

Article

Unification Theories: Examples and Applications

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Abstract: We consider several unification problems in mathematics. We refer to transcendental numbers. Furthermore, we present some ways to unify the main non-associative algebras (Lie algebras and Jordan algebras) and associative algebras.

Keywords: transcendental numbers; Euler formula; Yang–Baxter equation; Jordan algebras; Lie algebras; associative algebras; coalgebras

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1. Introduction

Andre Weil explains (see [1]) that the areas of modern mathematics need to be unified in a simple and general theory. In this way, we will see more clearly which are the main problems in mathematics (as opposed to the less important themes and results).

In physics, unification theories have contributed to a better understanding of certain phenomena. The unified approaches suggested in this communication refer to the theory of functions and algebraic structures, two of the pillars of modern physical tools.

The purpose of this paper is three-fold: (i) we present new results on transcendental numbers; (ii) we obtain new results in the theory of the unification of non-associative structures; (iii) we re-initiate a debate on unification(s) in mathematics.

In the next section, we will give several examples of unification problems. This section is related to the paper [2] on transcendental numbers. Some knowledge of Hopf algebra theory is needed in order to understand some results from this section.

The third section presents structures that unify (non-)associative structures. The main non-associative structures are Lie algebras and Jordan algebras. Arguably less studied, Jordan algebras have applications in physics, differential geometry, ring geometries, quantum groups, analysis, biology, etc. (see [3]). There are several ways to unify Lie algebras, Jordan algebras and associative algebras. We will also refer to cases when the unification of (non-)associative structures could be realized just in the conclusions of theorems.

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All tensor products will be defined over the field k .

2. Examples of Unification Problems

In this section, we will give several examples of unification problems related to transcendental numbers.

The following two identities with transcendental numbers:

$$\int_{-\infty}^{+\infty} e^{-x^2} dx = \sqrt{\pi}, \quad (1)$$

$$\int_{-\infty}^{+\infty} e^{-ix^2} dx = \sqrt{\frac{\pi}{2}}(1-i), \quad (2)$$

were unified and proven (by contour integration) in [4].

The next formulas (from [2]) can be also unified:

$$e^{\pi i} + 1 = 0, \quad (3)$$

$$|e^i - \pi| > e \quad (4)$$

and:

$$|e^{1-z} + e^{\bar{z}}| > \pi \quad \forall z \in \mathbb{C}. \quad (5)$$

Remark 1. Let us consider the two-variable complex function $f : \mathbb{C} \times \mathbb{C} \rightarrow \mathbb{R}$, $f(z, w) = |e^z + e^w|$, which gives the length of the sum of the vectors e^z and e^w . The Formulas (3)–(5) can be unified using the function $f(z, w)$:

$$f(\pi i, 0) = 0, \quad f(1 - z, \bar{z}) > \pi \quad \forall z \in \mathbb{C}, \quad f(i, \pi i + \ln \pi) > e.$$

Remark 2. The function $f(z, w)$ can be expressed in another form using the formula $|e^{x+i\alpha} + e^{y+i\beta}| = \rho \sqrt{1 + \sin(2\theta) \cos(\alpha - \beta)}$, where $\rho = \sqrt{e^{2x} + e^{2y}}$ and $\theta = \cos^{-1} \left(\frac{e^x}{\rho} \right)$. The relations from Remark 1 can be reinterpreted using this formula. For example, in the first formula: $\rho = \sqrt{2}$, $\theta = \frac{\pi}{4}$, $\alpha = \pi$, $\beta = 0$; so, $f(\pi i, 0) = \sqrt{2} \sqrt{1 + \sin(\frac{\pi}{2}) \cos(\pi)} = 0$.

Remark 3. While properties about the image of the function $f(z, w)$ unify the Formulas (3)–(5), the formula $e^{x+iy} = e^x(\cos y + i \sin y)$ could be considered a common part (or an essential part) of all of them.

This latest formula can be related to a certain subcoalgebra of the trigonometric coalgebra (see [5]). Indeed, the properties of the trigonometric functions \cos and \sin lead to the trigonometric coalgebra, given by the maps: $\Delta(c) = c \otimes c - s \otimes s$, $\Delta(s) = s \otimes c + c \otimes s$, $\varepsilon(c) = 1$, $\varepsilon(s) = 0$. Euler's relation leads to the subcoalgebra generated by $c + is$.

The dual case states that $1 + ix$ generates an ideal in the \mathbb{C} algebra $\frac{\mathbb{C}[X]}{X^2+1} = \mathbb{C}[x]$, where $x^2 = -1$. In other words, Euler's relation implies that $\forall a, b \in \mathbb{C}$, there exists $c \in \mathbb{C}$ such that $(a + bx)(1 + ix) = c(1 + ix)$ (this can be checked directly).

