\{x_n\}$ as in the proof of Theorem 4. Clearly $\{x_n\}$ is a Cauchy sequence in X, then there exists $v \in X$ in order that

$$\lim_{n\to\infty} x_n = v \tag{27}$$

As a result of the condition of Equation (5), there exists a subsequence $\{x_{n_l}\}$ of $\{x_n\}$ in order that $x_{n_l} \preceq v$ for all l. Therefore, x_{n_l} and v are comparable. In addition, from Equation (13) on taking limit as $n \to \infty$ and using Equation (27), we get

$$\lim_{n\to\to\infty} x_n = v.$$

$$\lim_{n\to\to\infty} Sx_{2n_l} = \lim_{n\to\to\infty} x_{2n_l+1} = v, \lim_{n\to\to\infty} Tx_{2n_l+1} = \lim_{n\to\to\infty} x_{2n_l+2} = v. \tag{28}$$

From the definition of α yields that $\alpha(x_{n_l}, v) \geq 1$ for all l. Now by applying Equation (6), we have

$$\begin{aligned} & \psi(\sigma(x_{2n_l+1}, Tv)) \\ = \ & \psi(\sigma(Sx_{2n}, Tv)) \\ \leq \ & \lambda\beta(\psi(M_{x_{2n_l},v})\phi(M_{x_{2n_l},v}) \\ < \ & \lambda\phi(M_{x_{2n_l},v}) \\ < \ & \lambda\psi(M_{x_{2n_l},v}) \end{aligned} \tag{29}$$

where

$$M_{x_{2n_l},v} = \max\{\sigma(x_{2n}, v), \sigma(x_{2n}, Sx_{2n}), \sigma(v, Tv), \sigma(Sx_{2n}, v), \sigma(x_{2n}, Tv)\}$$

Letting $l \to +\infty$ and using Equations (27) and (28), we have

$$\lim_{l\to\infty} M_{x_{2n_l},v} = \max\{\sigma(v, Sv), \sigma(v, Tv)\} \tag{30}$$

Case **I**: Assume that $\lim_{l\to\infty} M_{x_{2n_l},v} = \sigma(v, Tv)$.
From Equation (30) and letting $l \to \infty$ in Equation (29). Then, we have

$$\psi(\sigma(v, Tv)) < \lambda\psi(\sigma(v, Tv)).$$

Regarding the concept of ψ, we deduce that $\sigma(v, Tv) < \lambda\sigma(v, Tv)$ which is a discrepancy. Hence, we get that $\sigma(v, Tv) = 0$. As a result of (σ_1), we have $v = Tv$.

Case **II**: Assume that $\lim_{l\to\infty} M_{x_{2n_l},v} = \sigma(v, Sv)$. Then, arguing like above, we get $v = Sv$. Thus, $v = Sv = Tv$. Uniqueness of the fixed point is follows from the Theorem 4. This completes the proof. □

If we set $S = T$ and $M(x, y) = \max\{\sigma(x, y), \sigma(x, Tx), \sigma(y, Ty), \sigma(Tx, y), \sigma(x, Ty)\}$ in Theorems 4 and 5, then we obtain the following corollaries.

Corollary 1. *Let (X, σ) be a partially ordered metric-like space and $\alpha: X \times X \to [0, \infty)$ a function. Assume that $S: X \to X$ holds the following:*

(1) *there exists $\psi \in \Psi, \beta \in \mathfrak{S}$ and a continuous function $\phi : [0, \infty) \to [0, \infty)$ are continuous functions with $\phi(t) < \psi(t)$ for all $t > 0$ such that*

$$\alpha(x, y)\psi(\sigma(Sx, Sy)) \leq \lambda\beta(\psi(M_{x,y}))\phi(M_{x,y}), \tag{31}$$

holds for all comparable elements $x, y \in X$ and $0 \leq \lambda < 1$,

(2) *S is triangular α-admissible and there exists an $x_0 \in X$ such that $\alpha(x_0, Sx_0) \geq 1$,*

(3) *$Sx \preceq S(Sx)$ for all $x, y \in X$,*

(4) *T is σ-continuous mappings.*

Then, S has an unique fixed point $v \in X$ with $\sigma(v, v) = 0$.

