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Harmonic Coordinates for the Nonlinear Finsler Laplacian and Some Regularity Results for Berwald Metrics

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Abstract: We prove existence of harmonic coordinates for the nonlinear Laplacian of a Finsler manifold and apply them in a proof of the Myers–Steenrod theorem for Finsler manifolds. Different from the Riemannian case, these coordinates are not suitable for studying optimal regularity of the fundamental tensor, nevertheless, we obtain some partial results in this direction when the Finsler metric is Berwald.

Keywords: Finsler Laplacian; harmonic coordinates; regularity; Berwald

MSC: 53B40; 58J60

1. Introduction

The existence of harmonic coordinates is well-known both on Riemannian and Lorentzian manifolds. Actually, apart from isothermal coordinates on a surface, the problem of existence of harmonic coordinates on a Lorentzian manifold (*wave harmonic*) was considered before the Riemannian case. Indeed, A. Einstein, T. De Donder, and C. Lanczos considered harmonic coordinates in the study of the Cauchy problem for the Einstein field equations, cf. [1]. In a Riemannian manifold with a smooth metric, a proof of the existence of harmonic coordinates is given in [2] (Lemma 1.2) but actually it can be found in the work of other authors such as [3] or [4] (p. 231). The motivation to consider harmonic coordinates comes from the fact that the expression of the Ricci tensor in such coordinates simplifies highly (see the introduction in [2]).

Different from the Riemannian case, the Finsler Laplacian is a quasi-linear operator and, although it is uniformly elliptic with smooth coefficients where $du \neq 0$, the lack of definition of the coefficients on the set where $du = 0$ makes the analogous Finslerian problem not completely similar to the Riemannian one. In a recent paper, T. Liimatainen and M. Salo [5] considered the problem of existence of "harmonic" coordinates for nonlinear degenerate elliptic operators on Riemannian manifold, including the p-Laplace operator. We will show that this result extends to the Finslerian Laplace operator as well.

Theorem 1. *Let (M, F, μ) be a smooth Finsler manifold of dimension m, endowed with a smooth volume form μ, such that $F \in C^{k+1}(TM \setminus 0)$; $k \geq 2$ (respectively, $F \in C^{\infty}(TM \setminus 0)$—$M$ is endowed with an analytic structure, μ is also analytic, and $F \in C^{\omega}(TM \setminus 0)$). Let $p \in M$, then, there exists a neighborhood V of p and a map $\Psi : V \to \Psi(V) \subset \mathbb{R}^m$ such that $\Psi = (u^1, \ldots, u^m)$ is a $C^{k-1,\alpha}$; $\alpha \in (0,1)$ depending on V (respectively, C^{∞}; analytic) diffeomorphism; and $\Delta u^i = 0$, for all $i = 1, \ldots, m$, where Δ is the nonlinear Laplacian operator associated to F.*

Theorem 1 is proved in Section 3, where we also show (Proposition 3) that harmonic (for the Finsler nonlinear Laplacian) coordinates can be used to prove the Myers–Steenrod theorem about regularity of distance-preserving bijection between Finsler manifolds. In the Riemannian case, this was established first by M. Taylor in [6].

For a semi-Riemannian metric h, the expression in local coordinates of the Ricci tensor is given by

$$\left(\mathrm{Ric}(h)\right)_{ij} = -\frac{1}{2}h^{rs}\frac{\partial^2 h_{ij}}{\partial x^r \partial x^s} + \frac{1}{2}\left(h_{ri}\frac{\partial H^r}{\partial x^j} + h_{rj}\frac{\partial H^r}{\partial x^i}\right) + \textit{lower order terms},$$

(see Lemma 4.1 [2]) where $H^r := h^{ij}H^r_{ij}$ and H^r_{ij} are the Christoffel symbols of h. As recalled above, in harmonic coordinates, the higher order terms in this expression simplify to $-\frac{1}{2}h^{rs}\frac{\partial^2 h_{ij}}{\partial x^r \partial x^s}$ because

$$H^r = 0 \tag{1}$$

in such coordinates. When h is Riemannian, by regularity theory for elliptic PDEs system [7,8], this observation leads to optimal regularity results for the components of the metric h once a certain level of regularity of the Ricci tensor is known [2]. Roughly speaking, the fact that the nonlinear Laplacian is a differential operator on M while the fundamental tensor and the components of any Finslerian connection are objects defined on $TM \setminus 0$—harmonic coordinates for the nonlinear Finsler Laplacian do not give such information (see Remark 5). Nevertheless, in Section 4, we will consider these types of problems for Berwald metrics and we will obtain some partial result in this direction.

2. The Nonlinear Finsler Laplacian

Let M be a smooth (i.e., C^∞), oriented manifold of dimension m, and let us denote by TM and $TM \setminus 0$, respectively, the tangent bundle and the slit tangent bundle of M, i.e., $TM \setminus 0 := \{v \in TM : v \neq 0\}$. A Finsler metric on M is a non-negative function on TM such that for any $x \in M$, $F(x,\cdot)$ is a strongly convex Minkowski norm on T_xM, i.e.,

- $F(x, \lambda v) = \lambda F(x, v)$ for all $\lambda > 0$ and $v \in TM$, $F(x, v) = 0$ if and only if $v = 0$;
- the bilinear symmetric form on T_xM, depending on $(x, v) \in TM \setminus 0$,

$$g(x,v)[w_1, w_2] := \frac{1}{2}\frac{\partial^2}{\partial s \partial t}F^2(x, v + sw_1 + tw_2)|_{(s,t)=(0,0)} \tag{2}$$

is positive definite for all $v \in T_xM \setminus \{0\}$ and it is called the *fundamental tensor* of F.

