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Approximation Properties of an Extended Family of the Szász–Mirakjan Beta-Type Operators

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Received: 14 September 2019; Accepted: 30 September 2019; Published: 10 October 2019

Abstract: Approximation and some other basic properties of various linear and nonlinear operators are potentially useful in many different areas of researches in the mathematical, physical, and engineering sciences. Motivated essentially by this aspect of approximation theory, our present study systematically investigates the approximation and other associated properties of a class of the Szász-Mirakjan-type operators, which are introduced here by using an extension of the familiar Beta function. We propose to establish moments of these extended Szász-Mirakjan Beta-type operators and estimate various convergence results with the help of the second modulus of smoothness and the classical modulus of continuity. We also investigate convergence via functions which belong to the Lipschitz class. Finally, we prove a Voronovskaja-type approximation theorem for the extended Szász-Mirakjan Beta-type operators.

Keywords: gamma and beta functions; Szász-Mirakjan operators; Szász-Mirakjan Beta type operators; extended Gamma and Beta functions; confluent hypergeometric function; Modulus of smoothness; modulus of continuity; Lipschitz class; local approximation; Voronovskaja type approximation theorem

2010 Mathematics Subject Classification: Primary 33B15; 33C05; 41A25; 41A35; Secondary 33C20

1. Introduction, Definitions and Preliminaries

In approximation theory and its related fields, approximation and other basic properties of various linear and nonlinear operators are investigated, because mainly of the potential for their usefulness in many areas of researches in the mathematical, physical, and engineering sciences. Our study in this article is motivated essentially by the demonstrated applications of such results as those associated with various approximation operators. With this objective in view, we begin by providing the following definitions and other (chiefly historical) background material related to our presentation here.

For a given continuous function, $f \in C[0,\infty)$, and for $x \in [0,\infty)$, Otto Szász [1] defined a family of operators in the year 1950, which we recall here as follows,

$$S_n(f;x) = e^{-nx} \sum_{k=0}^{\infty} \frac{(nx)^k}{k!} f\left(\frac{k}{n}\right) \qquad (n \in \mathbb{N} := \{1,2,3,\cdots\}), \tag{1}$$

This family of operators was considered earlier in 1941 by G. M. Mirakjan (see [2]). There are several integral and other modifications, variations, and basic (or q-) extensions of the Szász-Mirakjan-type operators. These include the Bézier, Kantorovich, Durrmeyer, and other types of modifications and extensions of the Szász-Mirakjan operators (see, for details, [3–15]). In particular, Gupta and Noor [6] introduced an integral modification of the Szász-Mirakjan operators in Equation (1) by considering a weight function in terms of the Beta basis functions as given below.

$$T_n(f;x) = \sum_{k=1}^{\infty} s_{n,k}(x) \int_0^{\infty} b_{n,k}(t)\; f(t)\, dt + s_{n,0}(x)\, f(0), \tag{2}$$

where

$$s_{n,k}(x) = e^{-nx}\,\frac{(nx)^k}{k!},$$

$$b_{n,k}(t) = \frac{1}{B(k,n+1)}\,\frac{t^{k-1}}{(1+t)^{n+k+1}} = \frac{(n+1)_k}{(k-1)!}\,\frac{t^{k-1}}{(1+t)^{n+k+1}},$$

$$B(\alpha,\beta) = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)} = B(\beta,\alpha),$$

and $(\lambda)_\ell \ \ (\ell \in \mathbb{N}_0 := \mathbb{N} \cup \{0\})$ represents the Pochhammer symbol given by

$$(\lambda)_\ell := \frac{\Gamma(\lambda+\ell)}{\Gamma(\lambda)} = \begin{cases} 1 & (\ell = 0) \\ \lambda(\lambda+1)(\lambda+2)\cdots(\lambda+\ell-1) & (\ell \in \mathbb{N}) \end{cases}$$

in terms of the classical (Euler's) Gamma function $\Gamma(z)$ and the classical Beta function $B(\alpha,\beta)$.

Gupta and Noor [6] observed that the operators in Equation (2) reproduce not only the constant function, but linear functions as well. Owing to this valuable property of the operators in Equation (2), many authors investigated the different approximation properties of the summation-integral operators in Equation (2) (see, for example, [16–18]). Gupta and Noor [6] also derived some direct results for the operators T_n, a pointwise rate of convergence, a Voronovskaja-type asymptotic formula, and an error estimate in simultaneous approximation.

