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(L)-Semigroup Sums †

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† In this note, all spaces are Hausdorff, and the term map or mapping shall always mean continuous function.

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Abstract: An (L)-semigroup S is a compact n-manifold with connected boundary B together with a monoid structure on S such that B is a subsemigroup of S. The sum $S + T$ of two (L)-semigroups S and T having boundary B is the quotient space obtained from the union of $S \times \{0\}$ and $T \times \{1\}$ by identifying the point $(x, 0)$ in $S \times \{0\}$ with $(x, 1)$ in $T \times \{1\}$ for each x in B. It is shown that no (L)-semigroup sum of dimension less than or equal to five admits an H-space structure, nor does any (L)-semigroup sum obtained from (L)-semigroups having an Abelian boundary. In particular, such sums cannot be a retract of a topological group.

Keywords: topological group; Lie group; compact topological semigroup; H-space; mapping cylinder; fibre bundle

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1. Introduction

An H-space is a space X together with a continuous multiplication $m : X \times X \to X$ and an identity element $e \in X$ such that $m(e, x) = m(x, e) = x$ for all $x \in X$. If, in addition, the multiplication is associative, then X is called a topological monoid. A space together with an associative continuous multiplication is called a topological semigroup. A compact n-manifold S with connected boundary B together with a topological monoid structure such that B is a subsemigroup of S is called an (L)-semigroup in [1], p. 117. Such a topological monoid S can be considered as a mapping cylinder $MC(f)$ of a quotient morphism $f : X \to X/N$ of a compact connected Lie group X where N is a normal sphere subgroup of X (see [1–3]).

In [2], p. 315, it was shown that every commutative n-dimensional (L)-semigroup is a retract of a compact connected Lie group, and if $n \leq 4$, then every n-dimensional (L)-semigroup is a retract of a compact connected Lie group. In this note, it is shown that the sum of two commutative (L)-semigroups cannot be a retract of a topological group, nor can the sum of two n-dimensional (L)-semigroups if $n \leq 5$.

2. (L)-Semigroup Splitting

Let $\mathbb{I} = [0, 1]$ denote the unit interval endowed with the operation of multiplication of real numbers. If $f : X \to Y$ is a mapping between compact spaces, then the mapping cylinder $MC(f)$ is the quotient space obtained by taking the disjoint union of $X \times \mathbb{I}$ and Y and identifying each point $(x, 0) \in X \times \mathbb{I}$ with $f(x) \in Y$. There are natural embeddings $i_X : X \to MC(f)$ and $i_Y : Y \to MC(f)$, so X and Y may be regarded as disjoint closed subspaces of $MC(f)$, and it is easy to check that $i_Y(Y)$ is a strong deformation retract of $MC(f)$. In the special case when Y consists of a single point v, the mapping cylinder is called the cone over X, denoted by cone(X).

Let $\mathbb{S}^n$ denote the unit n-sphere in Euclidean n-space $\mathbb{R}^n$. Then, in the following result of Mostert and Shields [1], cone$(\mathbb{S}^n)$, $n = 0, 1, 3$, is homeomorphic to the unit one-ball in the real line $\mathbb{R}^1$, the unit disk $\mathbb{E}^2$ in the complex plane $\mathbb{C}$, the unit four-ball $\mathbb{E}^4$ in the quaternions $\mathbb{H}$, respectively, and is considered to be a topological monoid with the inherited multiplicative structure.

Proposition 1 (Mostert and Shields [1]; also see [2,3])**.** *Let X be a compact connected Lie group with a closed normal subgroup N such that N is isomorphic to $\mathbb{S}^n$, $n = 0, 1, 3$, and let $f : X \to X/N = Y$ be the quotient morphism. Then:*

(1) $S = MC(f)$ is a compact manifold with boundary $i_X(X)$ with S being a topological monoid such that $H(S) = i_X(X)$ is the group of units of S with identity $i_X(1_X)$ and $M(S) = i_Y(Y)$ is the minimal ideal of S with identity $i_Y(1_Y)$.
(2) $S = MC(f)$ is a locally-trivial fibre bundle over the Lie group $Y = X/N$ as base with fibre $F = \text{cone}(N)$, the unit n-ball for $n = 1, 2, 4$.