Corollary 2. *Let (X, σ) be a partially ordered metric-like space and $\alpha : X \times X \to [0, \infty)$ a function. Assume that $S : X \to X$ holds the following:*

(1) *there exists $\psi \in \Psi, \beta \in \mathfrak{S}$ and a continuous function $\phi : [0, \infty) \to [0, \infty)$ are continuous functions with $\phi(t) < \psi(t)$ for all $t > 0$ such that*

$$\alpha(x, y)\psi(\sigma(Sx, Sy)) \leq \lambda\beta(\psi(M_{x,y}))\phi(M_{x,y}), \tag{32}$$

holds for all comparable elements $x, y \in X$ and $0 \leq \lambda < 1$,

(2) *S is triangular α-admissible and there exists an $x_0 \in X$ such that $\alpha(x_0, Sx_0) \geq 1$,*
(3) *$Sx \preceq S(Sx)$ for all $x, y \in X$,*
(4) *$(\mathcal{C})$ holds.*

Then, S has an unique fixed point $v \in X$ with $\sigma(v, v) = 0$.

If we take $\alpha(x, y) = 1$ in Theorems 4 and 5, we have the following corollaries.

Corollary 3. *Let (X, σ) be a partially ordered metric-like space. Assume $S, T : X \to X$ are two mappings holding the following:*

(1) *there exists $\psi \in \Psi, \beta \in \mathfrak{S}$ and a continuous function $\phi : [0, \infty) \to [0, \infty)$ are continuous functions with $\phi(t) < \psi(t)$ for all $t > 0$ such that*

$$\psi(\sigma(Sx, Ty)) \leq \lambda\beta(\psi(M_{x,y}))\phi(M_{x,y}), \tag{33}$$

holds for all comparable elements $x, y \in X$ and $0 \leq \lambda < 1$, where

$$M_{x,y} = \max\{\sigma(x, y), \sigma(x, Sx), \sigma(y, Ty), \sigma(Sx, y), \sigma(x, Ty)\}.$$

(2) *the pair (S, T) is weakly increasing,*
(3) *S and T are σ-continuous mappings.*

Then, the pair S, T has an unique common fixed point $v \in X$ with $\sigma(v, v) = 0$.

Corollary 4. *Let (X, σ) be a partially ordered metric-like space, Assume $S, T : X \to X$ are two mappings holding the following:*

(1) *there exists $\psi \in \Psi, \beta \in \mathfrak{S}$ and a continuous function $\phi : [0, \infty) \to [0, \infty)$ are continuous functions with $\phi(t) < \psi(t)$ for all $t > 0$ such that*

$$\psi(\sigma(Tx, Ty)) \leq \lambda\beta(\psi(M_{x,y}))\phi(M_{x,y}), \tag{34}$$

holds for all comparable elements $x, y \in X$ and $0 \leq \lambda < 1$, where

$$M_{x,y} = \max\{\sigma(x, y), \sigma(x, Sx), \sigma(y, Ty), \sigma(Sx, y), \sigma(x, Ty)\},$$

(2) *the pair (S, T) is weakly increasing,*
(3) *the pair (S, T) is a generalized (α, ψ, ϕ)-quasi contraction non-self,*
(4) *$(\mathcal{C})$ holds.*

Then, the pair S, T has an unique common fixed point $v \in X$ with $\sigma(v, v) = 0$.

4. Consequences

If we put $M_{x,y} = \sigma(x,y)$, then, by Theorems 4 and 5, we get the following corollaries as an expansion of results from the literature.

Corollary 5. *Let (X,σ) be a partially ordered metric like space and $\alpha:X \times X \to [0,\infty)$ be a function. Suppose that $S,T:X \to X$ are two self-mappings holding the following:*

(1) *(S,T) is triangular α-admissible and there exists an $x_0 \in X$ such that $\alpha(x_0, Sx_0) \geq 1$,*
(2) *there exists $\psi \in \Psi, \beta \in \mathfrak{S}$ and a continuous function $\phi : [0,\infty) \to [0,\infty)$ are continuous functions with $\phi(t) < \psi(t)$ for all $t > 0$ in order that*

$$\psi(\sigma(Sx,Ty)) \leq \lambda\beta(\psi(\sigma(x,y))\phi(\sigma(x,y)), \tag{35}$$

satisfies for $x,y \in X$ and $0 \leq \lambda < 1$,
(3) *the pair (S,T) is weakly increasing,*
(4) *the pair (S,T) is σ-continuous mappings.*

Then, the pair (S,T) has an unique common fixed point $v \in X$ with $\sigma(v,v) = 0$.