According to [5], the role of such objects in number theory is unexplored at the moment.

Remark 4. The properties of the hyperbolic functions \cosh and \sinh lead to the following coalgebra, given by the maps: $\Delta(c) = c \otimes c + s \otimes s$, $\Delta(s) = s \otimes c + c \otimes s$, $\varepsilon(c) = 1$, $\varepsilon(s) = 0$. There exists a subcoalgebra generated by $c + s$, which can be related to Theorem 1 of [2], leading to some kind of "Euler formula" for hyperbolic functions.

Remark 5. The coalgebras from Remarks 3 and 4 can be unified as follows. For $a \in k$, we consider the coalgebra generated by c and s , $\Delta(c) = c \otimes c + a^2 s \otimes s$, $\Delta(s) = s \otimes c + c \otimes s$, $\varepsilon(c) = 1$, $\varepsilon(s) = 0$. There exists a subcoalgebra generated by $c + as$.

Remark 6. The following inequalities hold (see also [2]): $\pi > |e^i - \pi| > e$.

The last inequality was proven in [2]: $|e^i - \pi| > e \iff (\pi + e)(\pi - e) + 1 > 2\pi \cos 1$. Now, $\pi > 3.141$, $e < 2.719$; so, $(\pi + e)(\pi - e) + 1 > 3.47292$ and $\cos 1 < 1 - \frac{1}{2} + \frac{1}{4!} = \frac{13}{24}$, $2\pi \cos 1 < 3.142 \times \frac{13}{12} = 3.4038(3)$.

The inequality $\pi > |e^i - \pi|$ is equivalent to $2\pi \cos 1 > 1$, which follows from $\cos 1 > \cos 60^\circ = \frac{1}{2}$.

Remark 7. The next formula generalizes the Basel problem $\lim_{n \rightarrow \infty} \sum_{k=1}^n \frac{1}{k^2} = \frac{\pi^2}{6}$:

$$\sum_{k=1}^n \frac{1}{k^2} < \frac{2}{3} \left(\frac{n+1}{n} \right)^n \quad \forall n \in \mathbb{N}^*. \quad (6)$$

It could be an interesting problem to prove a similar formula for non-associative algebras. Likewise, one could try to generalize it for q -shifted factorials (see, for example, [6]).

3. The Unification of Non-Associative Structures

3.1. UJLA Structures

The UJLA (Unification of Jordan, Lie (and) Associative (algebras)) structures could be seen as structures that comprise the information encapsulated in associative algebras, Lie algebras and Jordan algebras. Thus, the category of the UJLA structures can be seen as a category that “includes” the categories of Jordan, Lie and associative algebras. A motivation for this unification is related to the study of bundles over Grassmannian manifolds.

Definition 1. For a k -space V , let $\eta : V \otimes V \rightarrow V$, $a \otimes b \mapsto ab$, be a linear map such that:

$$(ab)c + (bc)a + (ca)b = a(bc) + b(ca) + c(ab), \quad (7)$$

$$(a^2b)a = a^2(ba), \quad (ab)a^2 = a(ba^2), \quad (ba^2)a = (ba)a^2, \quad a^2(ab) = a(a^2b), \quad (8)$$

$\forall a, b, c \in V$. Then, (V, η) is called a UJLA structure.

Remark 8. If (A, θ) , where $\theta : A \otimes A \rightarrow A$, $\theta(a \otimes b) = ab$, is a (non-unital) associative algebra, then we define a UJLA structure (A, θ') , where $\theta'(a \otimes b) = \alpha ab + \beta ba$, for some $\alpha, \beta \in k$. For $\alpha = \beta = \frac{1}{2}$, (A, θ') is a Jordan algebra, and for $\alpha = 1 = -\beta$, (A, θ') is a Lie algebra.

Theorem 1. (Nichita [7]) Let (V, η) be a UJLA structure. Then, (V, η') , $\eta'(a \otimes b) = [a, b] = ab - ba$ is a Lie algebra.

Theorem 2. (Nichita [7]) Let (V, η) be a UJLA structure. Then, (V, η') , $\eta'(a \otimes b) = a \circ b = \frac{1}{2}(ab + ba)$ is a Jordan algebra.

Remark 9. The structures from the two above theorems are related by the relation:

$$[a, b \circ c] + [b, c \circ a] + [c, a \circ b] = 0.$$

Remark 10. The classification of UJLA structures is an open problem.

Remark 11. If the characteristic of k is two, then a Lie algebra is also a Jordan algebra.

Proof. Because the characteristic of k is two, the Lie algebra L is also commutative.

It follows easily that $[[x, x], x] = 0 \quad \forall x \in L$.

