Regional Enlarged Observability of Fractional Differential Equations with Riemann—Liouville Time Derivatives

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Abstract: We introduce the concept of regional enlarged observability for fractional evolution differential equations involving Riemann–Liouville derivatives. The Hilbert Uniqueness Method (HUM) is used to reconstruct the initial state between two prescribed functions, in an interested subregion of the whole domain, without the knowledge of the state.

1. Introduction

The exact birthday of fractional calculus, and the idea of non-integer differentiation, goes back to the 17th century, precisely to 30 September 1695, when L'Hôpital wrote a question to Leibniz about the meaning of $\frac{\partial^n}{\partial t^n}$ in the case $n = \frac{1}{2}$ ([1], pp. 301–302). Since then, many great mathematicians have investigated around this question. We can mention the studies of Euler (1730), Lagrange (1772), Laplace (1812), Fourier (1822), Abel (1823), Liouville (1832), Riemann (1847), Greer (1859), Grünwald (1867), Laurent (1884), Heaviside (1892), Pincherle (1902), Marchaud (1927), Love (1938), Widder (1941), Riesz (1949), Feller (1952), among others [2]. These mathematicians began to consider how to define a fractional derivative. In 1860s, Riemann and Liouville obtained now celebrated definitions of fractional operators, by extending the Cauchy integral formula. Such fractional operators have a major role in practical problems [3]. In particular, Heymans and Podlubny have shown that it is possible to attribute a physical meaning to initial conditions expressed in terms of Riemann–Liouville fractional derivatives on the field of viscoelasticity, which is more appropriate than standard initial conditions [4]. From a mathematical point of view, fractional calculus is a generalization of the traditional differential calculus to integrals and derivatives of non-integer order.

The fact that real systems are better described with non-integer order differential equations has attracted engineers' interest, making fractional calculus a tool used nowadays in almost every area of sciences. Indeed, in the last decades, fractional calculus has been recognized as one of the best tools to describe long-memory processes and materials, anomalous diffusion, long-range interactions, long-term behaviours, power laws, and allometric scaling [5]. Such models are those described by differential equations containing fractional order derivatives. On the other hand, fractional calculus is

especially efficient for modelling systems related to diffusion processes [6]. In [7,8], the heat transfer process was successfully modelled using a fractional model based on normal and anomalous diffusion equations. In [9–11], very accurate models for ultra-capacitors and electrical energy storage elements based on diffusion and the Helmholtz effect are presented. Simultaneously, fractional calculus has played a very important role in various fields such as physics, chemistry, mechanics, electricity, economics, signal and image processing, biophysics, and bioengineering [12–14]. Likewise, for control theory, fractional calculus has an enormous role [15–17]. The reader interested in applications of fractional calculus in control and mathematical modelling of systems and processes in physics, aerodynamics, electrodynamics of complex medium, viscoelasticity, heat conduction, and electricity mechanics, is referred to [2,18–24] and references therein. For numerical methods to fractional partial differential equations, see [25–27].

Despite its development, the theory of fractional differential equations, compared with the classical theory of differential equations, is a field of research only on its initial stage of development, calling great interest to many mathematicians [28–30]. For distributed parameter systems, several works deal with the problem of regional observability, which we study here in the fractional context, investigating the possibility to reconstruct the initial state or gradient only on a subregion ω of the evolution domain Ω [31–35]. For results on controllability, see [36–39]. The interest to study the concept of observability for fractional differential equations is not newee [17,40–42]. Here we investigate, for the first time in the literature, the concept of regional enlarged observability, that is, observability with constraints on the state for fractional diffusion equations. For that, we make use of the Hilbert Uniqueness Method (HUM) of Lions [43,44].

The paper is organized as follows. In Section 2, we present the problem of regional enlarged observability of fractional diffusion systems with the traditional first-order time derivative replaced by the Riemann–Liouville time fractional derivative. In Section 3, we give some preliminary results, which will be used throughout the paper. In Section 4, we characterize the enlarged observability of the system. Section 5 is focused on the regional reconstruction of the initial state in an internal subregion of the evolution domain. We present in Section 6 an example to demonstrate our main results. We end with Section 7 of conclusions and some possible directions of future research.

