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Extended Partial S_b-Metric Spaces

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Abstract: In this paper, we introduce the concept of extended partial S_b-metric spaces, which is a generalization of the extended S_b-metric spaces. Basically, in the triangle inequality, we add a control function with some very interesting properties. These new metric spaces generalize many results in the literature. Moreover, we prove some fixed point theorems under some different contractions, and some examples are given to illustrate our results.

Keywords: extended partial S_b-metric spaces; S_b-metric spaces; fixed point

1. Introduction

Fixed point theory has become the focus of many researchers lately due its applications in many fields see [1–12]). The concept of b-metric space was introduced by Bakhtin [13], which is a generalization of metric spaces.

Definition 1 ([13]). *A b-metric on a non empty set X is a function $d : X^2 \to [0,\infty)$ such that, for all $x, y, z \in X$ and $k \geq 1$, the following three conditions are satisfied:*
(i) $d(x,y) = 0$ if and only if $x = y$,
(ii) $d(x,y) = d(y,x)$,
(iii) $d(x,y) \leq k[d(x,z) + d(z,y)]$.
As usual, the pair (X,d) is called a b-metric spaces.

A three-dimensional metric space was introduced by Sedghi et al. [14], and it is called S-metric spaces. Later, and as a generalization of the S-metric spaces, S_b-metric spaces were introduced. In [15], extended S_b-metric spaces were introduced

Definition 2 ([15]). *Let X be a non empty set and a function $\theta : X^3 \to [1,\infty)$. If the function $S_\theta : X^3 \to [0,\infty)$ satisfies the following conditions for all $x, y, z, t \in X$:*

1. $S_\theta(x,y,z) = 0$ *implies* $x = y = z$;
2. $S_\theta(x,y,z) \leq \theta(x,y,z)[S_\theta(x,x,t) + S_\theta(y,y,t) + S_\theta(z,z,t)]$,

then the pair (X, S_θ) is called an extended S_b-metric spaces.

First, note that, if $\theta(x,y,z) = s \geq 1$, then we have an S_b-metric spaces, which leads us to conclude that every S_b-metric spaces is an extended S_b-metric spaces, but the converse is not always true.

Definition 3 ([15]). *Let (X, S_θ) be an extended S_b-metric space. Then,*

(i) A sequence $\{x_n\}$ is called convergent if and only if there exists $z \in X$ such that $S_\theta(x_n, x_n, z)$ goes to 0 as n goes toward ∞. In this case, we write $\lim_{n\longrightarrow\infty} x_n = z$.

(ii) A sequence $\{x_n\}$ is called a Cauchy sequence if and only if $S_\theta(x_n, x_n, x_m)$ goes to 0 as n, m goes toward ∞.

(iii) (X, S_θ) *is said to be a complete extended S_b-metric space if every Cauchy sequence $\{x_n\}$ converges to a point $x \in X$*
(iv) *Define the diameter of a subset Y of X by*

$$diam(Y) := Sup\{S_\theta(x,y,z) \mid x,y,z \in Y\}.$$

In the extended S_b-metric spaces, we define a ball as follows:

$$B(x,\epsilon) = \{y \in X \mid S_\theta(x,x,y) \leq \epsilon\}.$$

Next, we present some examples of extended S_b-metric spaces.

Example 1. *Let $X = C([a,b], (-\infty,\infty))$ be the set of all continuous real valued functions on $[a,b]$. Define*

$$S_\theta : X^3 \to [0,\infty);\ S_\theta(x(t),y(t),z(t)) = \sup_{t\in[a,b]} |\max\{x(t),y(t)\} - z(t)|^2,$$

and

$$\theta : X^3 \to [1,\infty);\ \theta(x(t),y(t),z(t)) = \max\{|x(t)|,|y(t)|\} + |z(t)| + 1.$$

It is not difficult to see that (X, S_θ) is a complete extended S_b-metric spaces.

2. Extended Partial S_b-metric spaces

In this section, as a generalization of the extended S_b-metric spaces, we introduce extended partial S_b-metric spaces, along with its topology.