Now, in the Jacobi identity, we take $z = x^2$: $[[x, y], x^2] + [[y, x^2], x] + [[x^2, x], y] = 0$.

From the above observations, it follows that $[[x, y], x^2] = [x, [y, x^2]]$. Therefore, L is also a Jordan algebra. \square

3.2. Yang–Baxter Equations

The authors of [8] argued that the Yang–Baxter equation leads to another unification of (non-)associative structures.

For V a k -space, we denote by $\tau : V \otimes V \rightarrow V \otimes V$ the twist map defined by $\tau(v \otimes w) = w \otimes v$ and by $I : V \rightarrow V$ the identity map of the space V ; for $R : V \otimes V \rightarrow V \otimes V$ a k -linear map, let $R^{12} = R \otimes I$, $R^{23} = I \otimes R$, $R^{13} = (I \otimes \tau)(R \otimes I)(I \otimes \tau)$.

Definition 2. A The Yang–Baxter operator is an invertible k -linear map, $R : V \otimes V \rightarrow V \otimes V$, which satisfies the braid condition (sometimes called the Yang–Baxter equation):

$$R^{12} \circ R^{23} \circ R^{12} = R^{23} \circ R^{12} \circ R^{23}. \quad (9)$$

If R satisfies (9), then both $R \circ \tau$ and $\tau \circ R$ satisfy the quantum Yang–Baxter equation (QYBE):

$$R^{12} \circ R^{13} \circ R^{23} = R^{23} \circ R^{13} \circ R^{12}. \quad (10)$$

Therefore, Equations (9) and (10) are equivalent.

Let A be a (unitary) associative k -algebra, and $\alpha, \beta, \gamma \in k$; the authors of [9] defined the k -linear map $R_{\alpha, \beta, \gamma}^A : A \otimes A \rightarrow A \otimes A$,

$$a \otimes b \mapsto \alpha ab \otimes 1 + \beta 1 \otimes ab - \gamma a \otimes b \quad (11)$$

which is a Yang–Baxter operator if and only if one of the following cases holds:

- (i) $\alpha = \gamma \neq 0, \beta \neq 0$;
- (ii) $\beta = \gamma \neq 0, \alpha \neq 0$;
- (iii) $\alpha = \beta = 0, \gamma \neq 0$.

An interesting property of (11), can be visualized in knot theory, where the link invariant associated with $R_{\alpha, \beta, \gamma}^A$ is the Alexander polynomial.

For $(L, [,]) a Lie algebra over $k, z \in Z(L) = \{z \in L : [z, x] = 0 \forall x \in L\}$, and $\alpha \in k$, the authors of the papers [10,11] defined the following Yang–Baxter operator: $\phi_{\alpha}^L : L \otimes L \rightarrow L \otimes L$,$

$$x \otimes y \mapsto \alpha [x, y] \otimes z + y \otimes x. \quad (12)$$

Remark 12. The Formulas (11) and (12) lead to the unification of associative algebras and Lie algebras in the framework of Yang–Baxter structures. At this moment, we do not have a satisfactory answer to the question how Jordan algebras fit in this framework (several partial answers were given).

3.3. Unification of the Conclusions of Theorems

Sometimes, it is not easy to find structures that unify theorems for (non-)associative structures, but we could unify just the conclusions of theorems, as we will see in the next theorems.

Theorem 3. If A is a Jordan algebra, a Lie algebra or an associative algebra and if $a, b \in A$, then:

$$D : A \rightarrow A, D(x) = a(bx) + b(ax) + (ax)b - a(xb) - (xb)a - (xa)b$$

is a derivation.

Proof. We consider three cases. If A is a Jordan algebra, then $D(x) = a(bx) + b(ax) + (ax)b - a(xb) - (xb)a - (xa)b = a(bx) - (xa)b = a(bx) - b(ax)$. According to [12], D is a derivation.

If A is a Lie algebra, then $D(x) = a(bx) + b(ax) + (ax)b - a(xb) - (xb)a - (xa)b = a(bx) - b(ax) = a(bx) + b(xa) = (ab)x$. Therefore, D is a derivation.

If A is an associative algebra, then $D(x) = a(bx) + b(ax) + (ax)b - a(xb) - (xb)a - (xa)b = (ab + ba)x - x(ab + ba)$. Therefore, D is a derivation. \square

Theorem 4. If A is a Jordan algebra, a Lie algebra or an associative algebra and if $a, b \in A$, then $D : A \rightarrow A$, $D(x) = a(bx) - (xa)b$ is a derivation.

Proof. The proof follows the same argumentation as the previous one, for which reason it is omitted. \square

Remark 13. We presented unifications of formulas, structures, categories and theorems. It is currently an open problem whether these types can be formulated in a unified form.

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