A compact topological monoid S of the above type is called an (L)-semigroup in the literature and S is nonorientable if $N = \mathbb{S}^0$ and orientable if $N = \mathbb{S}^n$, $n = 1, 3$. (Theorem C in [1]).

Let S and T be two (L)-semigroups with boundary B, and let $h : B \to B$ be an autohomeomorphism of B. The quotient space obtained by taking the union of $S \times \{0\}$ and $T \times \{1\}$ and identifying the point $(x, 0)$ in $S \times \{0\}$ with $(h(x), 1)$ in $T \times \{1\}$ for each $x \in B$ is a closed (i.e., compact without boundary) connected n-manifold. Any manifold M obtained in this fashion is said to admit an (L)-semigroup splitting. In the case when h is the identity mapping, we call M the sum of S and T and denote it by $S + T$. If $S = T$, then $S + S = 2S$, the double of the manifold S.

A space X is said to be homogeneous if for every $a, b, \in X$, there is an autohomeomorphism h of X such that $h(a) = b$.

Proposition 2. *If M admits an* (L)*-semigroup splitting, then M admits the structure of a topological monoid iff M is a Lie group.*

Proof. If M is a Lie group, then it is a topological monoid. Thus, suppose M is a topological monoid. A finite-dimensional homogeneous compact connected monoid admits the structure of a topological group [4]. If, in addition, it is locally contractible, then it must be a Lie group since a compact connected group is a Lie group iff it is locally contractible [5]. Since M is a closed connected n-manifold, the result follows. □

Proposition 3. *Let G be a compact connected Lie group. If M admits an* (L)*-semigroup splitting, then so does $M \times G$. In particular, if M is an* (L)*-semigroup sum, then so is $M \times G$.*

Proof. Let M, S, T, and $h : B \to B$ be defined as in the definition of an (L)-semigroup splitting. Then, $S \times G$ and $T \times G$ are (L)-semigroups with $B \times G$ as a boundary, and the correspondence $(x, g) \mapsto (h(x), g)$ determines an autohomeomorphism of $B \times G$. It follows that $M \times G$ admits an (L)-semigroup splitting if M does. In the case when $h = 1_B$, the identity mapping on B, we obtain $M \times G = (S \times G) + (T \times G)$. □

Remark 1. *It is well known that the fundamental group of an* H*-space is Abelian and that a covering space of an* H*-space admits an* H*-space structure (cf. p. 78 and p. 157 in [6]). According to a famous theorem of J.F.Adams [7], the only spheres that are* H*-space are $\mathbb{S}^n$, $n = 0, 1, 3, 7$, and it follows that $\mathbb{RP}^n$, $n = 0, 1, 3, 7$, are the only real projective n-spaces, which admit* H*-space structures. We also remark that if a product space is homogeneous, then it admits an* H*-space structure iff each factor does (Corollary 2.5 in [8]).*

Proposition 4. *Let B be a compact connected Abelian Lie group and let S, and T be* (L)*-semigroups with boundary B. Then, the sum $S + T$ does not admit an* H*-space structure.*