In recent years, some extensions of such well-known special functions as, for example, the classical Gamma and Beta functions, have been considered by several authors. For example, in 1994, Chaudhry and Zubair [19] introduced the following extension of the Gamma function,

$$\Gamma_p(x) := \int_0^{\infty} t^{x-1}\, \exp\left(-t - \frac{p}{t}\right) dt \qquad (\Re(p) > 0). \tag{3}$$

Subsequently, in 1997, Chaudhry et al. [20] presented the following extension of Euler's Beta function,

$$B_p(x,y) := \int_0^1 t^{x-1}\, (1-t)^{y-1}\, \exp\left(-\frac{p}{t(1-t)}\right) dt \tag{4}$$

$$(\Re(p) > 0;\ \Re(x) > 0;\ \Re(y) > 0).$$

Obviously, each of the definitions in Equations (3) and (4) also remains valid when $p = 0$, in which case we have the following relationships,

$$\Gamma_0(x) = \Gamma(x) \qquad \text{and} \qquad B_0(x,y) = B(x,y).$$

Özergin et al. [21] considered the following generalizations of the Gamma and Beta functions,

$$\Gamma_p^{(\alpha,\beta)}(x) := \int_0^\infty t^{x-1}\, {}_1F_1\left(\alpha;\beta;-t-\frac{p}{t}\right)\,dt$$

$$\big(\Re(\alpha)>0;\ \Re(\beta)>0;\ \Re(p)>0;\ \Re(x)>0\big)$$

and

$$B_p^{(\alpha,\beta)}(x,y) := \int_0^1 t^{x-1}\,(1-t)^{y-1}\, {}_1F_1\left(\alpha;\beta;-\frac{p}{t(1-t)}\right)\,dt$$

$$\big(\Re(\alpha)>0;\ \Re(\beta)>0;\ \Re(p)>0;\ \Re(x)>0;\ \Re(y)>0\big),$$

respectively. Here, as usual, ${}_1F_1$ denotes the (Kummer's) confluent hypergeometric function. It is obvious that

$$\Gamma_p^{(\alpha,\alpha)}(x) = \Gamma_p(x) \qquad \text{and} \qquad \Gamma_0^{(\alpha,\alpha)}(x) = \Gamma(x)$$

and that

$$B_p^{(\alpha,\alpha)}(x,y) = B_p(x,y) \qquad \text{and} \qquad B_0^{(\alpha,\alpha)}(x,y) = B(x,y)\,.$$

Finding different integral representations of the generalized Beta function is important and useful for later use. It is also useful to discuss the relationships between the classical Gamma and Beta functions and their generalizations. In fact, by definition, it is easily seen that

$$\int_0^\infty \Gamma_p(x)\,dp = \Gamma(x+1) \qquad \big(\Re(x)>-1\big).$$

and that (see, for example, [21])

$$\int_0^\infty B_p(x,y)\,dp = B(x+1,y+1) \tag{5}$$

$$\big(\Re(x)>-1;\ \Re(y)>-1\big).$$

Note that various further extensions and generalizations of the classical Gamma and Beta functions, as well as their corresponding hypergeometric and related functions, were introduced and studied by, among others, Lin et al. [22] and Srivastava et al. [23].

We now introduce the following generalization of Szász-Mirakjan Beta-type operators via the above extension of the Beta function as follows,

$$\begin{aligned} S_n^*(f,x) = {} & \sum_{k=0}^\infty e^{-nx}\,\frac{(nx)^k}{k!}\,\frac{1}{B(k+2,n+1)} \\ & \cdot \int_0^\infty\int_0^\infty \frac{t^k}{(1+t)^{n+k+1}}\,\exp\left(-\frac{p(1+t)^2}{t}\right) f(t)\,dt\,dp \end{aligned} \tag{6}$$

for $x\in[0,\infty)$, and for a function $f\in C_\nu[0,\infty)$, provided that the double integral in Equation (6) is convergent when $n>\nu$. Here, and in what follows, we have

$$C_\nu[0,\infty) := \big\{f : f\in C[0,\infty) \quad \text{and} \quad |f(t)| \leqq M(1+t)^\nu \quad (M>0;\ \nu>0)\big\}\,.$$

We note that, by setting $t = \frac{u}{1+u}$ in Equation (4), we get

$$B_p\,(x,y) = \int_0^\infty \frac{u^{x-1}}{(1+u)^{x+y}}\,\exp\left(-\frac{p(1+u)^2}{u}\right)\,du. \tag{7}$$

So, if we take in consideration Equations (5) and (7) in the definition Equation (6), then we can say that the operators, S_n^*, are a generalization of the operators, T_n, given by Equation (2).