Definition 4. *Let X be a non empty set and a function $\theta : X^3 \to [1,\infty)$. If the function $S_\theta : X^3 \to [0,\infty)$ satisfies the following conditions for all $x,y,z,t \in X$:*

1. $x = y = z$ *if and only if* $S_\theta(x,y,z) = S_\theta(x,x,x) = S_\theta(y,y,y) = S_\theta(z,z,z)$;
2. $S_\theta(x,x,x) \leq S_\theta(x,y,z)$,
3. $S_\theta(x,y,z) \leq \theta(x,y,z)[S_\theta(x,x,t) + S_\theta(y,y,t) + S_\theta(z,z,t)]$,

then the pair (X, S_θ) is called an extended partial S_b-metric spaces.

First, note that, if $\theta(x,y,z) = s \geq 1$, then we have a partial S_b-metric spaces, which leads us to conclude that every S_b-metric spaces is an extended S_b-metric spaces, but the converse is not always true.

Definition 5. *Let (X, S_θ) be a extended partial S_b-metric space. Then,*

- *A sequence $\{x_n\}$ is called convergent if and only if there exists $z \in X$ such that $S_\theta(x_n,x_n,z)$ goes to $S_\theta(z,z,z)$ as n goes toward ∞. In this case, we write $\lim_{n\longrightarrow\infty} x_n = z$.*
- *A sequence $\{x_n\}_{n=0}^{\infty}$ of elements in X is called S_θ-Cauchy if $\lim_{n,m} S_\theta(x_n,x_n,x_m)$ exists and is finite.*
- *The extended partial S_θ-metric spaces (X, S_θ) is called complete if, for each S_θ-Cauchy sequence $\{x_n\}_{n=0}^{\infty}$, there exists $z \in X$ such that*

$$S_\theta(z,z,z) = \lim_n S_\theta(z,z,x_n) = \lim_{n,m} S_\theta(x_n,x_n,x_m).$$

- *A sequence $\{x_n\}_n$ in an extended partial S_b-metric spaces (X, S_θ) is called 0-Cauchy if*

$$\lim_{n,m} S_\theta(x_n,x_n,x_m) = 0.$$

- *We say that (X, S_θ) is 0-complete if every 0-Cauchy in X converges to a point $x \in X$ such that $S_\theta(x,x,x) = 0$.*

Note that every extended S_b-metric spaces is an extended partial S_b-metric spaces, but the converse is not always true. The following example is an example of an extended partial S_b-metric spaces which is not extended S_b-metric spaces.

Example 2. *Let $X = C([a,b],(-\infty,\infty))$ be the set of all continuous real valued functions on $[a,b]$. Define*

$$S_\theta : X^3 \to [0,\infty);\ S_\theta(x(t),y(t),z(t)) = \sup_{t\in[a,b]} |\max\{x(t),y(t),z(t)\}|^2,$$

and

$$\theta : X^3 \to [1,\infty);\ \theta(x(t),y(t),z(t)) = \max\{|x(t)|,|y(t)|\} + |z(t)| + 1.$$

First, note that, for all $x(t), y(t), z(t)$ and $f(t) \in X$, we have

$$\begin{aligned} S_\theta(x(t),y(t),z(t)) &= \sup_{t\in[a,b]} |\max\{x(t),y(t),z(t)\}|^2 \\ &\le \sup_{t\in[a,b]} |\max\{x(t),x(t),f(t)\}|^2 + \sup_{t\in[a,b]} |\max\{y(t),y(t),f(t)\}|^2 \\ &+ \sup_{t\in[a,b]} |\max\{z(t),z(t),f(t)\}|^2 \\ &\le S_\theta(x(t),x(t),f(t)) + S_\theta(y(t),y(t),f(t)) + S_\theta(z(t),z(t),f(t)) \\ &\le \theta(x(t),y(t),z(t))[S_\theta(x(t),x(t),f(t)) + S_\theta(y(t),y(t),f(t)) + S_\theta(z(t),z(t),f(t))]. \end{aligned}$$

In the last inequality, we used the fact that, for all $x(t), y(t), z(t) \in X$, we have $\theta(x(t),y(t),z(t)) \ge 1$, Hence, (X, S_θ) is an extended partial S_b-metric spaces, but it is not an extended S_b-metric spaces, since the self distance is not zero.