Proof. Let $\mathbb{T}^n$ denote the n-torus, which is the product of n copies of the circle group $\mathbb{S}^1$. In the case when $B = \mathbb{T}^1 = \mathbb{S}^1$, the normal sphere subgroups are $\mathbb{S}^0$ and $\mathbb{S}^1$. For the two element subgroups $\mathbb{S}^0$ of $\mathbb{S}^1$, the quotient morphism $f : \mathbb{S}^1 \to \mathbb{S}^1 / \mathbb{S}^0$ yields $MC(f) = \mathbb{M}^2$, the classical Möbius band (see Example 2.3(b) in [2]). When the normal subgroup of $\mathbb{S}^1$ is $\mathbb{S}^1$, the quotient morphism $f : \mathbb{S}^1 \to \mathbb{S}^1 / \mathbb{S}^1 = \{1\}$ yields $MC(f) = \mathbb{E}^2$, the unit disk in the complex plane $\mathbb{C}$ (see Example 2.3(a) in [2]). Thus, the only two-dimensional (L)-semigroup splittings are $2\mathbb{E}^2 = \mathbb{S}^2$, $\mathbb{E}^2 + \mathbb{M}^2 = \mathbb{RP}^2$ and $2\mathbb{M}^2 = \mathbb{K}^2$, the Klein bottle. By Remarks 1, $\mathbb{S}^2$ and $\mathbb{RP}^2$ do not admit H-spaces structures, nor does $\mathbb{K}^2$ since its fundamental group $\Pi_1(\mathbb{K}^2)$ is not Abelian (this follows from the fact that the Abelianization of $\Pi_1(\mathbb{K}^2)$ is $\mathbb{Z} \oplus \mathbb{Z}_2$, the direct sum of the integers and a cyclic group of order two (see [6], p. 135), but $\Pi_1(\mathbb{K}^2)$ must contain a copy of $\Pi_1(\mathbb{T}^2) = \mathbb{Z} \oplus \mathbb{Z}$ since the two-torus $\mathbb{T}^2$ is a double covering space of the Klein bottle $\mathbb{K}^2$).

It follows from Proposition 2.3 that $\mathbb{S}^2 \times \mathbb{T}^n$, $\mathbb{RP}^3 \times \mathbb{T}^n$, and $\mathbb{K}^2 \times \mathbb{T}^n$, $n = 1, 2, \cdots$, are (L)-semigroup sums. Since $\mathbb{E}^2 \times \mathbb{T}^n$ and $\mathbb{M}^2 \times \mathbb{T}^n$ are the only $(n+2)$-dimensional (L)-semigroups with boundary $B = \mathbb{T}^{n+1}$ (see Corollaries 7.5.4 and 7.5.5 in [1]), it follows that the (L)-semigroup sum $S + T$ must be one of the manifolds $\mathbb{S}^2$, $\mathbb{RP}^2$, $\mathbb{K}^2$, $\mathbb{S}^2 \times \mathbb{T}^n$, $\mathbb{RP}^2 \times \mathbb{T}^n$ or $\mathbb{K}^2 \times \mathbb{T}^n$ for $n = 1, 2, \cdots$. However none of these manifolds admit on H-space structure since a homogeneous product space admits an H-space structure iff each of its factors does (see Corollary 2.5 in [8]). □

We remark that a retract of a homogeneous H-space admits an H-space structure (cf. Proposition 2.4 in [8]). Consequently, we have the following corollary.

Corollary 1. *Let B be a compact connected Abelian Lie group, and let S and T be* (L)*-semigroups with boundary B. Then, the sum $S + T$ is not a retract of a topological group.*

Proposition 5. *If M is a manifold that admits an* (L)*-semigroup splitting and is either two-dimensional or orientable and three-dimensional, then the following statements are equivalent:*

(1) M is a retract of a topological group.
(2) M admits an H-space structure.
(3) M is a Lie group.

Proof. In the two-dimensional case, the collection of (L)-semigroup sums coincides with the collection of spaces that admit (L)-semigroup splittings since the connected sum of two surfaces is independent of the homeomorphism h used to form the connected sum. Thus, the only surfaces that admit (L)-semigroup splittings are $\mathbb{S}^2$, $\mathbb{RP}^2$, and $\mathbb{K}^2$, and the result follows for surfaces.

The remark following the proof of Proposition 4 shows that (1) implies (2), and since the topological group is a retract of itself, (3) implies (1). Thus, it suffices to show that (2) $\Rightarrow$ (3). As was noted in the proof of Proposition 4, the only orientable three-dimensional (L)-semigroup is the solid torus $\mathbb{E}^2 \times \mathbb{S}^1$. It follows that M must be a (p, q)-lens space $L(p, q)$ where the degenerate cases $L(0, 1) = \mathbb{S}^2 \times \mathbb{S}^1$ and $L(1, q) = \mathbb{S}^3$ are included (see p. 234 in [9]). It follows from a theorem of William Browder (p. 140 in [10]) that only $L(1, q) = \mathbb{S}^3$ and $L(2, 1) = \mathbb{RP}^3 = SO(3)$ admit H-space structures. Since each of these spaces is a Lie group, the result follows. □