In this article, we investigate the moments of the general Szász-Mirakjan Beta-type operators S_n^* and find the rate of convergence with the help of the classical and second moduli of continuity. We also derive a Voronovskaja-type approximation theorem associated with these general operators, S_n^*.

2. A Set of Auxiliary Results

In this section, we give the moments of Szász-Mirakjan Beta-type operators, S_n^*, defined by Equation (6). We first recall for the S_n that

$$S_n\,(1,x) = 1, \quad S_n\,(t,x) = x \quad \text{and} \quad S_n\left(t^2,x\right) = x^2 + \frac{x}{n}, \tag{8}$$

just as in [1].

Lemma 1. *The moments of the Szász-Mirakjan Beta-type operators, S_n^*, defined by* (6) *are given by*

$$S_n^*(1,x) = 1, \tag{9}$$

$$S_n^*(t,x) = x + \frac{2}{n} \tag{10}$$

and

$$S_n^*(t^2,x) = \frac{n}{n-1}\,x^2 + \frac{6}{n-1}\,x + \frac{6}{n(n-1)}. \tag{11}$$

Proof. By using the known formulas in Equation (8), we find from the definition (6) that

$$\begin{aligned}
S_n^*(1,x) &= \sum_{k=0}^{\infty} e^{-nx}\,\frac{(nx)^k}{k!}\,\frac{1}{B(k+2,n+1)} \\
&\quad \cdot \int_0^\infty \int_0^\infty \frac{t^k}{(1+t)^{n+k+1}}\exp\left(-\frac{p(1+t)^2}{t}\right)\,dt\,dp \\
&= \sum_{k=0}^{\infty} e^{-nx}\,\frac{(nx)^k}{k!}\,\frac{1}{B(k+2,n+1)}\,\int_0^\infty B_p(k+1,n)\,dp \\
&= \sum_{k=0}^{\infty} e^{-nx}\,\frac{(nx)^k}{k!}\,\frac{B(k+2,n+1)}{B(k+2,n+1)} \\
&= S_n(1,x) = 1.
\end{aligned}$$

For $n > 1$, we have

$$\begin{aligned}
S_n^*(t,x) &= \sum_{k=0}^{\infty} e^{-nx} \frac{(nx)^k}{k!} \frac{1}{B(k+2,n+1)} \\
&\quad \cdot \int_0^{\infty} \int_0^{\infty} \frac{t^{k+1}}{(1+t)^{n+k+1}} \exp\left(-\frac{p(1+t)^2}{t}\right) dt\, dp \\
&= \sum_{k=0}^{\infty} e^{-nx} \frac{(nx)^k}{k!} \frac{1}{B(k+2,n+1)} \\
&\quad \cdot \int_0^{\infty} B_p(k+2,n-1)\, dp \\
&= \sum_{k=0}^{\infty} e^{-nx} \frac{(nx)^k}{k!} \frac{B(k+3,n)}{B(k+2,n+1)} \\
&= \sum_{k=0}^{\infty} e^{-nx} \frac{(nx)^k}{k!} \frac{\Gamma(k+3)\Gamma(n)}{\Gamma(n+k+3)} \frac{\Gamma(n+k+3)}{\Gamma(k+2)\Gamma(n+1)} \\
&= \sum_{k=0}^{\infty} e^{-nx} \frac{(nx)^k}{k!} \frac{k+2}{n} \\
&= S_n(t,x) + \frac{2}{n} S_n(1,x) = x + \frac{2}{n}
\end{aligned}$$