Let us recall that a function defined in some open subset U of $\mathbb{R}^m$ is of class $C^{k,\alpha}$, for $k \in \mathbb{N}$ and $\alpha \in (0,1]$, (respectively, C^∞ and C^ω), if all its derivative up to order k exist and are continuous in U and its k-th derivatives are Hölder continuous in U with exponent α (respectively, if the derivatives of any order exists and are continuous in U; and if it is real analytic in U).

We assume that $F \in C^{k+1}(TM \setminus 0)$, where $k \in \mathbb{N}$, $k \geq 2$ (respectively, $F \in C^\infty(TM \setminus 0)$; $F \in C^\omega(TM \setminus 0)$, provided that M is endowed with an analytic structure) in the natural charts of TM associated to an atlas of M—it is easy to prove, by using 2-homogeneity of the function $F^2(x,\cdot)$, that F^2 is C^1 on TM with Lipschitz derivatives on subsets of TM of the type $K \times \mathbb{R}^m$, with K compact in M.

Let us consider the function

$$F^*\colon T^*M \to [0, +\infty), \qquad F^*(x,\omega) := \max_{\substack{v \in T_xM \\ F(v)=1}} \omega(x)[v],$$

which is a *co-Finsler* metric, i.e., for any $x \in M$, $F^*(x,\omega)$ is a Minkowski norm on T^*_xM. It is well-known (see, e.g., [9] (p. 308)) that $F = F^* \circ \ell$, where $\ell : TM \to T^*M$ is the Legendre map:

$$\ell(x,v) = \left(x, \frac{1}{2}\frac{\partial}{\partial v}F^2(x,v)[\cdot]\right),$$

and $\frac{\partial}{\partial v}F^2(x,v)[\cdot]$ is the vertical derivative of F^2 evaluated at (x,v), i.e., $\frac{\partial}{\partial v}F^2(x,v)[u] := \frac{d}{dt}F(v+tu)|_{t=0}$. Thus, $F^* = F \circ \ell^{-1}$, and $F^* \in C^k(T^*M \setminus 0)$ (respectively, $F^* \in C^\infty(T^*M \setminus 0)$; $F^* \in C^\omega(T^*M \setminus 0)$). Its fundamental tensor, obtained as in (2), will be denoted by g^*. The components $(g^*)^{ij}(x,\omega)$ of $g^*(x,\omega)$, $(x,\omega) \in T^*M \setminus 0$, in natural local coordinate of T^*M define a square matrix which is the inverse of the one defined by the components $g_{ij}(\ell^{-1}(x,\omega))$ of $g(\ell^{-1}(x,\omega))$.

Henceforth, we will often omit the dependence on x in $F(x,v)$, $F^*(x,\omega)$, $g(x,v)$, $g^*(x,\omega)$, etc. (which is implicitly carried on by vectors or covectors), writing simply $F(v)$, $F^*(\omega)$, g_v, g^*_ω, etc.

For a differentiable function $f\colon M \to \mathbb{R}$, the *gradient* of f is defined as $\nabla f := \ell^{-1}(df)$. Hence, $F(\nabla f) = F^*(df)$ and, wherever $df \neq 0$, $df = \ell(\nabla f) = g_{\nabla f}(\nabla f, \cdot)$.

Given a smooth volume form μ on M, μ locally given as $\mu = \sigma dx^1 \wedge \ldots \wedge dx^m$, the *divergence* of a vector field $x \in \mathfrak{X}(M)$ is defined as the function $\operatorname{div}(X)$ such that $\operatorname{div}(X)\mu = \mathfrak{L}_X\mu$, where $\mathfrak{L}$ is the Lie derivative—in local coordinates this is the function $\frac{1}{\sigma}\partial_{x^i}(\sigma X^i)$. The Finslerian Laplacian of a smooth function on M is then defined as $\Delta f := \operatorname{div}(\nabla f)$, thus, in local coordinates it is given by

$$\Delta f = \frac{1}{\sigma}\partial_{x^i}\left(\sigma \frac{1}{2}\frac{\partial F^{*2}}{\partial \omega_i}(df)\right),$$

where $(x^i, \omega_i)_{i=1,\ldots,m}$ are natural local coordinates of T^*M and the Einstein summation convention has been used. Notice that, wherever $df \neq 0$, Δf is equal to

$$\Delta f = \frac{1}{\sigma}\partial_{x^i}\left(\sigma (g^*_{df})^{ij}\partial_{x^j} f\right),$$

thus, Δ is a quasi-linear operator, and when F is the norm of a Riemannian metric h and $\sigma = \sqrt{\det h}$—it becomes linear and equal to the Laplace–Beltrami operator of h.

Let $\Omega \subset M$ be an open relatively compact subset with smooth a boundary, and let $H^k_{\mathrm{loc}}(\Omega)$, $k \in \mathbb{N} \setminus \{0\}$ be the Sobolev space of functions defined on Ω that are of H^k class on the open subsets Ω' with compact closure contained in Ω. $\Omega' \subset\subset \Omega$ $H^k_{\mathrm{loc}}(\Omega)$ can be defined only in terms of the differentiable structure of M, see [10] (Section 4.7). Let us also denote by $H^1(\Omega)$ and $H^1_0(\Omega)$ the usual Sobolev spaces on a smooth compact manifold with boundary (see [10] (Sections 4.4 and 4.5)).