2. Problem Statement

In this section we formulate the concept of regional enlarged observability for a Riemann–Liouville time fractional diffusion system of order $\alpha \in (0, 1]$. Let Ω be an open bounded subset of $\mathbb{R}^n$ ($n = 1, 2, 3$), with a regular boundary $\partial \Omega$. For $T > 0$, let us denote $Q_T = \Omega \times [0, T]$ and $\Sigma_T = \partial \Omega \times [0, T]$. We consider the following time fractional order diffusion system:

$$\begin{cases} {}_0D_t^\alpha y(x,t) = Ay(x,t) & \text{in} \quad Q_T, \\ y(\xi, t) = 0 & \text{on} \quad \Sigma_T, \\ \lim_{t \to 0^+} {}_0I_t^{1-\alpha} y(x,t) = y_0(x) & \text{in} \quad \Omega, \end{cases} \tag{1}$$

where ${}_0D_t^\alpha$ and ${}_0I_t^\alpha$ denote, respectively, the (left) Riemann–Liouville fractional derivative and integral with respect to time t, with $\alpha \in \mathbb{R}$ such that $0 < \alpha \leq 1$. For details on these operators, see, e.g., [20,24,25,45]. Here we just recall their definition:

$$ {}_0D_t^\alpha y(x,t) = \frac{\partial}{\partial t} {}_0I_t^{1-\alpha} y(x,t) $$

and

$$ {}_0I_t^\alpha y(x,t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} y(x,s)ds, $$

where $\Gamma(\alpha)$ denotes Euler's Gamma function. The second order operator A in (1) is linear with dense domain, such that the coefficients do not depend on time t and generates a strongly continuous

semi-group $(S(t))_{t\geq 0}$ on the Hilbert space $L^2(\Omega)$. We assume that the initial state $y_0 \in L^2(\Omega)$ is unknown. The observation space is $\mathcal{O} = L^2(0, T; \mathbb{R}^q)$.

Without loss of generality, we denote $y(\cdot, t) := y(t)$. The measurements are obtained by the output function given by

$$z(t) = Cy(t), \quad t \in [0, T], \tag{2}$$

where C is called the observation operator, which is a linear operator (possibly unbounded), depending on the structure and the number $q \in \mathbb{N}$ of the considered sensors, with dense domain $D(C) \subseteq L^2(\Omega)$ and range in $\mathcal{O}$.

3. Preliminaries

In this section, we recall some results for Riemann–Liouville time fractional differential systems and some notions and results to be used thereafter.

Lemma 1 (See [36,46,47]). *For any $u_0 \in L^2(\Omega)$, $0 < \alpha \leq 1$, we say that the function $u \in L^2(0, T; L^2(\Omega))$ is a mild solution of the system*

$$\begin{cases} {}_0D_t^\alpha u(t) = Au(t), & t \in [0, T], \\ \lim_{t \to 0^+} {}_0I_t^{1-\alpha} u(t) = u_0, \end{cases}$$

if u satisfies the equation

$$u(t) = H_\alpha(t)u_0,$$

where

$$H_\alpha(t) = \alpha t^{\alpha-1} \int_0^\infty \theta \xi_\alpha(\theta) S(t^\alpha \theta) d\theta,$$

$$\xi_\alpha(\theta) = \frac{1}{\alpha} \theta^{-1-\frac{1}{\alpha}} \varpi_\alpha(\theta^{-\frac{1}{\alpha}}),$$

$$\varpi_\alpha(\theta) = \frac{1}{\pi} \sum_{n=1}^\infty (-1)^{n-1} \theta^{-n\alpha-1} \frac{\Gamma(n\alpha+1)}{n!} \sin(n\pi\alpha), \quad \theta \in (0, \infty),$$

with ξ_α the probability density function defined on $(0, \infty)$, satisfying

$$\xi_\alpha(\theta) \geq 0, \quad \theta \in (0, \infty) \quad and \quad \int_0^\infty \xi_\alpha(\theta) d\theta = 1.$$

Moreover,

$$\int_0^\infty \theta^\nu \xi_\alpha(\theta) d\theta = \frac{\Gamma(1+\nu)}{\Gamma(1+\alpha\nu)}, \quad \nu \geq 0.$$

Note that a mild solution of system (1) can be written as

$$y(x, t) = H_\alpha(t)y_0, \quad t \in [0, T].$$

In order to prove our results, the following lemma is used.

Lemma 2 (See [48]). *Let the reflection operator $\mathcal{Q}$ on the interval $[0, T]$ be defined by*

$$\mathcal{Q}f(t) := f(T-t),$$

for some function f that is differentiable and integrable in the Riemann–Liouville sense. Then the following relations hold:

$$\mathcal{Q}_0D_t^\alpha f(t) = {}_tD_T^\alpha \mathcal{Q}f(t), \qquad \mathcal{Q}_0I_t^\alpha f(t) = {}_tI_T^\alpha \mathcal{Q}f(t)$$

and

$$_0D_t^\alpha \mathcal{Q}f(t) = \mathcal{Q}_t D_T^\alpha f(t), \qquad _0I_t^\alpha \mathcal{Q}f(t) = \mathcal{Q}_t I_T^\alpha f(t).$$

Follows some notions of admissibility of the output operator C. The output function of the autonomous system (1) is expressed by

$$z(t) = CH_\alpha(t)y_0 = K_\alpha(t)y_0, \quad t \in [0, T],$$

where $K_\alpha : L^2(\Omega) \longrightarrow \mathcal{O}$ is a linear operator. To obtain the adjoint operator of K_α, we have two cases.