Lemma 1. *Let X be a closed n-manifold, which is the total space of a locally-trivial $\mathbb{S}^2$ fibre bundle over a compact Lie group G. Then, X does not admit an* H*-space structure.*

Proof. Suppose X does admit an H-space structure, and consider the fibre bundle $\mathbb{S}^2 \to X \to G$. This sequence extends to a fibration sequence $\cdots \Omega G \to \mathbb{S}^2 \to X \to G$ (cf. [11], p. 409). Since X is a (compact metric) ANR-space (see [12]), it has the homotopy type of a finite complex ([13], Corollary 44.2), and it follows from a theorem of W.Browder ([14]) that $\Pi_2(X) = 0$, where $\Pi_2(X)$ denotes the second homotopy group of X. Exactness yields a surjection from $\Pi_2(\Omega G)$ onto $\Pi_2(\mathbb{S}^2)$. An element of $\Pi_2(\Omega G)$ mapping to a generator of $\Pi_2(\mathbb{S}^2)$ is represented by a map $\mathbb{S}^2 \to \Omega G$ whose composition with the map $\Omega(G) \to \mathbb{S}^2$ is homotopic to the identity mapping $1_{\mathbb{S}^2}$ on $\mathbb{S}^2$. Consequently, there is a homotopy retraction $r : \Omega G \to \mathbb{S}^2$ (i.e., $r| \mathbb{S}^2$ is homotopic to $1_{\mathbb{S}^2}$). Since a loop space admits an H-space structure, we may assume that ΩG is an H-space with identity e, and we may assume that $e \in \mathbb{S}^2$ (since ΩG is a homogeneous space when viewed as a loop group).

Define a mapping $m : \mathbb{S}^2 \times \mathbb{S}^2 \to \mathbb{S}^2$ by $m(x, y) = r(xy)$ for $x, y \in \mathbb{S}^2$, where xy denotes the product of x and y in the H-space ΩG. The maps $\mathbb{S}^2 \to \mathbb{S}^2$ given by $x \mapsto m(x, e)$ and $x \mapsto m(e, x)$ are homotopic to the identity mapping $1_{\mathbb{S}^2}$, and therefore, e is a homotopy identity of $\mathbb{S}^2$. For CW complexes the existence of a homotopy identity can be used as the definition of an H-space (see [11], p. 291). Consequently, $\mathbb{S}^2$ admits an H-space structure, and this contradiction completes the proof of the lemma. □

Proposition 6. *Let $S = MC(f)$ be an* (L)*-semigroup as defined in Proposition 1 where X is a compact connected Lie group and $f : X \to X/N = Y$ is a quotient morphism with N being a closed normal subgroup of X, which is isomorphic to $\mathbb{S}^1$. Then, the double $2S$ does not admit an* H*-space structure.*

Proof. By Proposition 1 $S = MC(f)$ is a locally-trivial $\mathbb{E}^2$ bundle over the compact connected Lie group Y, and it follows that its double is a locally-trivial $\mathbb{S}^2$ bundle over Y. Consequently, by Lemma 1, $2S$ does not admit an H-space structure. □

Corollary 2. *Let $f : U(n) \to U(n)/\mathbb{Z}U(n) = PU(n)$ denote the quotient morphism where $U(n)$ is the unitary group, $\mathbb{Z}(U(n))$ is its centre, and $PU(n)$ is the projective unitary group. Then, if $n > 1$ and $S = MC(f)$, the double $2S$ does not admit an* (H)*-space structure.*

Proof. The elements of $U(n)$ are the complex $n \times n$ unitary matrices, and its centre $\mathbb{Z}(U(n))$ is isomorphic to $\mathbb{S}^1$ since its elements are diagonal matrices equal to $e^{i\theta}$ multiplied by the identity matrix. It follows from Proposition 6 that $2S$ does not admit an H-space structure. □

Theorem 1. *No* (L)*-semigroup sum of dimension $n \leq 5$ admits an* H*-space structure.*

Proof. Proposition 4 shows that the result is true for all n-dimensional (L)-semigroup sums of the form $S + L$ where both S and L have a compact connected Abelian Lie group boundary B. Thus, we need only consider admissible n-dimensional non-Abelian boundaries B with $n = 3, 4$. Hence, B must be one of $\mathbb{S}^3$, $\mathbb{S}^1 \times \mathbb{S}^3$, $\mathbb{S}^1 \times SO(3)$ and $U(2)$ (we note that $SO(3)$ does not qualify as an admissible boundary for an (L)-semigroup since it does not contain normal subgroups of the form $\mathbb{S}^n$, $n = 0, 1, 3$).