Let $E(u) := \frac{1}{2}\int_\Omega (F^{*2}(du)d\mu = \frac{1}{2}\int_\Omega F^2(\nabla u)d\mu \in [0, +\infty)$ be the Dirichlet functional of (M,F). Let $\varphi \in H^1(\Omega)$, then, the critical points u of E on $\{\varphi\} + H^1_0(\Omega)$ are the weak solutions of

$$\begin{cases} \Delta u = 0 & \text{in } \Omega; \\ u = \varphi & \text{on } \partial\Omega, \end{cases} \tag{3}$$

i.e., for all $\eta \in H^1_0(\Omega)$ it holds:

$$\frac{1}{2}\int_\Omega \frac{\partial F^{*2}}{\partial \omega}(du)[d\eta]d\mu = 0, \qquad u - \varphi \in H^1_0(\Omega).$$

Let (V,ϕ) be a coordinate system in M, with $\phi : V \subset M \to U \subset \mathbb{R}^m$, $\phi(p) = (x^1(p), \ldots, x^m(p))$, and $\bar{U}$ compact. In the coordinates $(x^1, \ldots, x^m)$, (up to the factor $1/\sigma$), the equation $\Delta u = 0$ corresponds to $\operatorname{div}(\mathcal{A}(x,Du) = 0$, where $\mathcal{A}\colon U \times \mathbb{R}^m \to \mathbb{R}^m$ is the map whose components are given by $\mathcal{A}^i(x,\omega) := \sigma(x)\frac{1}{2}\frac{\partial F^{*2}}{\partial \omega_i}(\omega)$ and Du is the vector whose components are $(\partial_{x^1}u, \ldots, \partial_{x^m}u)$. Observe that $\mathcal{A} \in C^0(\bar{U} \times \mathbb{R}^m) \cap C^{k-1}(\bar{U} \times \mathbb{R}^m \setminus \{0\})$ (respectively, $\mathcal{A} \in C^0(\bar{U} \times \mathbb{R}^m) \cap C^\infty(\bar{U} \times \mathbb{R}^m \setminus \{0\})$; $\mathcal{A} \in C^0(\bar{U} \times \mathbb{R}^m) \cap C^\omega(\bar{U} \times \mathbb{R}^m \setminus \{0\})$). Thus, Finslerian harmonic functions are locally $\mathcal{A}$-harmonic in the sense of [5]. We notice that $\mathcal{A}$ satisfies the following properties: There exists $C \geq 0$ such that

1. for all $(x, \omega) \in U \times \mathbb{R}^m \setminus \{0\}$:
$$\|\mathcal{A}(x,\omega)\| + \|\partial_x \mathcal{A}(x,\omega)\| + \|\omega\| \, \|\partial_\omega \mathcal{A}(x,\omega)\| \leq C\|\omega\|; \tag{4}$$
2. for all $(x, \omega) \in U \times (\mathbb{R}^m \setminus \{0\})$ and all $h \in \mathbb{R}^m$:
$$\partial_\omega \mathcal{A}(x,\omega)[h,h] \geq \frac{1}{C}\|h\|^2; \tag{5}$$
3. for all $x \in U$ and $\omega_1, \omega_2 \in \mathbb{R}^m$:
$$\big(\mathcal{A}(x,\omega_2) - \mathcal{A}(x,\omega_1)\big)(\omega_2 - \omega_1) \geq \frac{1}{C}\|\omega_2 - \omega_1\|^2; \tag{6}$$

in particular, (6) comes from strong convexity of F^{*2} on $TM \setminus 0$ (i.e., by (5)) and by a continuity argument when $\omega_2 = -\lambda\omega_1$, for some $\lambda > 0$.

By the theory of monotone operators or by a minimization argument based on the fact that E satisfies the Palais–Smale condition (see [11] (pp. 729–730)) we have that for all $\varphi \in H^1(\Omega)$, there exists a minimum of E on $\{\varphi\} \times H^1_0(\Omega)$, which is then a weak solution of (3).

Now, as in [11,12], the following proposition holds:

Proposition 1. *Any weak solution of* (3) *belongs to* $H^2_{\mathrm{loc}}(\Omega) \cap C^{1,\alpha}_{\mathrm{loc}}(\Omega)$*, for some* $\alpha \in (0,1)$.

Remark 1. *The Hölder constant* α *in the above proposition depends on the open relatively compact subset* $\Omega' \subset\subset \Omega$*, where* u *is seen as a local weak solution of* $\Delta u = 0$*, i.e.,*
$$\frac{1}{2}\int_\Omega \frac{\partial F^{*2}}{\partial \omega}(du)[d\eta]d\mu = 0, \qquad \textit{for all } \eta \in C^\infty_c(\Omega').$$

From the above proposition and classical results for uniformly elliptic operators, we can obtain higher regularity, where $du \neq 0$. In fact, if $du \neq 0$ on an open subset $U \subset\subset \Omega$, then, being $u \in H^2(U)$, the equation $\Delta u = 0$ is equivalent to
$$\sum_{i,j=1}^m (g^*_{du})^{ij} \frac{\partial^2 u}{\partial x^i \partial x^j} = -\sum_{i=1}^m \left(\frac{1}{\sigma}\frac{\partial \sigma}{\partial x^i}\frac{1}{2}\frac{\partial F^{*2}}{\partial \omega_i}(du) + \frac{1}{2}\frac{\partial^2 F^{*2}}{\partial x^i \partial \omega_i}(du) \right), \quad \text{a.e. on } U. \tag{7}$$

This can be interpreted as a linear elliptic equation:
$$a^{ij}(x)\partial_{x^i x^j} u = f(x),$$
where $a^{ij}(x) := (g^*_{du(x)})^{ij}$, and $f(x) = -\sum_{i=1}^m \left(\frac{1}{\sigma(x)}\frac{\partial \sigma}{\partial x^i}(x)\frac{1}{2}\frac{\partial F^{*2}}{\partial \omega_i}(du(x)) + \frac{1}{2}\frac{\partial^2 F^{*2}}{\partial x^i \partial \omega_i}(du(x)) \right)$. Thus, when $k \geq 3$, the coefficients a^{ij} and f are at least α-Hölder continuous, then, $u \in C^{2,\alpha}(U)$. By a bootstrap argument, we then get the following proposition (see, e.g., [13] (Appendix J, Theorem 40)):

Proposition 2. *Let* $U \subset\subset \Omega$ *be an open subset such that* $du \neq 0$ *on* U*, then, any weak solution of* (3) *belongs to* $C^{k-1,\alpha}(U)$*, for some* α *depending on* U*. Moreover, it is* $C^\infty(U)$ *(respectively,* $C^\omega(U)$*) if* $k = \infty$*, (respectively, if* M *is endowed with an analytic structure,* $k = \omega$*, and* σ *is also analytic).*

3. Harmonic Coordinates in Finsler Manifolds

Let us consider a smooth atlas of the manifold M and a point $p \in M$. Let (U, ϕ) be a chart of the atlas, with components $(x^1, \ldots, x^m)$, such that $p \in U$, $\phi(p) = 0$, also let us assume that the open ball $B_\epsilon(0)$, for some $\epsilon \in (0,1)$, is contained in $\phi(U)$.