In the extended partial S_b-metric spaces, we define a ball as follows:

$$B(x,\epsilon) = \{y \in X \mid S_\theta(x,x,y) \le \epsilon\}.$$

Definition 6. *An extended partial S_b-metric spaces (X, S_θ) is said to be symmetric if it satisfies the following condition:*

$$S_\theta(x,x,y) = S_\theta(y,y,x) \text{ for all } x,y \in X.$$

Theorem 1. *Let (X, S_θ) be a complete symmetric extended partial S_b-metric spaces such that S_θ is continuous, and let T be a continuous self mapping on X satisfying the following condition:*

$$S_\theta(Tx,Ty,Tz) \le kS_\theta(x,y,z) \text{ for all } x,y,z \in X,$$

where $0 < k < 1$ and for every $x_0 \in X$ we have $\lim_{n,m\to\infty} \theta(T^n x, T^n x, T^m) < \frac{1}{k}$. Then, T has a unique fixed point say u. In addition, for every $y \in X$, we have $\lim_{n\to\infty} T^n y = u$.

Proof. Since X is a nonempty set, pick $x_0 \in X$ and define the sequence $\{x_n\}$ as follows:

$$x_1 = Tx_0, x_2 = Tx_1 = T^2x_0, \cdots, x_n = T^n x_0, \cdots$$

Note that, by (1), we have

$$S_\theta(x_n,x_n,x_{n+1}) \le kS_\theta(x_{n-1},x_{n-1},x_n) \le \cdots \le k^n S_\theta(x_0,x_0,x_1).$$

Now, pick two natural numbers $n < m$. Hence, by the triangle inequality of the extended partial S_b-metric space, we deduce

$$\begin{aligned}S_\theta(x_n, x_n, x_m) &\le \theta(x_n, x_n, x_m)2k^n S_\theta(x_0, x_0, x_1) + \theta(x_n, x_n, x_m)\theta(x_{n+1}, x_{n+1}, x_m)2k^{n+1} S_\theta(x_0, x_0, x_1)\\ &+ \cdots + \theta(x_n, x_n, x_m)\cdots\theta(x_{m-1}, x_{m-1}, x_m)2k^{m-1} S_\theta(x_0, x_0, x_1)\\ &\le S_\theta(x_0, x_0, x_1)[\theta(x_1, x_1, x_m)\theta(x_2, x_2, x_m)\cdots\theta(x_{n-1}, x_{n-1}, x_m)\theta(x_n, x_n, x_m)2k^n\\ &+ \theta(x_1, x_1, x_m)\theta(x_2, x_2, x_m)\cdots\theta(x_n, x_n, x_m)\theta(x_{n+1}, x_{n+1}, x_m)2k^{n+1}\\ &+ \cdots + \theta(x_1, x_1, x_m)\theta(x_2, x_2, x_m)\cdots\theta(x_{m-2}, x_{m-2}, x_m)\theta(x_{m-1}, x_{m-1}, x_m)2k^{m-1}].\end{aligned}$$

By the hypothesis of the theorem, we have

$$\lim_{n,m\to\infty} \theta(x_n, x_n, x_m)(k) < 1.$$

Therefore, by the Ratio test, the series $\sum_{n=1}^{\infty} 2k^n \prod_{i=1}^{n} \theta(x_i, x_i, x_m)$ converges. Now, let

$$A = \sum_{n=1}^{\infty} 2k^n \prod_{i=1}^{n} \theta(x_i, x_i, x_m) \text{ and } A_n = \sum_{j=1}^{n} 2k^j \prod_{i=1}^{j} \theta(x_i, x_i, x_m).$$