Case 1. C is bounded (e.g., zone sensors). Let $C : L^2(\Omega) \longrightarrow \mathcal{O}$ and C^* be its adjoint. We get that the adjoint operator of K_α can be given by

$$\begin{aligned} K_\alpha^* : \mathcal{O} &\longrightarrow L^2(\Omega) \\ z^* &\longrightarrow \int_0^T H_\alpha^*(s)C^*z^*(s)ds. \end{aligned}$$

Case 2. C is unbounded (e.g., pointwise sensors). In this case, we have

$$C : D(C) \subseteq L^2(\Omega) \longrightarrow \mathcal{O}$$

with C^* denoting its adjoint. In order to give a sense to (2), we make the assumption that C is an admissible observation operator in the sense of Definition 1.

Definition 1. *The operator C of system (1) and (2) is an admissible observation operator if there exists a constant $M > 0$ such that*

$$\int_0^T \|CH_\alpha(s)y_0\|^2 \, ds \le M \|y_0\|^2$$

for any $y_0 \in D(C)$.

Note that the admissibility of C guarantees that we can extend the mapping

$$y_0 \longmapsto CH_\alpha(t)y_0 = K_\alpha(t)y_0$$

to a bounded linear operator from $L^2(\Omega)$ to $\mathcal{O}$. For more details, see, e.g., [49–51]. Then the adjoint of the operator K_α can be defined as

$$\begin{aligned} K_\alpha^* : D(K_\alpha^*) \subseteq \mathcal{O} &\longrightarrow L^2(\Omega) \\ z^* &\longrightarrow \int_0^T H_\alpha^*(s)C^*z^*(s)ds. \end{aligned}$$

4. Enlarged Observability and Characterization

Let ω be a subregion of Ω with a positive Lebesgue measure. We define the restriction operator χ_ω and its adjoint χ_ω^* by

$$\begin{aligned} \chi_\omega : L^2(\Omega) &\longrightarrow L^2(\omega) \\ y &\longrightarrow \chi_\omega y = y_{|\omega} \end{aligned}$$

and

$$(\chi_\omega^* y)(x) = \begin{cases} y(x) & \text{if } x \in \omega, \\ 0 & \text{if } x \in \Omega\backslash\omega. \end{cases}$$

Similarly to the discussions in [32,49,52], it follows that a necessary and sufficient condition for the regional exact observability of the system described by (1) and (2) in ω at time t is that $Im(\chi_\omega K_\alpha^*) = L^2(\omega)$.

Let $\beta(\cdot)$ and $\gamma(\cdot)$ be two functions defined in $L^2(\omega)$ such that $\beta(\cdot) \leq \gamma(\cdot)$ almost everywhere (a.e.) in ω. Throughout the paper, we set

$$[\beta(\cdot), \gamma(\cdot)] = \left\{ y \in L^2(\omega) \,|\, \beta(\cdot) \leq y(\cdot) \leq \gamma(\cdot) \quad \text{a.e. in } \omega \right\}.$$

We consider

$$y_0 = \begin{cases} y_0^1 & \text{in } [\beta(\cdot), \gamma(\cdot)], \\ y_0^2 & \text{in } L^2(\Omega) \backslash [\beta(\cdot), \gamma(\cdot)]. \end{cases}$$

The study of regional enlarged observability for the Riemann–Liouville time fractional order diffusion system amounts to solving the following problem.

Problem 1. *Given the system (1) together with the output (2) in ω at time $t \in [0, T]$, is it possible to reconstruct y_0^1 between two prescribed functions $\beta(\cdot)$ and $\gamma(\cdot)$ in ω?*

Before proving our first result, we need two important definitions.