and, for $n > 2$, we find that

$$\begin{aligned}
S_n^*(t^2,x) &= \sum_{k=0}^{\infty} e^{-nx} \frac{(nx)^k}{k!} \frac{1}{B(k+2,n+1)} \\
&\quad \cdot \int_0^{\infty} \int_0^{\infty} \frac{t^{k+2}}{(1+t)^{n+k+1}} \exp\left(-\frac{p(1+t)^2}{t}\right) dt\, dp \\
&= \sum_{k=0}^{\infty} e^{-nx} \frac{(nx)^k}{k!} \frac{1}{B(k+2,n+1)} \\
&\quad \cdot \int_{p=0}^{\infty} B_p(k+3,n-2)\, dp \\
&= \sum_{k=0}^{\infty} e^{-nx} \frac{(nx)^k}{k!} \frac{B(k+4,n-1)}{B(k+2,n+1)} \\
&= \sum_{k=0}^{\infty} e^{-nx} \frac{(nx)^k}{k!} \frac{\Gamma(k+4)\Gamma(n-1)}{\Gamma(n+k+3)} \frac{\Gamma(n+k+3)}{\Gamma(k+2)\Gamma(n+1)} \\
&= \sum_{k=0}^{\infty} e^{-nx} \frac{(nx)^k}{k!} \frac{(k+3)\,(k+2)}{n(n-1)} \\
&= \frac{n}{n-1} S_n(t^2,x) + \frac{5}{n-1} S_n(t,x) + \frac{6}{n(n-1)} S_n(1,x) \\
&= \frac{n}{n-1} x^2 + \frac{6}{n-1} x + \frac{6}{n(n-1)}.
\end{aligned}$$

The proof of Lemma 1 is thus completed. □

Lemma 2. *The central moments of the Szász-Mirakjan Beta-type operators, S_n^*, defined by Equation (6) are given by*

$$S_n^*(t-x,x) = \frac{2}{n} \tag{12}$$

and

$$S_n^*\left((t-x)^2, x\right) = \frac{1}{n-1}\, x^2 + \frac{2(n+2)}{n(n-1)}\, x + \frac{6}{n(n-1)} =: \varepsilon_n(x). \tag{13}$$

Proof. The assertions (12) and (13) of Lemma 2 follow easily from those of Lemma 1, so we omit the details involved. □

3. Local Approximation

Let $C_{\mathcal{B}}[0,\infty)$ be the set of all real-valued continuous and bounded functions f on $[0,\infty)$, which is endowed with the norm given by

$$\|f\| = \sup_{x\in[0,\infty)} |f(x)|.$$

Then Peetre's K-functional is defined by

$$K_2(f;\delta) = \inf\left\{ \|f-g\| + \delta\,\|g''\| : g \in C_{\mathcal{B}}^2[0,\infty)\right\},$$

where

$$C_{\mathcal{B}}^2[0,\infty) := \left\{ g : g \in C_{\mathcal{B}}[0,\infty) \quad \text{and} \quad g', g'' \in C_{\mathcal{B}}[0,\infty)\right\}.$$

There exists a positive constant $C > 0$ such that (see, for example, [24])

$$K_2(f,\delta) \leqq C\omega_2\left(f, \sqrt{\delta}\right), \tag{14}$$

where $\delta > 0$ and ω_2 denotes the second-order modulus of smoothness for $f \in C_{\mathcal{B}}[0,\infty)$, which is defined by

$$\omega_2\left(f;\sqrt{\delta}\right) = \sup_{0<h\leqq\delta}\ \sup_{x\in[0,\infty)} |f(x+2h) - 2f(x+h) + f(x)|.$$

The usual modulus of continuity for $f \in C_{\mathcal{B}}[0,\infty)$ is given by

$$\omega(f;\delta) = \sup_{0<h\leqq\delta}\ \sup_{x\in[0,\infty)} |f(x+h) - f(x)|.$$

Lemma 3 below provides an auxiliary inequality which is useful in proving our next theorem (see Theorem 1).

Lemma 3. *For all $g \in C_{\mathcal{B}}^2[0,\infty)$, it is asserted that*

$$\left|\widetilde{S}_n^*(g,x) - g(x)\right| \leqq \delta_n(x)\,\|g''(x)\|, \tag{15}$$

where

$$\delta_n(x) = \frac{1}{n-1}\, x^2 + \frac{2(n+2)}{n(n-1)}\, x + \frac{2(5n-2)}{n^2(n-1)} \tag{16}$$

and

$$\widetilde{S}_n^*(f,x) = S_n^*(f,x) + f(x) - f\left(x + \frac{2}{n}\right) \tag{17}$$

for $f \in C_{\mathcal{B}}[0,\infty)$.