Corollary 6. *Let (X,σ) be a partially ordered metric-like space. Assume $S,T:X \to X$ are two mappings holding the following:*

(1) *(S,T) is triangular α-admissible and there exists an $x_0 \in X$ such that $\alpha(x_0, Sx_0) \geq 1$,*
(2) *there exists $\psi \in \Psi, \beta \in \mathfrak{S}$ and a continuous function $\phi : [0,\infty) \to [0,\infty)$ are continuous functions with $\phi(t) \leq \psi(t)$ for all $t > 0$ in order that*

$$\psi(\sigma(Sx,Ty)) \leq \lambda\beta(\psi(\sigma(x,y))\phi(\sigma(x,y)), \tag{36}$$

satisfies for $x,y \in X$ and $0 \leq \lambda \leq 1$,
(3) *the pair (S,T) is weakly increasing,*
(4) *$(\mathcal{C})$ holds.*

Then, the pair (S,T) has an unique common fixed point $v \in X$ with $\sigma(v,v) = 0$.

Corollary 7. *Let (X,σ) be a partially ordered metric-like space. Assume $\alpha:X \times X \to [0,\infty)$ is a function and $S:X \to X$ is a mapping holding the following:*

(1) *S is triangular α-admissible and there exists an $x_0 \in X$ such that $\alpha(x_0, Sx_0) \geq 1$.*
(2) *there exists $\psi \in \Psi, \beta \in \mathfrak{S}$ and a continuous function $\phi : [0,\infty) \to [0,\infty)$ are continuous functions with $\phi(t) < \psi(t)$ for all $t > 0$ in order that*

$$\alpha(x,y)\psi(\sigma(Sx,Sy)) \leq \lambda\beta(\psi(\sigma(x,y))\phi(\sigma(x,y)), \tag{37}$$

holds for all comparable elements $x,y \in X$ and $0 \leq \lambda < 1$,
(3) *$S \preceq S(Sx)$,*
(4) *the pair (S,T) is σ-continuous mappings.*

Then, S has an unique fixed point $v \in X$ with $\sigma(v,v) = 0$.

Corollary 8. *Let (X,σ) be a partially ordered metric-like space. Assume $\alpha:X \times X \to [0,\infty)$ is a function and $S:X \to X$ is a mapping holding the following:*

(1) *S is triangular α-admissible and there exists an $x_0 \in X$ such that $\alpha(x_0, Sx_0) \geq 1$,*

(2) *there exists $\psi \in \Psi, \beta \in \mathfrak{S}$ and a continuous function $\phi : [0,\infty) \to [0,\infty)$ are continuous functions with $\phi(t) < \psi(t)$ for all $t > 0$ in order that*

$$\alpha(x,y)\psi(\sigma(Sx,Sy)) \leq \lambda\beta(\psi(\sigma(x,y))\phi(\sigma(x,y)), \tag{38}$$

satisfies for $x, y \in X$ and $0 \leq \lambda < 1$,

(3) *$S \preceq S(Sx)$,*

(4) *$(\mathcal{C})$ holds.*

Then S has an unique fixed point $v \in X$ with $\sigma(v,v) = 0$.