Definition 2. *The system (1) together with the output (2) is said to be exactly $[\beta(\cdot), \gamma(\cdot)]$–observable in ω if*

$$Im(\chi_\omega K_\alpha^*) \cap [\beta(\cdot), \gamma(\cdot)] \neq \varnothing.$$

Definition 3. *The sensor (D, f) is said to be exactly $[\beta(\cdot), \gamma(\cdot)]$–strategic in ω if the observed system is exactly $[\beta(\cdot), \gamma(\cdot)]$–observable in ω.*

Remark 1. *If $\alpha = 1$, then system (1) is reduced to the normal diffusion process recently considered in [53]. The results of [53] are a particular case of our results.*

Remark 2. *If the system (1) together with the output (2) is exactly $[\beta(\cdot), \gamma(\cdot)]$–observable in ω_1, then it is exactly $[\beta(\cdot), \gamma(\cdot)]$–observable in any subregion $\omega_2 \subset \omega_1$.*

Theorem 1. *The following two statements are equivalent:*

1. *The system (1) together with the output (2) is exactly $[\beta(\cdot), \gamma(\cdot)]$–observable in ω.*
2. *$Ker(K_\alpha \chi_\omega^*) \cap [\beta(\cdot), \gamma(\cdot)] = \{0\}$.*

Proof. We begin by proving that statement 1 implies 2. For that we show that

$$Im(\chi_\omega K_\alpha^*) \cap [\beta(\cdot), \gamma(\cdot)] \neq \varnothing \implies Ker(K_\alpha \chi_\omega^*) \cap [\beta(\cdot), \gamma(\cdot)] = \{0\}.$$

Suppose that

$$Ker(K_\alpha \chi_\omega^*) \cap [\beta(\cdot), \gamma(\cdot)] \neq \{0\}.$$

Let us consider $y \in Ker(K_\alpha \chi_\omega^*) \cap [\beta(\cdot), \gamma(\cdot)]$ such that $y \neq 0$. Then, $y \in Ker(K_\alpha \chi_\omega^*)$ and $y \in [\beta(\cdot), \gamma(\cdot)]$. We have $Ker(K_\alpha \chi_\omega^*) = Im(\chi_\omega K_\alpha^*)^\perp$, so that $y \in Im(\chi_\omega K_\alpha^*)^\perp$, $y \neq 0$. Therefore, $y \notin Im(\chi_\omega K_\alpha^*)$, and

$$Ker(K_\alpha \chi_\omega^*) \cap [\beta(\cdot), \gamma(\cdot)] \subset L^2(\omega) \backslash Im(\chi_\omega K_\alpha^*),$$
$$Im(\chi_\omega K_\alpha^*) \subset \left[L^2(\omega) \backslash Ker(K_\alpha \chi_\omega^*) \right] \cup \left[L^2(\omega) \backslash [\beta(\cdot), \gamma(\cdot)] \right].$$

We have

$$Im(\chi_\omega K_\alpha^*) \subset L^2(\omega) \backslash Ker(K_\alpha \chi_\omega^*).$$

Accordingly,

$$Im(\chi_\omega K_\alpha^*) \cap Ker(K_\alpha \chi_\omega^*) = \varnothing$$

and

$$Im(\chi_\omega K_\alpha^*) \cap Im(\chi_\omega K_\alpha^*)^\perp = \emptyset,$$

which is absurd. Since

$$Im(\chi_\omega K_\alpha^*) \subset L^2(\omega) \setminus [\beta(\cdot), \gamma(\cdot)],$$

it follows that

$$Im(\chi_\omega K_\alpha^*) \cap [\beta(\cdot), \gamma(\cdot)] = \emptyset,$$

which is also absurd. Consequently,

$$Ker(K_\alpha \chi_\omega^*) \cap [\beta(\cdot), \gamma(\cdot)] = \{0\}.$$

We now prove the reverse implicationtatement 2 implies 1. For that we show that

$$Ker(K_\alpha \chi_\omega^*) \cap [\beta(\cdot), \gamma(\cdot)] = \{0\} \implies Im(\chi_\omega K_\alpha^*) \cap [\beta(\cdot), \gamma(\cdot)] \neq \emptyset.$$

Suppose that

$$Ker(K_\alpha \chi_\omega^*) \cap [\beta(\cdot), \gamma(\cdot)] = \{0\}.$$

Let us consider

$$y \in Ker(K_\alpha \chi_\omega^*) \cap [\beta(\cdot), \gamma(\cdot)].$$

Then, $y \in Ker(K_\alpha \chi_\omega^*)$ and $y \in [\beta(\cdot), \gamma(\cdot)]$ such that $y = 0$. We have

$$Ker(K_\alpha \chi_\omega^*) = Im(\chi_\omega K_\alpha^*)^\perp,$$

so $y \in Im(\chi_\omega K_\alpha^*)^\perp$ such that $y = 0$. Hence,

$$y \in Im(\chi_\omega K_\alpha^*) \text{ and } y \in [\beta(\cdot), \gamma(\cdot)]$$

and

$$Im(\chi_\omega K_\alpha^*) \cap [\beta(\cdot), \gamma(\cdot)] \neq \emptyset,$$

which shows that (1) and (2) is exactly $[\beta(\cdot), \gamma(\cdot)]$–observable in ω. □