In [2], it is shown that the (L)-semigroups with boundary $\mathbb{S}^3$ are $\mathbb{E}^4$ and the four-dimensional Möbius manifold $\mathbb{M}^4$ (which is homeomorphic to $\mathbb{RP}^4$ with the interior of a four-dimensional Euclidean ball removed). It follows (see [2]) that the (L)-semigroups with boundaries $\mathbb{S}^3$, $\mathbb{S}^1 \times \mathbb{S}^3$, $\mathbb{S}^1 \times SO(3)$ are $\mathbb{E}^4$, $\mathbb{M}^4$, $\mathbb{E}^2 \times \mathbb{S}^3$, $\mathbb{M}^2 \times \mathbb{S}^3$, $\mathbb{S}^1 \times \mathbb{E}^4$, $\mathbb{S}^1 \times \mathbb{M}^4$, $\mathbb{E}^2 \times SO(3)$, $\mathbb{M}^2 \times SO(3)$, and the corresponding (L)-semigroup sums are $\mathbb{S}^4$, $\mathbb{RP}^4$, $2\mathbb{M}^4$, $\mathbb{S}^2 \times \mathbb{S}^3$, $\mathbb{RP}^4 \times \mathbb{S}^3$, $\mathbb{K}^2 \times \mathbb{S}^3$, $\mathbb{S}^1 \times \mathbb{S}^4$, $\mathbb{S}^1 \times \mathbb{RP}^4$, $\mathbb{S}^1 \times (2\mathbb{M}^4)$, $\mathbb{S}^2 \times SO(3)$, $\mathbb{RP}^2 \times SO(3)$, $\mathbb{K}^2 \times SO(3)$. Since a retract of a homogeneous H-space admits an H-space structure (cf. [8], Prop. 2.4), it follows that no product containing a copy of $\mathbb{S}^2$, $\mathbb{S}^4$, $\mathbb{RP}^2$, $\mathbb{RP}^4$ of $\mathbb{K}^2$ as a factor can admit an H-space structure. This leaves only $2\mathbb{M}^4$ for consideration. However, its fundamental group $\Pi_1(2\mathbb{M}^4)$ is the free product of $\Pi_1(\mathbb{RP}^4) = \mathbb{Z}_2$ with itself, which is non-Abelian, so $2\mathbb{M}^4$ does not admit an H-space structure. Finally, the only five-dimensional (L)-semigroup sum with boundary $U(2)$ is the manifold $2U(2)$ in Corollary 2, which does not admit an H-space structure. □

Corollary 3. *No* (L)*-semigroup sum of dimension $n \leq 5$ is a retract of a topological group.*

Proof. It was noted above that every retract of a homogeneous H-space admits an H-space structure. Since a topological group is an H-space, the result follows from Theorem 1. □

In [15], a space homeomorphic to a retract of a topological group is called a GR-space (often referred to as a retral space in the literature). Clearly AR-spaces and topological groups themselves are GR-spaces, and in [2], it was shown that $\mathbb{M}^2$ and $\mathbb{M}^4$ are GR-spaces. Since GR-spaces are preserved by topological products, it follows that products of $\mathbb{E}^2$, $\mathbb{E}^4$, $\mathbb{M}^2$, $\mathbb{M}^4$, and topological groups are GR-spaces. This will include all the (L)-semigroups mentioned in this note excluding (L)-semigroups with boundary $U(n)$, $n \geq 2$. This suggests two questions.

(a) Is every (L)-semigroup a retract of a topological group?
(b) Does every (L)-semigroup sum fail to admit an H-space structure?

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