Example 3. *Let $X = \{0,1,2\}$ and specify the partial order $\preceq$ on X in order that*

$$\preceq := \{(0,0),(1,1),(2,2),(0,2),(2,1),(0,1)\}.$$

Take into consideration that the function $S : X \to X$ specified as

$$S = \begin{pmatrix} 0 & 1 & 2 \\ 1 & 1 & 0 \end{pmatrix}, \tag{39}$$

which increasing with respect to $\preceq$. Let $x_0 = 0$. Hence, $S(x_0) = 1$ and $S(S(X_0)) = S(1) = 1$. Characterize to begin with the metric like space σ on X by $\sigma(0,1) = 1, \sigma(0,2) = \frac{5}{2}, \sigma(1,2) = \frac{3}{2}$ and $\sigma(x,x) = 0$. Then, (X,σ) is a complete metric-like space. Let $\beta \in \mathcal{S}$ is given by $\beta(t) = \frac{e^t}{2}$, $\psi(t) = t$, $\lambda = \frac{1}{2}$ and $\phi(t) = \frac{2}{3}t$. Define a function α:$X \times X \to [0,\infty)$ in order that

$$\alpha(x,y) = \begin{cases} 1 \text{ if } x \in \{0,1,2\} \\ 0 \text{ if } \text{ otherwise.} \end{cases}$$

Note that $S \in X$ and is continuous. S is α-admissible mapping. Indeed, $\alpha(Sx,Sy) = 1$. If $(x,y) = (0,1)$, then $\alpha(0,1) = 1$ and

$$\begin{aligned} M_{0,1} &= \max\{\sigma(0,1),\sigma(0,S0),\sigma(1,S1),\sigma(S0,1),\sigma(0,S1)\} \\ &= \max\{\sigma(0,1),\sigma(0,1),\sigma(1,1),\sigma(1,1),\sigma(0,1)\} \\ &= \max\{1,1,0,1,0\} = 1. \end{aligned}$$

$\sigma(S0,S1) = \sigma(1,1) = 0$. Now

$$0 = \alpha(0,1)\psi(\sigma(\sigma(S0,S1))) \leq \beta(\psi(M_{0,1}))\phi(M_{0,1}) = \frac{1}{2}\beta(1) \simeq \phi(1) = \frac{1}{2} \times \frac{e}{2} \times \frac{2}{3} = \frac{e}{6}$$

holds.

If $(x,y) = (0,2)$, then $\alpha(0,2) = 1$ and

$$\begin{aligned} M_{0,2} &= \max\{\sigma(0,2),\sigma(0,S0),\sigma(2,S2),\sigma(S0,2),\sigma(0,S2)\} \\ &= \max\{\sigma(0,2),\sigma(0,1),\sigma(2,0),\sigma(1,2),\sigma(0,0)\} \\ &= \max\{\frac{5}{2},1,\frac{5}{2},\frac{3}{2},0\} = \frac{5}{2}. \end{aligned}$$

$\sigma(S0,S2) = \sigma(1,0) = \frac{5}{2}$. Now

$$\frac{5}{2} = \alpha(0,2)\psi(\sigma(\sigma(S0,S2))) \leq \beta(\psi(M_{0,2}))\phi(M_{0,2}) = \frac{1}{2}\beta(\frac{e^{\frac{5}{2}}}{2}) \times \frac{2}{3} \times \frac{5}{2} = \frac{5e^{\frac{5}{2}}}{12}$$

holds. Similarly, for the case $(x = 1, y = 2)$, it is simple to examine that the contractive condition in Corollary 1 is satisfied.

All conditions (1)–(4) of Corollary 1 are satisfied. Hence S has a unique fixed point $x = 1$.

5. Application

The aim of this section is to give the existence of fixed points of an integral equation, where we can apply the obtained result of Corollary 1 to get a common solution.

We consider X with the partial order $\preceq$ presented by:

$$x \preceq y \Leftrightarrow x(t) \preceq y(t) \ \ for\, all\, t \in [0,1].$$

Let $X = C(I, \mathbb{R})$ be the set of continuous functions specified on $I = [0,1]$. The metric-like space $\sigma : X \times X \to [0, \infty)$ presented by

$$\sigma(x,y) = \sup_{t\in[0,1]} \mid x(t) - y(t) \mid,$$

for all $x, y \in X$. Since (X, σ) is a complete metric-like space. We consider the integral equation

$$x(t) = g(t) + \int_0^1 P(t,r) f(r, x(r)) dr; \ \ t \in [0,1] \tag{40}$$

for all $x \in X$.