5. The HUM Approach

The purpose of this section is to present an approach that allows us to reconstruct the initial state of the system (1) between two prescribed functions $\beta(\cdot)$ and $\gamma(\cdot)$ in ω. Our approach constitutes an extension of the Hilbert Uniqueness Method (HUM) developed by Lions [43,44]. In what follows, $\mathcal{G}$ is defined by

$$\mathcal{G} = \left\{ g \in L^2(\Omega) \mid g = 0 \text{ in } L^2(\Omega) \setminus [\beta(\cdot), \gamma(\cdot)] \right\}. \tag{3}$$

5.1. Pointwise Sensors

Let us consider system (1) observed by a pointwise sensor (b, δ_b), where $b \in \overline{\Omega}$ is the sensor location and δ is the Dirac mass concentrated in b. For details on pointwise sensors we refer the reader to [33]. Here the output function is given by

$$z(t) = \varphi(b, T - t), \quad t \in [0, T]. \tag{4}$$

For $\varphi_0 \in \mathcal{G}$, we consider the following system:

$$\begin{cases} {}_0D_t^{\alpha}\varphi(x,t) = A\varphi(x,t) & \text{in} \quad Q_T, \\ \varphi(\xi,t) = 0 & \text{on} \quad \Sigma_T, \\ \lim_{t \to 0^+} {}_0I_t^{1-\alpha}\varphi(x,t) = \varphi_0(x) & \text{in} \quad \Omega. \end{cases} \tag{5}$$

Without loss of generality, we denote $\varphi(x,t) := \varphi(t)$. System (5) admits a unique solution $\varphi \in L^2(0,T; H_0^1(\Omega)) \cap C(\Omega \times [0,T])$ given by $\varphi(t) = H_\alpha(t)\varphi_0$. We consider a semi-norm on $\mathcal{G}$ defined by

$$\varphi_0 \longmapsto \|\varphi_0\|_{\mathcal{G}}^2 = \int_0^T \|C\varphi(T-t)\|^2 \, dt. \tag{6}$$

The following result holds.

Lemma 3. *If the system (1) together with the output (4) is exactly $[\beta(\cdot), \gamma(\cdot)]$–observable in ω, then (6) defines a norm on $\mathcal{G}$.*

Proof. Consider $\varphi_0 \in \mathcal{G}$. Then,

$$\|\varphi_0\|_{\mathcal{G}} \quad \Longrightarrow \quad C\varphi(T-t) = 0 \quad \text{for all} \quad t \in [0,T].$$

We have

$$\varphi_0 \in L^2(\Omega) \quad \Longrightarrow \quad \chi_\omega \varphi_0 \in L^2(\omega)$$

or

$$K_\alpha(t)\chi_\omega^* \chi_\omega \varphi_0 = CH_\alpha(t)\chi_\omega^* \chi_\omega \varphi_0 = 0.$$

Hence,

$$\chi_\omega \varphi_0 \in Ker(K_\alpha \chi_\omega^*).$$

For $\chi_\omega \varphi_0 \in [\beta(\cdot), \gamma(\cdot)]$, one has $\chi_\omega \varphi_0 \in Ker(K_\alpha \chi_\omega^*) \cap [\beta(\cdot), \gamma(\cdot)]$ and, because the system is exactly $[\beta(\cdot), \gamma(\cdot)]$–observable in ω, $\chi_\omega \varphi_0 = 0$. Consequently, $\varphi_0 = 0$ and (6) is a norm. $\square$

Consider the system

$$\begin{cases} \mathcal{Q}\,{}_tD_T^{\alpha}\Psi(x,t) = A^*\mathcal{Q}\Psi(x,t) + C^*C\mathcal{Q}\varphi(x,t) & \text{in} \quad Q_T, \\ \Psi(\xi,t) = 0 & \text{on} \quad \Sigma_T, \\ \lim_{t \to T^-} \mathcal{Q}\,{}_tI_T^{1-\alpha}\Psi(x,t) = 0 & \text{in} \quad \Omega, \end{cases} \tag{7}$$

controlled by the solution of (5). For $\varphi_0 \in \mathcal{G}$, we define the operator $\Lambda : \mathcal{G} \longrightarrow \mathcal{G}^*$ by

$$\Lambda\varphi_0 = \mathcal{P}(\Psi(0)),$$

where $\mathcal{P} = \chi_\omega^* \chi_\omega$ and $\Psi(0) = \Psi(x,0)$. Let us now consider the system

$$\begin{cases} \mathcal{Q}\,{}_tD_T^{\alpha}\Theta(x,t) = A^*\mathcal{Q}\Theta(x,t) + C^*\mathcal{Q}z(t) & \text{in} \quad Q_T, \\ \Theta(\xi,t) = 0 & \text{on} \quad \Sigma_T, \\ \lim_{t \to T^-} \mathcal{Q}\,{}_tI_T^{1-\alpha}\Theta(x,t) = 0 & \text{in} \quad \Omega. \end{cases} \tag{8}$$