Proof. First of all, we find from (17) that

$$\widetilde{S}_n^*(t-x,x) = S_n^*(t-x,x) - \frac{2}{n} = \frac{2}{n} - \frac{2}{n} = 0. \tag{18}$$

Now, by using the Taylor's formula, we have

$$g(t) - g(x) = (t-x)g'(x) + \int_x^t (t-u)g''(u)\,du,$$

which, in view of Equation (18), yields

$$\begin{aligned}\widetilde{S}_n^*(g,x) - g(x) &= \widetilde{S}_n^*(t-x,x)g'(x) + \widetilde{S}_n^*\left(\int_x^t (t-u)g''(u)\,du, x\right)\\ &= S_n^*\left(\int_x^t (t-u)g''(u)du, x\right) - \int_x^{x+\frac{2}{n}} \left(x+\frac{2}{n}-u\right) g''(u)\,du.\end{aligned}$$

On the other hand, as

$$\begin{aligned}\left|\int_x^t (t-u)g''(u)\,du\right| &\leqq \int_x^t |t-u|\cdot|g''(u)|\,du\\ &\leqq \|g''\| \int_x^t |t-u|\,du\\ &\leqq (t-x)^2\|g''\|\end{aligned}$$

and

$$\left|\int_x^{x+\frac{2}{n}} \left(x+\frac{2}{n}-u\right) g''(u)\,du\right| \leqq \left(\frac{2}{n}\right)^2 \|g''\|,$$

we conclude that

$$\begin{aligned}\left|\widetilde{S}_n^*(g,x) - g(x)\right| &\leqq S_n^*((t-x)^2,x)\|g''\| + \frac{4}{n^2}\|g''\|\\ &= \left(\frac{1}{n-1}x^2 + \frac{2(n+2)}{n(n-1)}x + \frac{6}{n(n-1)} + \frac{4}{n^2}\right)\|g''\|\\ &= \left(\frac{1}{n-1}x^2 + \frac{2(n+2)}{n(n-1)}x + \frac{2(5n-2)}{n^2(n-1)}\right)\|g''\| = \delta_n(x)\|g''\|.\end{aligned}$$

This is the result asserted by Lemma 3. □

We now state and prove our main results in this section.

Theorem 1. *Let $f \in C_B[0,\infty)$. Then, for every $x \in [0,\infty)$, there exists a constant $L > 0$, such that*

$$|S_n^*(f,x) - f(x)| \leqq L\omega_2\left(f; \sqrt{\delta_n(x)}\right) + \omega\left(f; \frac{2}{n}\right),$$

where $\omega_2(f;\delta)$ is the second-order modulus of smoothness, $\omega(f;\delta)$ is the usual modulus of continuity, and $\delta_n(x)$ is given by Equation (16).

Proof. We observe from Equation (17) that

$$\begin{aligned}|S_n^*(f,x) - f(x)| &\leqq \left|\widetilde{S}_n^*(f,x) - f(x)\right| + \left|f(x) - f\left(x+\frac{2}{n}\right)\right|\\ &\leqq \left|\widetilde{S}_n^*(f-g,x) - (f-g)(x)\right| + \left|f(x) - f\left(x+\frac{2}{n}\right)\right|\\ &\quad + \left|\widetilde{S}_n^*(g,x) - g(x)\right|\end{aligned}$$

for $g \in C_{\mathcal{B}}^2[0,\infty)$. Thus, by applying Lemma 3 for $g \in C_{\mathcal{B}}^2[0,\infty)$, we get

$$|S_n^*(f,x) - f(x)| \leqq 4\,\|f-g\| + \delta_n(x)\,\|g''\| + \omega\left(f;\frac{2}{n}\right),$$

which, by taking the infimum on the right-hand side over all $g \in C_{\mathcal{B}}^2[0,\infty)$ and using (14), yields

$$\begin{aligned}|S_n^*(f,x) - f(x)| &\leqq 4K_2(f;\delta_n(x)) + \omega\left(f;\frac{2}{n}\right)\\ &\leqq L\omega_2\left(f;\sqrt{\delta_n(x)}\right) + \omega\left(f;\frac{2}{n}\right).\end{aligned}$$

where $L = 4M > 0$. This evidently completes the demonstration of Theorem 1. □

Theorem 2. *Let E be any bounded subset of the interval $[0,\infty)$, and suppose that $0 < \alpha \leqq 1$. If $f \in C_{\mathcal{B}}[0,\infty)$ is locally $\mathrm{Lip}_M(\alpha)$, that is, if the following inequality holds true,*

$$|f(y) - f(x)| \leqq M\,|y-x|^{\alpha} \qquad (y \in E;\ x \in [0,\infty)),$$

then, for each $x \in [0,\infty)$,

$$|S_n^*(f,x) - f(x)| \leqq M\big([\varepsilon_n(x)]^{\frac{\alpha}{2}} + 2[d(x,E)]^{\alpha}\big), \tag{19}$$