We suppose that f:$[0,1] \times \mathbb{R} \to \mathbb{R}$ and g:$[0,1] \to \mathbb{R}$ are two continuous functions. Suppose that P:$[0,1] \times [0,1] \to [0, \infty)$ in order that

$$Sx(t) = g(t) + \int_0^1 P(t,r) f(r, x(r)) dr; \ \ t \in [0,1] \tag{41}$$

for all $x \in X$. Then, a solution of Equation (40) is a fixed point of S.

Now, We will prove the following Theorem with our obtained results.

Theorem 6. *Assume that the following conditions are satisfied:*

(i) *There exists ζ:$X \times X \to [0,1)$ such that for all $r \in [0,1]$ and for all $x, y \in X$*

$$0 \leq \mid f(r, x(r)) - f(r, y(r)) \mid \leq \zeta(x,y) \mid x(r) - y(r) \mid,$$

(ii) *there exists β:$[0, \infty) \to [0,1)$ such that*

$$\lim_{n\to\infty} \beta(t_n) = 1 \ \Rightarrow \ \lim_{n\to\infty} t_n = 0,$$

and

$$\| \int_0^1 P(t,r) \zeta(x,y) dr \|_\infty \leq (\frac{1}{4} \beta(\| x - y \|_\infty)).$$

Then the integral Equation (41) has a unique solution in X.

Proof. By conditions (i) and (ii), we get

$$\begin{aligned}
\mid S(x)(t) - S(y)(t) \mid &= \left| \int_0^1 P(t,r)[f(r,x(r)) - f(r,x(r))]dr \right| \\
&\leq \int_0^1 P(t,r) \mid f(r,x(r)) - f(r,y(r)) \mid dr \\
&\leq \int_0^1 P(t,r)\zeta(x,y) \mid f(r,x(r)) - f(r,y(r)) \mid dr \\
&\leq \int_0^1 P(t,r)\zeta(x,y) \parallel x - y \parallel_\infty dr \\
&\leq \sigma(x,y) \int_0^1 P(t,r)\zeta(x,y)dr \\
&\leq \frac{1}{4}\beta(\sigma(x,y))\sigma(x,y) \\
&= \frac{1}{2}\beta(\sigma(x,y))\frac{1}{2}\sigma(x,y) \\
&= \frac{1}{2}\beta(\sigma(x,y))\phi(\sigma(x,y)).
\end{aligned}$$

At that point, we have

$$\parallel S(x)(t) - S(y)(t) \parallel_\infty \leq \frac{1}{2}\beta(\sigma(x,y))\phi(\sigma(x,y)).$$

for all $x, y \in X$.

Thus, we obtain

$$\sigma(Sx, Sy) \leq \frac{1}{2}\beta(\sigma(x,y))\phi(\sigma(x,y)), \ forall x, y \in X.$$

Lastly, we specify $\beta : X \times X \to [0, \infty)$ such that

$$\alpha(x,y) = \begin{cases} 1 & \text{if } x, y \in X, \\ 0 & \text{if } otherwise. \end{cases}$$

Then, we have

$$\alpha(x,y)\sigma(Sx, Sy) \leq \frac{1}{2}\beta(\sigma(x,y))\sigma(x,y).$$

Obviously, $\alpha(x,y) = 1$ and $\alpha(Sx, Sy) = 1$ for all $x, y, z \in X$. Therefore, S is triangular $\alpha-$admissible mapping.

Hence, the hypotheses of Corollary 1 hold with $\psi(t) = t, \lambda = \frac{1}{2}$ and $\phi(t) = \frac{t}{2}$. Thus, S has a unique fixed point, that is, the integral Equation (40) has a unique solution in X. □

6. Conclusions

We have introduced some common fixed point results for generalized (α, ψ, ϕ)-quasi contraction self-mapping in partially ordered metric-like spaces. We have generalized weakly contractive mapping as we used quasi contraction self-mapping, α-admissible mapping, triangular α-admissible mapping and ψ, ϕ as strictly increasing and continuous functions. We have provided an example and application to show the superiority of our results over corresponding (common) fixed point results. Alternatively, we suggest finding new results by replacing the single-valued mapping with multi-valued mapping. Furthermore, we suggest generalizing more results in other spaces like b-metric space, metric-like space, and others. Otherwise, we suggest using our main results for non-self-mapping to establish the existence of an optimal approximate solution.

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