If φ_0 is chosen such that $\Theta(0) = \Psi(0)$ in ω, then system (8) can be seen as the adjoint of system (1) and our problem of enlarged observability is reduced to solve the equation

$$\Lambda\varphi_0 = \mathcal{P}(\Theta(0)). \tag{9}$$

Theorem 2. *If system (1) together with the output (4) is exactly $[\beta(\cdot), \gamma(\cdot)]$–observable in ω, then equation (9) admits a unique solution $\varphi_0 \in \mathcal{G}$, which coincides with the initial state y_0^1 observed between $\beta(\cdot)$ and $\gamma(\cdot)$ in ω. Moreover, $y_0^1 = \chi_\omega \varphi_0$.*

Proof. By Lemma 1, we see that $\|\cdot\|_{\mathcal{G}}$ is a norm of the space $\mathcal{G}$ provided that the system (1) together with the output (4) is exactly $[\beta(\cdot), \gamma(\cdot)]$–observable in ω. Now, we show that (9) admits a unique solution in $\mathcal{G}$. For any $\varphi_0 \in \mathcal{G}$, Equation (9) admits a unique solution if Λ is an isomorphism. Then,

$$
\begin{aligned}
\langle \Lambda \varphi_0, \varphi_0 \rangle_{L^2(\Omega)} &= \langle \mathcal{P}\Psi(0), \varphi_0 \rangle_{L^2(\Omega)} \\
&= \langle \chi_\omega^* \chi_\omega \Psi(0), \varphi_0 \rangle_{L^2(\Omega)} \\
&= \langle \Psi(0), \varphi_0 \rangle_{L^2(\omega)}
\end{aligned}
$$

or $\Psi(t)$ is the solution of system (7), that is,

$$
\Psi(t) = H_\alpha^*(T - t)\Psi(T) + \int_t^T H_\alpha^*(T - \tau)C^*C\varphi(T - \tau)d\tau
$$

and

$$
\Psi(0) = \int_0^T H_\alpha^*(T - \tau)C^*C\varphi(T - \tau)d\tau,
$$

where

$$
H_\alpha^*(t) = \alpha t^{\alpha - 1} \int_0^\infty \theta \xi_\alpha(\theta) S^*(t^\alpha \theta) d\theta
$$

with $(S^*(t))_{t \geq 0}$ the strongly continuous semi-group generated by A^*. We obtain that

$$
\begin{aligned}
\langle \Lambda \varphi_0, \varphi_0 \rangle_{L^2(\Omega)} &= \langle \Psi(0), \varphi_0 \rangle \\
&= \left\langle \int_0^T H_\alpha^*(T - \tau)C^*C\varphi(T - \tau)d\tau, \varphi_0 \right\rangle \\
&= \int_0^T \langle C\varphi(T - \tau), CH_\alpha(T - \tau)\varphi_0 \rangle d\tau \\
&= \int_0^T \|C\varphi(T - \tau)\|^2 d\tau \\
&= \|\varphi_0\|_{\mathcal{G}}^2,
\end{aligned}
$$

concluding that Λ is an isomorphism. Consequently, Equation (9) has a unique solution that is also the initial state to be estimated between $\beta(\cdot)$ and $\gamma(\cdot)$ in the subregion ω given by

$$
y_0^1 = \chi_\omega \varphi_0.
$$

The proof is complete. □

5.2. Zone Sensors

Let us come back to system (1) and suppose that the measurements are given by an internal zone sensor defined by (D, f) with $D \subset \Omega$ and $f \in L^2(D)$. The system is augmented with the output function

$$
z(t) = \int_D y(x, T - t)f(x)dx. \tag{10}
$$

In this case, we consider (5), $\mathcal{G}$ given by (3), and we define a semi-norm on $\mathcal{G}$ by

$$
\|\varphi_0\|_{\mathcal{G}}^2 = \int_0^T \langle \varphi(T - t), f \rangle_{L^2(D)}^2 dt \tag{11}
$$

with

$$
\begin{cases}
\mathcal{Q}\,_tD_T^\alpha \Psi(x,t) = A^* \mathcal{Q}\Psi(x,t) + \langle \mathcal{Q}\varphi(t), f \rangle_{L^2(D)} \chi_D f(x) & \text{in} \quad Q_T, \\
\Psi(\xi,t) = 0 & \text{on} \quad \Sigma_T, \\
\lim_{t \to T^-} \mathcal{Q}\,_tI_T^{1-\alpha}\Psi(x,t) = 0 & \text{in} \quad \Omega.
\end{cases}
$$