Hence, for $m > n$, we deduce that

$$S_\theta(x_n, x_n, x_m) \le S_\theta(x_0, x_0, x_1)[A_{m-1} - A_{n-1}].$$

Taking the limit as $n, m \to \infty$, we conclude that $\{x_n\}$ is a Cauchy sequence. Since X is complete, $\{x_n\}$ converges to some $u \in X$, such that

$$S_\theta(u, u, u) = \lim_{n\to\infty} S_\theta(u, u, x_n) = \lim_{n,m\to\infty} S_\theta(x_n, x_n, x_m).$$

Now, using the fact that T and S_θ are continuous, we deduce that

$$S_\theta(x_{n+1}, x_{n+1}, x_{n+1}) = S_\theta(x_{n+1}, x_{n+1}, Tx_n) = S_\theta(Tx_n, Tx_n, Tx_n).$$

Taking the limit in the above equalities, we can easily conclude that $Tu = u$. Hence, u is a fixed point of T. To show uniqueness, assume that there exists $v \ne u \in X$ such that $Tu = u$ and $Tv = v$. Thus,

$$\begin{aligned}S_\theta(u, u, v) &= S_\theta(Tu, Tu, Tv)\\ &\le kS_\theta(u, u, v)\\ &< S_\theta(u, u, v),\end{aligned}$$

which leads us to a contradiction. Therefore, T has a unique fixed point in X as desired. □

Now, we present the following example as an application of Theorem 1.

Example 3. *Let $X = C([a, b], (-\infty, \infty))$ be the set of all continuous real valued functions on $[a, b]$. Define*

$$S_\theta : X^3 \to [0, \infty);\ S_\theta(x(t), y(t), z(t)) = \sup_{t\in[a,b]} |\max\{x(t), y(t), z(t)\}|^2,$$

and

$$\theta : X^3 \to [1, \infty);\ \theta(x(t), y(t), z(t)) = \max\{|x(t)|, |y(t)|\} + |z(t)| + 1.$$

Now, let T be a self mapping on X defined by

$$Tx = \frac{x}{2}\,.$$

Note that

$$S_\theta(Tx, Ty, Tz) \leq \frac{1}{3} S_\theta(x, y, z)\,.$$

In this case, we have $k = \frac{1}{3}$. On the other hand, it is not difficult to see that, for every $x \in X$, we have

$$T^n x = \frac{x}{2^n}.$$

Thus, it is not difficult to see that

$$\lim_{n,m\to\infty} \theta(T^n x, T^n x, T^m x) < \frac{3}{2}.$$

Therefore, all the conditions of Theorem 1 are satisfied and hence T has a unique fixed point which is in this case 0.

Theorem 2. *Let (X, S_θ) be a symmetric complete extended partial S_b-metric spaces such that S_θ is continuous and T be a continuous self mapping on X satisfying*

$$S_\theta(Tx, Ty, Tz) \leq \psi[S_\theta(x, y, z)] \textit{ for all } x, y, z \in X,$$

where ψ is a comparison function (i.e., $\psi : [0, +\infty) \longrightarrow [0, +\infty)$ is an increasing function such that $\lim_{n\to\infty} \psi^n(t) = 0$ for each fixed $t > 0$.) In addition, assume that there exists $s > 1$ such that, for every $x_0 \in X$ and $x \in X$, we have

$$\lim_{n\to\infty} \theta(x_n, x_n, x) < \frac{s}{2}.$$

Then, T has a unique fixed point in X.

Proof. Let $x \in X$ and $\epsilon > 0$. Let n be a natural number such that $\psi^n(\epsilon) < \dfrac{\epsilon}{2s}$.