Proof of Theorem 1. Let $(u^i)_{i\in\{1,\ldots,m\}}$ be weak solutions of the m Dirichlet problems:

$$\begin{cases} \operatorname{div}\left(\mathcal{A}(x,Du^i)\right)=0 & \text{in } B_\epsilon(0) \\ u^i = x^i & \text{on } \partial B_\epsilon(0). \end{cases}$$

Following [14] (Section 3.9) and [5] (Theorem 2.4), we can rescale the above problems by considering $\tilde{u}^i(\tilde{x}) := \frac{1}{\epsilon}u^i(\epsilon\tilde{x})$ and $\mathcal{A}_\epsilon(\tilde{x},\omega) := \mathcal{A}(\epsilon\tilde{x},\omega)$, so that the problems are transferred on $B_1(0)$ with Dirichlet data $\tilde{x}^i := \frac{x^i}{\epsilon}$. Notice that $\mathcal{A}_\epsilon$ satisfies (4)–(6) uniformly with respect to $\epsilon \in (0,1)$. Hence, there exists a solution $\tilde{u}^i \in H^1\left(B_1(0)\right)$ of

$$\begin{cases} \operatorname{div}\left(\mathcal{A}_\epsilon(\tilde{x},D\tilde{u}^i)\right)=0 & \text{in } B_1(0) \\ \tilde{u}^i = \tilde{x}^i & \text{on } \partial B_1(0). \end{cases} \tag{8}$$

Moreover, there exists $\delta \in (0,1)$ such that $\|\tilde{u}^i\|_{C^{1,\alpha}\left(B_\delta(0)\right)} \le C_1$, for all $i \in \{1,\ldots,m\}$, uniformly with respect to ϵ (notice that, since α depends only on $m,\ C,\ \delta$, and $\|\tilde{u}^i\|_{H^1\left(B_1(0)\right)}$, which is uniformly bounded with respect to to ϵ, α is independent of ϵ as well).

Let us denote $\tilde{x}^i$ by $\tilde{u}^i_0$ and let $\tilde{v}^i := \tilde{u}^i - \tilde{u}^i_0$. From (6), we have

$$\int_{B_1(0)} |D\tilde{v}^i|^2 d\tilde{x} \le C\int_{B_1(0)} \left(\mathcal{A}_\epsilon(\tilde{x},D\tilde{u}^i) - \mathcal{A}_\epsilon(\tilde{x},D\tilde{u}^i_0)\right)(D\tilde{v}^i)d\tilde{x}.$$

Recalling that $\tilde{u}^i$ solves (8) and using $\tilde{v}^i$ as a test function, we obtain

$$\begin{aligned} &\int_{B_1(0)} \left(\mathcal{A}_\epsilon(\tilde{x},D\tilde{u}^i) - \mathcal{A}_\epsilon(\tilde{x},D\tilde{u}^i_0)\right)(D\tilde{v}^i)d\tilde{x} = \\ &-\int_{B_1(0)} \mathcal{A}_\epsilon(\tilde{x},D\tilde{u}^i_0)(D\tilde{v}^i)d\tilde{x} = \\ &-\int_{B_1(0)} \left(\mathcal{A}_\epsilon(\tilde{x},D\tilde{u}^i_0) - \mathcal{A}_0(0,D\tilde{u}^i_0)\right)(D\tilde{v}^i)d\tilde{x}, \end{aligned}$$

where the last equality is a consequence of being $\tilde{u}^i_0$, $\mathcal{A}_0$-harmonic. As $\mathcal{A}$ is locally Lipschitz and $D\tilde{u}^i_0$ is a constant vector, also using Hölder's inequality, we then get

$$-\int_{B_1(0)} \left(\mathcal{A}_\epsilon(\tilde{x},D\tilde{u}^i_0) - \mathcal{A}_0(0,D\tilde{u}^i_0)\right)(D\tilde{v}^i)d\tilde{x} \le \epsilon C_2\left(\int_{B_1(0)} |D\tilde{v}^i|^2 d\tilde{x}\right)^{1/2},$$

thus, $\|D\tilde{v}^i\|^2_{L^2\left(B_1(0)\right)} \le \epsilon C_2\|D\tilde{v}^i\|_{L^2\left(B_1(0)\right)}$, i.e., $\|D\tilde{v}^i\|_{L^2\left(B_1(0)\right)} \le \epsilon C_2$. Since there exists also a constant $C_4 \ge 0$ such that

$$\|D\tilde{v}^i\|_{C^{0,\alpha}\left(B_\delta(0)\right)} \le C_4,$$

by [5] (Lemma A.1)) we get that there exists $\delta' \in (0,\delta)$, such that $\|D\tilde{v}^i\|_{L^\infty\left(B_{\delta'}(0)\right)} = o(1)$ as $\epsilon \to 0$, for all $i \in \{0,\ldots,m\}$. Therefore, if $\tilde{\Phi}(\tilde{x}) := \left(\tilde{u}^1(\tilde{x}),\ldots,\tilde{u}^m(\tilde{x})\right)$, we have $\|D\tilde{\Phi}(0) - \mathrm{Id}\| = o(1)$ as $\epsilon \to 0$, and then $D\Phi(0) = D\tilde{\Phi}(0)$ $(\Phi(x) := \left(u^1(x),\ldots,u^m(x)\right))$ is invertible. Moreover, up to considering a smaller δ', we have also that $Du^i(x) \neq 0$ for all $x \in B_{\delta'}(0)$ and all $i \in \{1,\ldots,m\}$. Thus, from Proposition 2 Φ is a $C^{k-1,\alpha}$ (respectively, C^∞; analytic) diffeomorphism on $B_{\delta'}(0)$ and for $V_p := \phi^{-1}(B_{\delta'}(0))$, we have that $\Psi := \Phi\circ\phi|V$ is a $C^{k-1,\alpha}$ (respectively, C^∞; analytic) diffeomorphism whose components $u^i\circ\phi$ are harmonic. □