We introduce the operator

$$
\begin{aligned}
\Lambda : \mathcal{G} &\longrightarrow \mathcal{G}^* \\
\varphi_0 &\longrightarrow \Lambda\varphi_0 = \mathcal{P}(\Psi(0)),
\end{aligned}
\tag{12}
$$

where $\mathcal{P} = \chi_\omega^* \chi_\omega$ and $\Psi(0) = \Psi(x,0)$. Let us consider the system

$$
\begin{cases}
\mathcal{Q}\,_tD_T^\alpha \Theta(x,t) = A^* \mathcal{Q}\Theta(x,t) + \langle \mathcal{Q}z(t), f \rangle_{L^2(D)} \chi_D f(x) & \text{in} \quad Q_T, \\
\Theta(\xi,t) = 0 & \text{on} \quad \Sigma_T, \\
\lim_{t \to T^-} \mathcal{Q}\,_tI_T^{1-\alpha}\Theta(x,t) = 0 & \text{in} \quad \Omega.
\end{cases}
\tag{13}
$$

If φ_0 is chosen such that $\Theta(0) = \Psi(0)$ in ω, then (13) can be seen as the adjoint of system (1) and our problem of enlarged observability consists to solve the equation

$$
\Lambda\varphi_0 = \mathcal{P}(\Theta(0)).
\tag{14}
$$

Theorem 3. *If system (1) together with the output (10) is exactly $[\beta(\cdot), \gamma(\cdot)]$–observable in ω, then Equation (14) has a unique solution $\varphi_0 \in \mathcal{G}$, which coincides with the initial state y_0^1 observed between $\beta(\cdot)$ and $\gamma(\cdot)$ in ω.*

Proof. The proof is similar to the proof of Theorem 2. $\square$

6. Example

Let us consider the following one-dimensional time fractional differential system of order $\alpha \in (0,1]$ in $\Omega_1 = [0,1]$, excited by a pointwise sensor:

$$
\begin{cases}
_0D_t^\alpha y(x,t) = \dfrac{\partial^2 y(x,t)}{\partial x^2} & \text{in} \quad [0,1] \times [0,T], \\
y(0,t) = y(1,t) = 0 & \text{in} \quad [0,T], \\
\lim_{t \to 0^+} {}_0I_t^{1-\alpha} y(x,t) = y_0(x) & \text{in} \quad [0,1],
\end{cases}
\tag{15}
$$

augmented with the output function

$$
z(t) = Cy(x,t) = y(b,t),
\tag{16}
$$

where $b = \dfrac{1}{2} \in \Omega_1$. The operator $A = \dfrac{\partial^2}{\partial x^2}$ has a complete set of eigenfunctions (φ_k) in $L^2(\Omega_1)$ associated with the eigenvalues (λ_k), given by

$$
\varphi_k(x) = \sqrt{2}\sin(k\pi x) \quad \text{and} \quad \lambda_k = -k^2\pi^2
$$

with

$$
S(t)y(x) = \sum_{k=1}^{\infty} e^{\lambda_k t} \langle y, \varphi_k \rangle_{L^2(\Omega_1)} \varphi_k(x).
$$

Then,

$$H_\alpha(t)y(x) = \sum_{k=1}^{\infty} t^{\alpha-1} E_{\alpha,\alpha}(\lambda_k t^\alpha) \langle y, \varphi_k \rangle_{L^2(\Omega_1)} \varphi_k(x),$$

where $E_{\alpha,\alpha'}(z) := \sum_{k=0}^{\infty} \dfrac{z^k}{\Gamma(\alpha k + \alpha')}$, **Re** $\alpha > 0$, $\alpha', z \in \mathbb{C}$, is the generalized Mittag–Leffler function in two parameters (see, e.g., [54]). Let $y_0(x) = \sin(2\pi x)$ be the initial state to be observed. Then, for $\omega_1 = \left[\dfrac{1}{6}, \dfrac{1}{3}\right]$, the following result holds.