Let $F = T^n$ and $x_k = F^k(x)$ for $k \in \mathbb{N}$. Then, for $x, y \in X$ and $\alpha = \psi^n$, we have

$$\begin{aligned} S_\theta(Fx, Fx, Fy) &\leq \psi^n(S_\theta(x, x, y)) \\ &= \alpha(S_\theta(x, x, y)). \end{aligned}$$

Hence, for $k \in \mathbb{N}$ $S_\theta(x_{k+1}, x_{k+1}, x_k)$ goes to 0 as k goes toward ∞. Therefore, let k be such that

$$S_\theta(x_{k+1}, x_{k+1}, x_k) < \frac{\epsilon}{2s}.$$

Note that $x_k \in B(x_k, \epsilon)$. Therefore, $B(x_k, \epsilon) \neq \emptyset$. Hence, for all $z \in B(x_k, \epsilon)$, we have

$$\begin{aligned} S_\theta(Fz, Fz, Fx_k) &\leq \alpha(S_\theta(x_k, x_k, z)) \\ &\leq \alpha(\epsilon) = \psi^n(\epsilon) < \frac{\epsilon}{2s} < \frac{\epsilon}{s}. \end{aligned}$$

Since $S_\theta(Fx_k, Fx_k, Fx_k) = S_\theta(x_{k+1}, x_{k+1}, x_k) < \dfrac{\epsilon}{2s}$, thus

$$\begin{aligned} S_\theta(x_k, x_k, Fz) &\leq \theta(x_k, x_k, Fz)[S_\theta(x_k, x_k, x_{k+1}) + S_\theta(x_k, x_k, x_{k+1}) + S_\theta(Fz, Fz, x_{k+1})] \\ &= \theta(x_k, x_k, Fz)[2S_\theta(x_k, x_k, x_{k+1}) + S_\theta(Fz, Fz, x_{k+1})] \\ &\leq \theta(x_k, x_k, Fz)[2\frac{\epsilon}{2s} + \frac{\epsilon}{s}]. \end{aligned}$$

Now, taking the limit of the above inequality as $k \to \infty$, we get

$$S_\theta(x_k, x_k, F_z) \le \epsilon.$$

Hence, F maps $B(x_k, \epsilon)$ to itself. Since $x_k \in B(x_k, \epsilon)$, we have $Fx_k \in B(x_k, \epsilon)$. By repeating this process, we get

$$F^m_{x_k} \in B(x_k, \epsilon) \text{ for all } m \in \mathbb{N}.$$

That is, $x_l \in B(x_k, \epsilon)$ for all $l \ge k$. Hence,

$$S_\theta(x_m, x_m, x_l) < \epsilon \text{ for all } m, l > k.$$

Therefore, $\{x_k\}$ is a Cauchy sequence and, by the completeness of X, there exists $u \in X$ such that x_k converges to u as k goes toward ∞. Moreover, $u = \lim_{k\to\infty} x_{k+1} = \lim_{k\to\infty} x_k = F(u)$.

Thus, F has u as a fixed point. We prove now the uniqueness of the fixed point for F. Since $\alpha(t) = \psi^n(t) < t$ for any $t > 0$, let u and u_1 be two fixed points of F:

$$\begin{aligned} S_\theta(u, u, u_1) &= S_\theta(F_u, F_u, F_{u_1}) \\ &\le \psi^n(S_\theta(u, u, u_1)) \\ &= \alpha(S_\theta(u, u, u_1)) \\ &< S_\theta(u, u, u_1). \end{aligned}$$

Thus, $S_\theta(u, u, u_1) = 0$, that is $u = u_1$ and hence, F has a unique fixed point in X. On the other hand, $T^{nk+r}(x) = F^k(T^r(x))$ goes to u as k goes toward ∞. Hence, $T^m x$ goes to u as m goes toward ∞ for every x. That is $u = \lim_{m\to\infty} Tx_m = T(u)$. Therefore, T has a fixed point. □

Theorem 3. *Let (X, S_θ) be a complete symmetric extended partial S_b-metric spaces such that S_θ is continuous, and let T be a continuous self mapping on X satisfying the following condition:*

$$S_\theta(Tx, Ty, Tz) \le \lambda[S_\theta(x, x, Tx) + S_\theta(y, y, Ty) + S_\theta(z, z, Tz)] \text{ for all } x, y, z \in X, \tag{1}$$

where $\lambda \in [0, \frac{1}{3})$, $\lambda \ne \dfrac{1}{3\theta(T^n x, T^n x, T^m)}$ for every $x_0 \in X$. Then, T has a unique fixed point $u \in X$ and $S_\theta(u, u, u) = 0$.