Harmonic coordinates were successfully used by M. Taylor [6] in a new proof of the Myers–Steenrod theorem about regularity of isometries between Riemannian manifolds (including the case when the metrics are only Hölder continuous). In the Finsler setting, Myers–Steenrod theorem has been obtained with different methods, in [15,16] for smooth (and strongly convex) Finsler metrics; and in [17] for Hölder continuos Finsler metrics. We show here that harmonic coordinates can be used to prove the Myers–Steenrod in the Finsler setting too, provided that the Finsler metrics are smooth enough.

Let (M_1, F_1, μ_1) and (M_2, F_2, μ_2) be two oriented Finsler manifolds of the same dimension m, endowed with the volume forms μ_1 and μ_2. Let us assume that μ_1 and μ_2 are locally Lipschitz with locally bounded differential, meaning that in the local expressions of μ_1 and μ_2, $\mu_1 = \sigma_1 dx^1 \wedge \ldots \wedge dx^m$, $\mu_2 = \sigma_2 dx^1 \wedge \ldots \wedge dx^m$, σ_1 and σ_2 are Lipschitz functions and their derivatives (which are defined a.e. by Rademacher's theorem); and $\partial_{x^i}\sigma_1$ and $\partial_{x^i}\sigma_2$, for each $i \in \{1, \ldots, m\}$, are L^∞ functions. Let d_i, $i = 1, 2$, be the (nonsymmetric) distances associated to F_i. Let $\mathcal{I} : M_1 \to M_2$ be a distance-preserving bijection. Clearly, $\mathcal{I}$ is an isometry of the symmetric distance $\tilde{d}_i(x_1, x_2) := d_i(x, y) + d_i(y, x)$, thus, in particular, it is a bi-Lipschitz map (i.e., it is Lipschitz with Lipschitz inverse) with respect to the distances $\tilde{d}_i$; and it is locally Lipschitz with respect to the distances associated to any Riemannian metric on M_1 and M_2. Hence, $\mathcal{I}$ and its inverse are differentiable a.e. on M_1 and M_2, respectively. In the next lemma we deal with the relations existing between the Finsler metrics F_1 and F_2, the inverse maps of their Legendre maps ℓ_1 and ℓ_2, and their co-Finsler metrics F_1^* and F_2^*, in presence of an isometry $\mathcal{I}$. For a fixed x in M_1 or M_2, let us denote by $\mathcal{J}_{i,x}$, $i = 1, 2$, the diffeomorphisms between $T_x M_i$ and $T_x^* M_i$, given by $\mathcal{J}_{i,x}(v) := \frac{1}{2}\frac{\partial}{\partial v} F_i^2(x, v)[\cdot]$.

Lemma 1. *Let $\mathcal{I}$ be a distance preserving bijection between (M_1, F_1, μ_1) and (M_2, F_2, μ_2). Then, for a.e. $x \in M_1$, we have:*

(a) $F_1 = \mathcal{I}^*(F_2)$, *(i.e.,* $F_1(x, v) = F_2(\mathcal{I}(x), d\mathcal{I}(x)[v])$*);*
(b) $\ell_1^{-1}(x, \omega) = \big(x, d\mathcal{I}^{-1}(\mathcal{I}(x))[\mathcal{J}^{-1}_{2,\mathcal{I}(x)}(\omega \circ d\mathcal{I}^{-1})]\big)$*;*
(c) $F_1^* = \mathcal{I}^*(F_2^*)$, *(i.e.,* $F_1^*(x, \omega) = F_2^*\big(\mathcal{I}(x), \omega \circ d\mathcal{I}^{-1}\big)$*).*

Proof. It is well-known that any Finsler metric F on a manifold M can be computed by using the associated distance d as $F(x, v) = \lim_{t \to 0^+} \frac{1}{t} d\big(\gamma(0), \gamma(t)\big)$, where γ is a smooth curve on M such that $\gamma(0) = x$ and $\dot{\gamma}(0) = v$—hence, (a) immediately follows from this property and the fact that $\mathcal{I}$ is a distance preserving map. From (a), we get

$$\mathcal{J}_{1,x}(v) = \frac{1}{2}\frac{\partial}{\partial v}\Big(F_2^2(\mathcal{I}(x), d\mathcal{I}(x)[v])\Big) = \frac{1}{2}\frac{\partial}{\partial w} F_2^2(\mathcal{I}(x), d\mathcal{I}(x)[v])\,[d\mathcal{I}(x)[\cdot]]$$
$$= \frac{1}{2}\mathcal{J}_{2,\mathcal{I}(x)}(d\mathcal{I}(x)[v])\,[d\mathcal{I}(x)[\cdot]],$$

then, we deduce (b). Finally, (c) follows from

$$F_1^*(x, \omega) = F_1\big(\ell_1^{-1}(x, \omega)\big) = F_2\Big(\mathcal{I}(x), d\mathcal{I}(x)\Big[d\mathcal{I}^{-1}(\mathcal{I}(x))[\mathcal{J}^{-1}_{2,\mathcal{I}(x)}(\omega \circ d\mathcal{I}^{-1})]\Big]\Big)$$
$$= F_2\big(\mathcal{I}(x), \mathcal{J}^{-1}_{2,\mathcal{I}(x)}(\omega \circ d\mathcal{I}^{-1})\big) = F_2^*\big(\mathcal{I}(x), \omega \circ d\mathcal{I}^{-1}\big).$$