$\varepsilon_n(x)$ is given by Equation (13), M is a constant depending on α and f, and $d(x,E)$ is the distance between x and E defined as follows:

$$d(x,E) = \inf\{\,|y-x| : y \in E\}.$$

Proof. Let $\overline{E}$ denote the closure of E in $[0,\infty)$. Then there exists a point $x_0 \in \overline{E}$ such that

$$|x - x_0| = d(x,E).$$

By the above-mentioned definition of $\mathrm{Lip}_M(\alpha)$, we get

$$\begin{aligned}|S_n^*(f,x) - f(x)| &\leqq S_n^*(\,|f(y)-f(x)|\,,x)\\ &\leqq S_n^*(\,|f(y)-f(x_0)|\,,x) + S_n^*(|f(x)-f(x_0)|\,,x)\\ &\leqq M\{S_n^*(|y-x_0|^{\alpha},x) + |x-x_0|^{\alpha}\}\\ &\leqq M\{S_n^*(|y-x|^{\alpha} + |x-x_0|^{\alpha},x) + |x-x_0|^{\alpha}\}\\ &\leqq M\{S_n^*(|y-x_0|^{\alpha},x) + 2\,|x-x_0|^{\alpha}\}.\end{aligned}$$

Now, if we use the Hölder inequality with

$$p = \frac{2}{\alpha} \qquad \text{and} \qquad q = \frac{2}{2-\alpha},$$

we find that

$$\begin{aligned}|S_n^*(f,x) - f(x)| &\leqq M\left(\left[S_n^*(\,(y-x_0)^2,x)\right]^{\frac{\alpha}{2}} + 2[d(x,E)]^{\alpha}\right)\\ &= M\left([\varepsilon_n(x)]^{\frac{\alpha}{2}} + 2[d(x,E)]^{\alpha}\right).\end{aligned}$$

We have thus completed our demonstration of the result asserted by Theorem 2. □

4. A Voronovskaja-Type Approximation Theorem

By applying Equations (5) to (7), as well as Lemma 1, we first prove the following result.

Lemma 4. *It is asserted that*

$$\begin{aligned} S_n^*(t^3,x) = & \frac{n^2}{(n-1)(n-2)}\,x^3 + \frac{12n}{(n-1)(n-2)}\,x^2 \\ & + \frac{36}{(n-1)(n-2)}x + \frac{24}{n(n-1)(n-2)} \end{aligned} \tag{20}$$

and

$$\begin{aligned} S_n^*(t^4,x) = & \frac{n^3}{(n-1)(n-2)(n-3)}\,x^4 + \frac{20n^2}{(n-1)(n-2)(n-3)}\,x^3 \\ & + \frac{120n}{(n-1)(n-2)(n-3)}\,x^2 + \frac{240}{(n-1)(n-2)(n-3)}\,x \\ & + \frac{120}{n(n-1)(n-2)(n-3)}. \end{aligned} \tag{21}$$

Furthermore, the following result holds true,

$$\begin{aligned} S_n^*\left((t-x)^4, x\right) = & \frac{3(n+6)}{(n-1)(n-2)(n-3)}\,x^4 + \frac{4(3n^2+32n+12)}{n(n-1)(n-2)(n-3)}\,x^3 \\ & + \frac{12(n^2+21n+18)}{n(n-1)(n-2)(n-3)}\,x^2 + \frac{144(n+2)}{n(n-1)(n-2)(n-3)}\,x \\ & + \frac{120}{n(n-1)(n-2)(n-3)}. \end{aligned} \tag{22}$$

Proof. We begin by recalling the following moments of the Szász-Mirakjan operators,

$$S_n\left(t^3,x\right) = x^3 + \frac{3x^2}{n} + \frac{x}{n^2} \tag{23}$$

and

$$S_n\left(t^4,x\right) = x^4 + \frac{6x^3}{n} + \frac{7x^2}{n^2} + \frac{x}{n^3}. \tag{24}$$

Using Equations (8) and the above formulas (23) and (24), we thus find for $n > 3$ that