Proof. We first prove that if T has a fixed point, then it is unique. We must show that, if $u \in X$ is a fixed point of T, that is, $Tu = u$ then $S_\theta(u, u, u) = 0$.
From (1), we obtain

$$\begin{aligned} S_\theta(u, u, u) = S_\theta(Tu, Tu, Tu) &\le \lambda[S_\theta(u, u, Tu) + S_\theta(u, u, Tu) + S_\theta(u, u, Tu)] \\ &= 3\lambda S_\theta(u, u, Tu) \text{ since } \lambda \in [0, \tfrac{1}{3}), \text{ we have} \\ &< S_\theta(u, u, u), \end{aligned}$$

which implies that we must have $S_\theta(u, u, u) = 0$ Suppose $u, v \in X$ be two fixed points, that is, $Tu = u$ and $Tv = v$. Then, we have $S_\theta(u, u, u) = S_\theta(v, v, v) = 0$. Relation (1) gives

$$\begin{aligned} S_\theta(u, u, v) &= S_\theta(Tu, Tu, Tv) \\ &\le \lambda[S_\theta(u, u, Tu) + S_\theta(u, u, Tu) + S_\theta(v, v, Tv)] \\ &= 2\lambda S_\theta(u, u, u) + \lambda S_\theta(v, v, v) \\ &= 0. \end{aligned}$$

Therefore, $u = v$. Thereby, we get the uniqueness of the fixed point if it exists. For the existence of the fixed point, let $x_0 \in X$ arbitrary, set $x_n = T^n x_0$ and $S_{b_n} = S_\theta(x_n, x_n, x_{n+1})$. We can assume $S_{b_n} > 0$ for all $n \in \mathbb{N}$; otherwise, x_n is a fixed point of T for at least one $n \geq 0$. For all n, we obtain from (1)

$$\begin{aligned} S_{b_n} &= S_\theta(x_n, x_n, x_{n+1}) = S_\theta(Tx_{n-1}, Tx_{n-1}, Tx_n) \\ &\leq \lambda[2S_\theta(x_{n-1}, x_{n-1}, Tx_{n-1}) + S_\theta(x_n, x_n, Tx_n)] \\ &= \lambda[2S_\theta(x_{n-1}, x_{n-1}, x_n) + S_\theta(x_n, x_n, x_{n+1})] \\ &= \lambda[2S_{b_{n-1}} + S_{b_n}]. \end{aligned}$$

Therefore, $(1-\lambda)S_{b_n} \leq 2\lambda S_{b_{n-1}}$. Thus,

$$S_{b_n} \leq \frac{2\lambda}{1-\lambda} S_{b_{n-1}}, \quad \lambda \in [0, \frac{1}{3}). \tag{2}$$

Let $\beta = \dfrac{2\lambda}{1-\lambda} < 1$. By repeating this process, we obtain

$$S_{b_n} \leq \beta^n b_0.$$

Therefore, $\lim_{n\to\infty} S_{b_n} = 0$. Let us prove that $\{x_n\}$ is a Cauchy sequence. It follows from (1) that, for $n, m \in \mathbb{N}$:

$$\begin{aligned} S_\theta(x_n, x_n, x_m) &= S_\theta(Tx_{n-1}, Tx_{n-1}, Tx_{m-1}) \\ &\leq \lambda[2S_\theta(x_{n-1}, x_{n-1}, Tx_{n-1}) + S_\theta(x_{m-1}, x_{m-1}, Tx_{m-1})] \\ &= \lambda[2S_\theta(x_{n-1}, x_{n-1}, x_n) + S_\theta(x_{m-1}, x_{m-1}, x_m)] \\ &= \lambda[2S_{b_{n-1}} + S_{b_{m-1}}]. \end{aligned}$$