□

Proposition 3. *Let (M_1, F_1, μ_1) and (M_2, F_2, μ_2) be two oriented Finsler manifolds, where μ_1 and μ_2 are locally Lipschitz with locally bounded differential (in the sense specified above) volume forms. Let $\mathcal{I}\colon M_1 \to M_2$ be a distance preserving bijective map, if $\mu_1 = \mathcal{I}^*(\mu_2)$ (i.e., locally $\sigma_1 = |\mathrm{Jac}(\mathcal{I})|\sigma_2 \circ \mathcal{I}$, where $\mathrm{Jac}(\mathcal{I})$ is the Jacobian of $\mathcal{I}$) and F_i, $i = 1, 2$ are at least C^3 on $TM_i \setminus 0$, then $\mathcal{I}$ is a C^1 diffeomorphism.*

Proof. Given that $\mathcal{I}$ a bi-Lipshitz map, it is enough to prove that $\mathcal{I}$ is locally C^1 with locally C^1 inverse. Under the assumptions on F_i and μ_i, $i = 1, 2$, we have that (4)–(6) hold, then, given an open relatively compact subset $\Omega_1 \subset M_1$ with Lipschitz boundary, we have the existence of a minimum of the functional E_1 on $\{\varphi\} \times H_0^1(\Omega_1)$, for any $\varphi \in H^1(\Omega_1)$—clearly, the same existence of minima holds for E_2 on analogous subsets $\Omega_2 \subset\subset M_2$. Moreover, the same assumptions on F_i and μ_i ensure that such minima are locally weak harmonic and, arguing as in [12] (Theorem 4.6 and Theorem 4.9), they are H^2 and $C^{1,\alpha}$ on subsets $U_1 \subset\subset \Omega_1$ (respectively, $U_2 \subset\subset \Omega_2$). Therefore, Theorem 1 still holds under the assumptions of Proposition 3, and it provides $C^{1,\alpha}$ diffeomorphisms Ψ_i, $i = 1, 2$, whose components are harmonic. Now, let $p \in M_1$ and (U_2, Ψ_2) be a chart of harmonic coordinates of (M_2, F_2, μ_2) centred at $\mathcal{I}(p)$, $\Psi_2 = (u_2^1, \dots, u_2^m)$. Let us show that, for each $j \in \{1, \dots, m\}\}$, $u_1^j := u_2^j \circ \mathcal{I}$ is weakly harmonic on $U_1 = \phi^{-1}(U_2)$. First, we notice that for any function $u \in H^1(U_2)$, $u \circ \mathcal{I} \in H^1(U_1)$, because $\mathcal{I}$ is Lipschitz. Moreover, from (c) of Lemma 1 and the change of variable formula for integrals under bi-Lipschitz transformations, we have

$$E_1(u \circ \mathcal{I}|_{U_1}) = \frac{1}{2}\int_{U_1} F_2^{*2}\Big(\mathcal{I}(x), \big(du \circ d\mathcal{I} \circ d\mathcal{I}^{-1}\big)\big(I(x)\big)\Big)\mathcal{I}^*(d\mu_2) = \frac{1}{2}\int_{U_2} F_2^{*2}(x, du(x))d\mu_2.$$

Hence, given that u_2^j is a minimum of E_2 on $\{u_2^j\} \times H_0^1(U_2)$, we deduce that u_1^j is a minimum of E_1 on $\{u_1^j\} \times H_0^1(U_1)$, and then it is a weakly harmonic function. Therefore, $\Psi_1 := (u_1^1, \dots, u_1^m)$ is a $C^{1,\alpha}$ diffeomorphism, and $\mathcal{I}|_{U_1} = \Psi_2^{-1} \circ \Psi_1$ and its inverse are both $C^{1,\alpha}$ map as well. □

Remark 2. *If, for each $i \in \{1, 2\}$, F_i is of class C^{k+1} on $TM \setminus 0$, with $k \geq 3$; and μ_i is of class C^{k-1}, as in Theorem 1 (recall, in particular, (7)), we can deduce that $\mathcal{I}$ is a C^{k-1} diffeomorphism provided that μ_1 and μ_2 are related by $\mu_1 = \mathcal{I}^*(\mu_2)$.*

4. Regularity Results for Berwald Metrics

Let us recall the following result from [2], which gives optimal regularity of a Riemannian metric in harmonic coordinates, in connection with the regularity of the Ricci tensor (the meaning of "optimal" here is illustrated in all its facets in [2]).

Theorem 2 (Deturck–Kazdan)**.** *Let h be a C^2 Riemannian metric, and $\mathrm{Ric}(h)$ be its Ricci tensor. If, in harmonic coordinates of h, $\mathrm{Ric}(h)$ is of class $C^{k,\alpha}$, for $k \geq 0$, (respectively, C^∞ and C^ω), then, in these coordinates, h is of class $C^{k+2,\alpha}$ (respectively, C^∞ and C^ω).*

Let us consider now a Finsler manifold (M, F) such that F is C^4 on $TM \setminus 0$. A role similar to the one of the Ricci tensor in the result above will be played by the *Riemann curvature* of F. This is a family of linear transformations of the tangent spaces defined in the following way (see [18] (p. 97)): let $G^i(x, y)$, $y \in T_xM \setminus \{0\}$, and $x \in M$ be the spray coefficients of F:

$$G^i(x, y) := \frac{1}{4}g^{ij}(x, y)\left(\frac{\partial^2 F^2}{\partial x^k \partial y^j}(x, y)y^k - \frac{\partial F^2}{\partial x^j}(x, y)\right),$$

where $g^{ij}(x, y)$ are the components of the inverse of the matrix representing the fundamental tensor g at the point $(x, y) \in TM \setminus 0$.