Proposition 1. *There is a state for which the system (15) and (16) is not weakly observable in Ω_1 but it is exactly $[\beta_1(\cdot), \gamma_1(\cdot)]$–observable in ω_1.*

Proof. To show that system (15) and (16) is not weakly observable in Ω_1, it is sufficient to verify that $y_0 \in Ker(K_\alpha)$. We have

$$\begin{aligned}
K_\alpha y_0(x) &= \sum_{i=1}^{\infty} t^{\alpha-1} E_{\alpha,\alpha}(\lambda_i t^\alpha) \langle y_0, \varphi_i \rangle_{L^2(\Omega_1)} \varphi_i(b) \\
&= 2\sum_{i=1}^{\infty} t^{\alpha-1} E_{\alpha,\alpha}(\lambda_i t^\alpha) \sin\left(\frac{i\pi}{2}\right) \int_0^1 \sin(2\pi x)\sin(i\pi x)dx \\
&= 0.
\end{aligned}$$

Hence, $K_\alpha y_0(x) = 0$. Consequently, the state y_0 is not weakly observable in Ω_1. On the other hand, one has

$$\begin{aligned}
K_\alpha \chi_{\omega_1}^* \chi_{\omega_1} y_0(x) &= \sum_{i=1}^{\infty} t^{\alpha-1} E_{\alpha,\alpha}(\lambda_i t^\alpha) \left\langle \chi_{\omega_1}^* \chi_{\omega_1} y_0, \varphi_i \right\rangle_{L^2(\Omega_1)} \varphi_i(b) \\
&= \sum_{i=1}^{\infty} t^{\alpha-1} E_{\alpha,\alpha}(\lambda_i t^\alpha) \langle y_0, \varphi_i \rangle_{L^2(\omega_1)} \varphi_i(b) \\
&= 2t^{\alpha-1} E_{\alpha,\alpha}(-\pi^2 t^\alpha) \int_{\frac{1}{6}}^{\frac{1}{3}} \sin(2\pi x)\sin(\pi x)dx \\
&= \frac{(3\sqrt{3}-1)t^{\alpha-1} E_{\alpha,\alpha}(-\pi^2 t^\alpha)}{6\pi} \\
&\neq 0,
\end{aligned}$$

which means that the state y_0 is weakly observable in ω_1. Moreover, for

$$\beta_1(x) = \left| y_{|\omega_1}^0(x) \right| - 1 < y_{|\omega_1}^0(x)$$

and

$$\gamma_1(x) = \left| y_{|\omega_1}^0(x) \right| + 1 > y_{|\omega_1}^0(x), \quad \forall x \in \omega_1,$$

we have $\chi_{\omega_1} y_0(x) \in [\beta_1(\cdot), \gamma_1(\cdot)]$ and system (15) and (16) is exactly $[\beta_1(\cdot), \gamma_1(\cdot)]$-observable in ω_1. The proof is complete. $\square$

Let $\mathcal{G}_1$ be the set defined by

$$\mathcal{G}_1 = \left\{ g \in L^2(\Omega_1) \mid g = 0 \text{ in } L^2(\Omega_1)\backslash[\beta_1(\cdot), \gamma_1(\cdot)] \right\}.$$

From Lemma 3, we see that

$$\varphi_0 \longmapsto \|\varphi_0\|_{\mathcal{G}_1}^2 = \int_0^T \|CH_\alpha(T-t)\varphi_0\|^2 dt$$

defines a norm on $\mathcal{G}_1$. Consider the system

$$
\begin{cases}
\mathcal{Q}\,_tD_T^\alpha \Theta(x,t) = A^*\mathcal{Q}\Theta(x,t) + \delta(x-b)z(T-t) & \text{in} \quad \Omega_1 \times [0,T], \\
\Theta(\xi,t) = 0 & \text{on} \quad \partial\Omega_1 \times [0,T], \\
\lim_{t \to T^-} \mathcal{Q}\,_tI_T^{1-\alpha}\Theta(x,t) = 0 & \text{in} \quad \Omega_1.
\end{cases}
$$

It follows from Theorem 2 that the equation $\Lambda : \varphi_0 \longmapsto \mathcal{P}(\Theta(0))$ has a unique solution in $\mathcal{G}_1$, which is also the initial state y_0 observed between $\beta_1(\cdot)$ and $\gamma_1(\cdot)$ in the subregion ω_1.

7. Conclusions

In this paper we have investigated the notion of regional enlarged observability for a time fractional diffusion system with Riemann–Liouville fraction derivative of order $\alpha \in (0,1]$. We developed an approach that leads to the reconstruction of the initial state between two prescribed functions only in an internal subregion ω of the whole domain Ω. We claim that the results here obtained can be useful to real problems of engineering.

As future work, we plan to study problems of regional boundary enlarged observability and regional gradient enlarged observability of fractional order distributed parameter systems, and provide illustrative numerical examples.

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