Thus, for every $\epsilon > 0$, as $\lim_{n\to\infty} S_{b_n} = 0$, we can find $n_0 \in \mathbb{N}$ such that $S_{b_{n-1}} < \dfrac{\epsilon}{4}$ and $S_{b_{m-1}} < \dfrac{\epsilon}{2}$ for all $n, m > n_0$. Then, we obtain $2S_{b_{n-1}} + S_{b_{m-1}} \leq 2\dfrac{\epsilon}{4} + \dfrac{\epsilon}{2} = \epsilon$. As $\lambda < 1$, it follows that $S_\theta(x_n, x_n, x_m) < \epsilon$ for all $n, m > n_0$. Thus, $\{x_n\}$ is a Cauchy sequence in X and $\lim_{n\to\infty} S_\theta(x_n, x_n, x_m) = 0$. By completeness of X, there exists $u \in X$ such that

$$\lim_{n,m\to\infty} S_\theta(x_n, x_n, u) = S_\theta(u, u, u) = 0.$$

Now, we shall prove that $Tu = u$. For any $n \in \mathbb{N}$,

$$\begin{aligned} S_\theta(u, u, Tu) &\leq \theta(u, u, Tu)[2S_\theta(u, u, x_{n+1}) + S_\theta(Tu, Tu, Tx_n)] \\ &\leq \theta(u, u, Tu)[2S_\theta(u, u, x_{n+1}) + \lambda(2S_\theta(u, u, Tu) + S_\theta(x_n, x_n, Tx_n))]. \end{aligned}$$

Therefore, $(1 - 2\theta(u, u, Tu)\lambda)S_\theta(u, u, Tu) \leq 2\theta(u, u, Tu)S_\theta(u, u, x_{n+1}) + \theta(u, u, Tu)\lambda S_\theta(x_n, x_n, Tx_n)$, giving

$$S_\theta(u, u, Tu) \leq \frac{2\theta(u, u, Tu)}{1 - 2\theta(u, u, Tu)\lambda} S_\theta(u, u, x_{n+1}) + \frac{\theta(u, u, Tu)\lambda}{1 - 2\theta(u, u, Tu)\lambda} S_\theta(x_n, x_n, Tx_n).$$

Since S_θ and T are continuous, we have $S_\theta(x_n, x_n, Tx_n)$ goes to $S_\theta(u, u, Tu)$, and n goes toward ∞. Therefore, we obtain

$$\begin{aligned} S_\theta(u,u,Tu) &\leq \frac{2\theta(u,u,Tu)}{1-2\theta(u,u,Tu)\lambda} S_\theta(u,u,u) + \frac{\theta(u,u,Tu)\lambda}{1-2\theta(u,u,Tu)\lambda} S_\theta(u,u,Tu) \\ (1 - \frac{\theta(u,u,Tu)\lambda}{1-2\theta(u,u,Tu)\lambda}) S_\theta(u,u,Tu) &\leq \frac{2\theta(u,u,Tu)}{1-2\theta(u,u,Tu)\lambda} S_\theta(u,u,u) \\ S_\theta(u,u,Tu) &\leq \frac{2\theta(u,u,Tu)}{1-3\theta(u,u,Tu)\lambda} S_\theta(u,u,u). \end{aligned}$$

As $\lambda \neq \frac{1}{3\theta(u,u,Tu)}$ and from (2), we obtain $S_\theta(u, u, Tu) = 0$ and then $Tu = u$. □

In closing, we would like to present to the reader the following open questions:

Question 1. *Is it possible to omit the continuity of T in Theorem 1, and obtain a unique fixed point?*

Question 2. *If we omit the symmetry condition of the extended partial S_b-metric spaces in Theorem 2, is it possible to prove the existence of a fixed point?*

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