Let

$$R_k^i(x, y) \quad := \quad 2\frac{\partial G^i}{\partial x^k}(x, y) - y^m\frac{\partial^2 G^i}{\partial x^m \partial y^k}(x, y) + 2G^m(x, y)\frac{\partial^2 G^i}{\partial y^m \partial y^k}(x, y) - \frac{\partial G^i}{\partial y^m}(x, y)\frac{\partial G^m}{\partial y^k}(x, y).$$

As above, we will omit the explicit dependence on x, by writing simply $R_k^i(y)$. The Riemann curvature of F at $y \in T_xM \setminus \{0\}$ is then the linear map $\mathbf{R}_y : T_xM \to T_xM$, given by $\mathbf{R}_y := R_k^i(y)\partial_{x^i} \otimes dx^k$. It can be shown (see [18] (Equations (8.11)–(8.12)) that

$$R_k^i(y) = R_{jkl}^i(y)y^j y^l,$$

where R^i_{jkl} are the components of the hh part of the curvature 2-forms of the Chern connection, which are equal, in natural local coordinate on TM, to

$$R^i_{jkl}(y) := \frac{\delta\Gamma^i_{jl}}{\delta x^k}(y) - \frac{\delta\Gamma^i_{jk}}{\delta x^l}(y) + \Gamma^m_{jl}(y)\Gamma^i_{mk}(y) - \Gamma^m_{jk}(y)\Gamma^i_{ml}(y), \tag{9}$$

where Γ^i_{jk} are the components of the Chern connection and $\frac{\delta}{\delta x^i}$ is the vector field on $TM \setminus 0$ defined by $\frac{\delta}{\delta x^i} := \frac{\partial}{\partial x^i} - N^m_i(y)\frac{\partial}{\partial y^m}$, where $N^m_i(y) := \frac{1}{2}\frac{\partial G^m}{\partial y^i}(y)$.

Finally, let us introduce the *Finsler Ricci scalar* as the contraction of the Riemann curvature $R(y) := R^i_i(y)$ (see [18] (Equation (6.10))).

We recall that, if F is the norm of a Riemannian metric h (i.e., $F(y) = \sqrt{h(y,y)}$), then, the components of the Chern connection coincide with those of the Levi–Civita connection, so they do not depend on y but only on $x \in M$, and the functions R^i_{jkl} are then equal to the components of the standard Riemannian curvature tensor of h.

Let us also recall (see [18] (p. 85)) that, to any Finsler manifold (M, F), we can associate a canonical *covariant derivative* of a vector field V on M in the direction $y \in TM$, defined in local coordinates as

$$D_y V := \left(dV^i(y) + V^j(x)N^i_j(y)\right)\partial_{x^i}|_x, \qquad x = \pi(y),$$

and extending it as 0 if $y = 0$.

There are several equivalent ways to introduce Berwald metrics (see e.g., [19]), we will say that a Finsler metric is said *Berwald* if the nonlinear connection N^i_j is actually a linear connection on M—thus, $N^i_j(y) = \Gamma^i_{jk}(x)y^k$ (see [20] (prop. 10.2.1)), so that the components of the Chern connection do not depend on y. From (9), the same holds for the components of the Riemannian curvature tensor R^i_{jkl}.

From a result by Z. Szabó [21], we know that there exists a Riemannian metric h such that its Levi–Civita connection is equal to the Chern connection of (M, F). Actually, such a Riemannian metric is not unique, and different ways to construct one do exist—in particular, a branch of these methods is based on averaging over the indicatrixes (or, equivalently, on the unit balls) of the Finsler metric $S_x := \{y \in T_xM : F(y) = 1\}$, $x \in M$ (see the nice review [22]). Moreover, the fundamental tensor of F can be used in this averaging procedure as shown first by C. Vincze [23]; then, in a slight different way, in [24] (based on [25]); and in other manners also in [22]. In particular, as described in [22], the Riemannian metric obtained in [25] is given, up to a constant conformal factor in the Berwald case, by

$$h_x(V_1, V_2) := \frac{\int_{S_x} g(x,y)[V_1, V_2]d\lambda}{\int_{S_x} d\lambda}, \tag{10}$$

where $d\lambda$ denotes the measure induced on S_x (seen as an hypersurface on $\mathbb{R}^m \cong T_xM$) by the Lebesgue measure on $\mathbb{R}^m$.

Proposition 4. *Let (M, F) be a Berwald manifold such that F is a C^4 function on $TM \setminus 0$. Assume that the Finsler Ricci scalar R of F is of class $C^{k,\alpha}$, $k \geq 0$, (respectively, C^∞ and C^ω) on $TW \setminus 0$, for some open set $W \subset M$. Then, the Riemannian metric h in (10) is of class $C^{k+2,\alpha}$ (respectively, C^∞ and C^ω) in a system of harmonic coordinates (U, ϕ), $U \subset W$, of the same metric.*

Proof. Given that F is of class C^4 on $TM \setminus 0$, the partial derivatives of $g(x,y)$, up to the second order, exist on $TM \setminus 0$ and are continuous—for each $y \in S_x$, $\frac{\partial F^2(y)}{\partial y}[y] = 2F^2(y) \neq 0$, thus, 1 is a regular value of the function F^2, and the indicatrix bundle $\{(x,y) \in TM : F(x,y) = 1\}$ is a C^4 embedded hypersurface in TM. Thus, both the area of S_x and the numerator in (10) are C^2 in x, then, h is a C^2 Riemannian metric on M. From (9), and the fact that F is Berwald, the components R^i_{jkl} are equal to the ones of the Riemannian curvature tensor of h, then we have

$$\mathrm{Ric}(h)_{\alpha\beta}(x) = R^m_{\alpha m\beta}(x) = \frac{1}{2}\frac{\partial^2}{\partial y^\alpha \partial y^\beta}\left(R^m_{jml}(x)y^j y^l\right) = \frac{1}{2}\frac{\partial^2 R}{\partial y^\alpha \partial y^\beta}(x,y). \tag{11}$$