$$\begin{aligned}
S_n^*(t^3,x) &= \sum_{k=0}^{\infty} e^{-nx}\,\frac{(nx)^k}{k!}\,\frac{1}{B(k+2,n+1)} \\
&\quad \cdot \int_0^\infty \int_0^\infty \frac{t^{k+3}}{(1+t)^{n+k+1}}\,\exp\left(-\frac{p(1+t)^2}{t}\right) dt\,dp \\
&= \sum_{k=0}^{\infty} e^{-nx}\,\frac{(nx)^k}{k!}\,\frac{1}{B(k+2,n+1)}\,\int_0^\infty B_p(k+4,n-3)\,dp \\
&= \sum_{k=0}^{\infty} e^{-nx}\,\frac{(nx)^k}{k!}\,\frac{B(k+5,n-2)}{B(k+2,n+1)} \\
&= \sum_{k=0}^{\infty} e^{-nx}\,\frac{(nx)^k}{k!}\,\frac{\Gamma(k+5)\Gamma(n-2)}{\Gamma(n+k+3)}\,\frac{\Gamma(n+k+3)}{\Gamma(k+2)\Gamma(n+1)} \\
&= \sum_{k=0}^{\infty} e^{-nx}\,\frac{(nx)^k}{k!}\,\frac{(k+4)\,(k+3)\,(k+2)}{n(n-1)(n-2)} \\
&= \frac{n^2}{(n-1)(n-2)}\,S_n(t^3,x) + \frac{9n}{(n-1)(n-2)}\,S_n(t^2,x) \\
&\quad + \frac{26}{(n-1)(n-2)}\,S_n(t,x) + \frac{24}{n(n-1)(n-2)}\,S_n(1,x) \\
&= \frac{n^2}{(n-1)(n-2)}\,\left(x^3+\frac{3x^2}{n}+\frac{x}{n^2}\right) + \frac{9n}{(n-1)(n-2)}\,\left(x^2+\frac{x}{n}\right) \\
&\quad + \frac{26}{(n-1)(n-2)}\,x + \frac{24}{n(n-1)(n-2)} \\
&= \frac{n^2}{(n-1)(n-2)}x^3 + \frac{12n}{(n-1)(n-2)}\,x^2 + \frac{36}{(n-1)(n-2)}\,x \\
&\quad + \frac{24}{n(n-1)(n-2)}.
\end{aligned}$$

On the other hand, for $n > 4$, we find that

$$\begin{aligned}
S_n^*(t^4,x) &= \sum_{k=0}^{\infty} e^{-nx}\,\frac{(nx)^k}{k!}\,\frac{1}{B(k+2,n+1)} \\
&\quad \cdot \int_0^\infty \int_0^\infty \frac{t^{k+4}}{(1+t)^{n+k+1}}\,\exp\left(-\frac{p(1+t)^2}{t}\right)\,dt\,dp \\
&= \sum_{k=0}^{\infty} e^{-nx}\,\frac{(nx)^k}{k!}\,\frac{1}{B(k+2,n+1)}\,\int_0^\infty B_p(k+5,n-4)\,dp \\
&= \sum_{k=0}^{\infty} e^{-nx}\,\frac{(nx)^k}{k!}\,\frac{B(k+6,n-3)}{B(k+2,n+1)} \\
&= \sum_{k=0}^{\infty} e^{-nx}\,\frac{(nx)^k}{k!}\,\frac{\Gamma(k+6)\Gamma(n-3)}{\Gamma(n+k+3)}\,\frac{\Gamma(n+k+3)}{\Gamma(k+2)\Gamma(n+1)} \\
&= \sum_{k=0}^{\infty} e^{-nx}\,\frac{(nx)^k}{k!}\,\frac{(k+5)(k+4)\,(k+3)\,(k+2)}{n(n-1)(n-2)(n-3)},
\end{aligned}$$

that is, that

$$\begin{aligned}
S_n^*(t^4,x) &= \frac{n^3}{(n-1)(n-2)(n-3)}\,S_n(t^4,x) + \frac{14n^2}{(n-1)(n-2)(n-3)}\,S_n(t^3,x) \\
&\quad + \frac{71n}{(n-1)(n-2)(n-3)}\,S_n(t^2,x) + \frac{154}{(n-1)(n-2)(n-3)}\,S_n(t,x) \\
&\quad + \frac{120}{n(n-1)(n-2)(n-3)}\,S_n(1,x) \\
&= \frac{n^3}{(n-1)(n-2)(n-3)}\,x^4 + \frac{20n^2}{(n-1)(n-2)(n-3)}\,x^3 \\
&\quad + \frac{120n}{(n-1)(n-2)(n-3)}\,x^2 + \frac{240}{(n-1)(n-2)(n-3)}\,x \\
&\quad + \frac{120}{n(n-1)(n-2)(n-3)},
\end{aligned}$$

which, together, complete the proof of Lemma 4. □

Theorem 3. *Let $f, f', f'' \in C_\nu[0,\infty)$ for $\nu \geqq 4$. Then, the following Voronovskaja-type approximation result holds true,*

$$\lim_{n\to\infty} \{n\,[S_n^*(f,x) - f(x)]\,\} = 2f'(x) + \left(\frac{x^2}{2} + x\right) f''(x). \tag{25}$$