Moreover, R is quadratic in the y^j variables, i.e., $R(x,y) = R^m_{jml}(x)y^j y^l$ and its second vertical derivatives $\frac{\partial^2 R}{\partial y^\alpha \partial y^\beta}$, being independent of y^j, are $C^{k,\alpha}$ (respectively, C^∞ and C^ω) functions on W. Thus, the result follows from (11) and Theorem 2. $\square$

Remark 3. *Clearly, an analogous result holds for any C^2 Riemannian metric, such that its Levi-Civita connection is equal to the canonical connection of the Berwald metric as the Binet–Legendre metric in [26].*

Remark 4. *Under the assumptions of Proposition 4, we get that components of the Chern connection of the Berwald metric F are $C^{k+1,\alpha}$ (respectively, C^∞ and C^ω) in harmonic coordinates of the metric h. In particular the geodesic vector field $y^i\partial_{x^i} - \Gamma^i_{jk}y^j y^k \partial_{y^i}$ is $C^{k+1,\alpha}$ (respectively, C^∞ and C^ω) in the corresponding natural coordinate system of TM.*

Remark 5. *Other notions of Finslerian Laplacian, which take into account the geometry of the tangent bundle more than the nonlinear Finsler Laplacian, could be considered in trying to obtain some regularity results for the fundamental tensor without averaging. Natural candidates are the horizontal Laplacians studied in [27] which, in the Berwald case, are equal (up to a minus sign) to*

$$\Delta_H f = \frac{1}{\sqrt{g}}\frac{\delta}{\delta x^i}\left(\sqrt{g}g^{ij}\frac{\delta f}{\delta x^j}\right),$$

where f is a smooth function defined on some open subset of TM and $\sqrt{g} := \sqrt{\det g(x,y)}$. We notice that this is also the definition of the horizontal Laplacian of any Finsler metric given [28]. Moreover, for a C^2 function $f: M \to \mathbb{R}$, $\Delta_H f$ is equal to the g-trace of the Finslerian Hessian of f, $\mathrm{Hess}f := \nabla(df)$, where ∇ is the Chern connection. In natural local coordinates of TM, for each $(x,y) \in TM\setminus 0$, and all $u,v \in T_xM$, $\mathrm{Hess}f(x,y)[u,v]$ is given by $(\mathrm{Hess}f)_{ij}(x,y)u^iv^j = \frac{\partial^2 f}{\partial x^i \partial x^j}(x)u^iv^j - \frac{\partial f}{\partial x^k}(x)\Gamma^k_{ij}(x,y)u^iv^j$ (see e.g., [29])—hence, the g-trace of $\mathrm{Hess}f$ is equal to $g^{ij}(x,y)\frac{\partial^2 f}{\partial x^i\partial x^j}(x) - \frac{\partial f}{\partial x^k}(x)\Gamma^k(x,y)$, where $\Gamma^k := g^{ij}\Gamma^k_{ij}$. This expression is equivalent to

$$g^{ij}(x,y)(\mathrm{Hess}f)_{ij}(x,y) = g^{ij}(x,y)\frac{\delta^2 f}{\delta x^i \delta x^j}(x) - \frac{\delta f}{\delta x^k}(x)\Gamma^k(x,y), \tag{12}$$

because $\frac{\delta f}{\delta x^k} = \frac{\partial f}{\partial x^k}$ for a function defined on M. For a C^2 function f on $TM\setminus 0$, taking into account that $\frac{1}{2}g^{lm}\frac{\delta g_{lm}}{\delta x^i} = \Gamma^l_{li}$ and $g^{ij}\Gamma^l_{li} + \frac{\delta g^{ij}}{\delta x^i} = -g^{pq}\Gamma^j_{pq}$, we have

$$\begin{aligned}\Delta_H f &= \frac{1}{2}g^{lm}\frac{\delta g_{lm}}{\delta x^i}g^{ij}\frac{\delta f}{\delta x^j} + \frac{\delta g^{ij}}{\delta x^i}\frac{\delta f}{\delta x^j} + g^{ij}\frac{\delta^2 f}{\delta x^i \delta x^j} \\ &= g^{ij}\Gamma^l_{li}\frac{\delta f}{\delta x^j} + \frac{\delta g^{ij}}{\delta x^i}\frac{\delta f}{\delta x^j} + g^{ij}\frac{\delta^2 f}{\delta x^i\delta x^j} \\ &= -g^{hk}\Gamma^j_{hk}\frac{\delta f}{\delta x^j} + g^{ij}\frac{\delta^2 f}{\delta x^i \delta x^j},\end{aligned}$$

which coincides with (12) when f is a function defined on M.

In particular, if f is a Δ_H-harmonic coordinate on M, then,

$$0 = \frac{1}{\sqrt{g}}\frac{\delta}{\delta x^i}\left(\sqrt{g}g^{ij}\right) = g^{hk}\Gamma^j_{hk} = \Gamma^j,$$

which is analogous to (1)*. Anyway, ellipticity (and quasi-diagonality)—that hold in the Riemannian case for the system* $R_{ij} := R^{m}_{imj} = A_{ij}$*, where* A_{ij} *are* $C^{k,\alpha}$ *(respectively,* C^{∞} *and* C^{ω}*) functions—would be spoiled, in* Δ_H*-harmonic coordinates of a Berwald metric, by the presence of second order terms of the type* $g^{rs} N^{l}_{s} \frac{\partial^2 g_{ij}}{\partial x^r y^l}$.

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