Proof. By Taylor's expansion of $f(t)$ at the point $t = x$, we have

$$f(t) = f(x) + f'(x)\,(t-x) + \frac{1}{2}\,f''(x)(t-x)^2 + \Psi(t,x)(t-x)^2, \tag{26}$$

where $\Psi(t,x)$ is remainder term, $\Psi(\cdot,x) \in C_\nu[0,\infty)$ and $\Psi(t,x) \to 0$ as $t \to x$.

Applying the Szász-Mirakjan Beta-type operators S_n^* to Equation (26) and, using Lemma 2, we obtain

$$\begin{aligned}
S_n^*(f,x) - f(x) &= f'(x)\,S_n^*(t-x,x) + \frac{1}{2}\,f''(x)\,S_n^*\big((t-x)^2,x\big) \\
&\quad + S_n^*\big(\Psi(t,x)\,(t-x)^2,x\big) \\
&= \frac{2}{n}\,f'(x) + \frac{1}{2}\left(\frac{1}{n-1}\,x^2 + \frac{2(n+2)}{n(n-1)}\,x + \frac{6}{n(n-1)}\right) f''(x) \\
&\quad + S_n^*\big(\Psi(t,x)\,(t-x)^2,x\big).
\end{aligned} \tag{27}$$

We now apply the Cauchy-Schwarz inequality to the third term on the right-hand side of Equation (27). We thus find that

$$n\left|S_n^*\big(\Psi(t,x)\,(t-x)^2,x\big)\right| \leqq \sqrt{n^2 S_n^*\big((t-x)^4,x\big)} \cdot \sqrt{S_n^*\big([\Psi(t,x)]^2,x\big)}.$$

Let

$$\eta(t,x) := [\Psi(t,x)]^2.$$

In this case, we observe that $\eta(x,x) = 0$ and also that $\eta(\cdot,x) \in C_\nu[0,\infty)$. Then, it follows that

$$\lim_{n\to\infty} \{S_n^*\big([\Psi(t,x)]^2,x\big)\} = \lim_{n\to\infty} \{S_n^*(\eta(t,x),x)\} = \eta(x,x) = 0$$

uniformly with respect to $x \in [0,b]$ $(b > 0)$ and the following limit,

$$\lim_{n\to\infty} \left\{ n^2 S_n^*\left((t-x)^4, x\right) \right\}$$

is finite. Consequently, we have

$$\lim_{n\to\infty} \left\{ n S_n^*\left(\Psi(t,x)\,(t-x)^2, x\right) \right\} = 0.$$

Thus, in the limit when $n \to \infty$ in Equation (27), we obtain

$$\lim_{n\to\infty} \left\{ n\left[S_n^*(f,x) - f(x)\right] \right\} = 2f'(x) + \left(\frac{x^2}{2} + x\right) f''(x).$$

The proof of Theorem 3 is thus completed. □

5. Concluding Remarks and Observations

We find it worthwhile to reiterate the fact that, in approximation theory and related fields, the approximation and some other basic properties of various linear and nonlinear operators are investigated because mainly of the potential for their usefulness in many areas of researches in the mathematical, physical, and engineering sciences. This article has been motivated essentially by the demonstrated applications of such results as those associated with various approximation operators.

In our present investigation, we have systematically studied a number of approximation properties of a class of the Szász-Mirakjan Beta-type operators, which we have introduced here by using an extension of the familiar Beta function $B(\alpha,\beta)$. We have established the moments of these extended Szász-Mirakjan Beta-type operators and estimated several convergence results with the help of the second modulus of smoothness and the classical modulus of continuity. We have also investigated convergence via functions belonging to the Lipschitz class. Finally, we have proved a Voronovskaja-type approximation theorem for the general Szász-Mirakjan Beta-type operators.

Using the other substantially more general forms of the classical Beta function $B(\alpha,\beta)$, which we have indicated in Section 1 of this article (see, for example, [22,23]), one can analogously develop further extensions and generalizations of the various results which we have presented here. In many of these suggested areas of further researches on the subject of this article, some other, possibly deeper, mathematical analytic tools and techniques will have to be called for.

Author Contributions: All three authors contributed equally to this investigation.

Funding: This research received no external funding.

Conflicts of Interest: The authors declare no conflict of interest.

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