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The Tubby Torus as a Quotient Group

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Abstract: Let E be any metrizable nuclear locally convex space and $\widehat{E}$ the Pontryagin dual group of E. Then the topological group $\widehat{E}$ has the tubby torus (that is, the countably infinite product of copies of the circle group) as a quotient group if and only if E does not have the weak topology. This extends results in the literature related to the Banach–Mazur separable quotient problem.

Keywords: torus; tubby torus; separable quotient problem; locally convex space; nuclear space; Banach space; pontryagin duality; weak topology

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1. Introduction and Preliminaries

The Separable Quotient problem for Banach spaces has its roots in the 1930s and is due to Stefan Banach and Stanisław Mazur. While a positive answer is known for various classes of Banach spaces [1], such as reflexive Banach spaces, weakly compactly generated Banach spaces, and more generally Banach-like spaces [2], the general problem remains unsolved.

Problem 1. *(Separable quotient problem for Banach spaces) Does every infinite-dimensional Banach space have a quotient Banach space which is separable and infinite-dimensional?*

The following problem stated in [3] is also unsolved, but a negative answer to it would give a negative answer to Problem 1.

Problem 2. *Does every infinite-dimensional Banach space have a quotient topological group which is homeomorphic to the countably infinite product, $\mathbb{R}^\omega$, of copies of $\mathbb{R}$?*

This suggests another question which we have not seen mentioned in the literature. We state the problem and answer it.

Question 1. *Does every infinite-dimensional Banach space have a quotient topological space which is homeomorphic to $\mathbb{R}^\omega$?*

Question 1 has a positive answer, although it uses very powerful machinery due to Toruńczyk. It is known [4] that every infinite-dimensional Fréchet space F (that is, a complete metrizable locally convex space) is homeomorphic to an infinite-dimensional Hilbert space H. So an infinite-dimensional Banach space B (indeed an infinite-dimensional Fréchet space) is homeomorphic to an infinite-dimensional Hilbert space H, which obviously has the infinite-dimensional separable Hilbert space ℓ_2 as a quotient. Further, by the separable case of Toruńczyk's theorem which is known as the Kadec–Anderson theorem, the separable Fréchet space $\mathbb{R}^\omega$ is homeomorphic to ℓ_2, from which the positive answer to Question 1 follows.

Noting that Problem 2 remains open, it is natural to ask if every infinite-dimensional Banach space has a quotient topological group which is a separable metrizable topological group which is infinite-dimensional as a topological space. This was answered in the positive by the following theorem.

Theorem 1. *[5] Every locally convex space E, which has a subspace which is an infinite-dimensional Fréchet space, has the tubby torus, $\mathbb{T}^\omega$, as a quotient group, where $\mathbb{T}$ is the compact circle group. In particular, this is the case if E is an infinite-dimensional Banach space.*

We should mention the following result.

Theorem 2. *[6] If E is any infinite-dimensional Fréchet space which is not a Banach space, then E has the locally convex space $\mathbb{R}^\omega$ as a quotient vector space.*

Corollary 1. *If E is any infinite-dimensional Fréchet space which is not a Banach space, then E has the tubby torus $\mathbb{T}^\omega$ as a quotient group.*

One might suspect that every infinite-dimensional locally convex space has the tubby torus as a quotient group. This is shown to be false in [5] for the free locally convex space φ on a countably infinite discrete space. Indeed in [7] it is shown that if X is a countably infinite k_ω-space, then the free topological vector space on X, which is a connected infinite-dimensional (in the topological sense) topological group, does not have the tubby torus as a quotient group or even any infinite-dimensional (in the topological sense) metrizable quotient group.

It was recently proved that free topological groups on infinite connected compact spaces also have the tubby torus as a quotient group.

Theorem 3. *[7] Let $F_G(X)$ and $A_G(X)$ be the Graev free topological group and the Graev free abelian topological group, respectively, on an infinite connected compact Hausdorff space. Then the connected topological groups $F_G(X)$ and $A_G(X)$ have the tubby torus $\mathbb{T}^\omega$ as a quotient group.*

It follows from Theorem 2.5 of [3] that every non-metrizable connected locally compact abelian group has the tubby torus as a quotient group. But as a connected locally compact abelian group G is isomorphic as a topological group to the product $\mathbb{R}^n \times K$, for some non-negative integer n and compact abelian group K, and $\mathbb{R}^n$ and all compact metrizable groups are separable, we see that if G is non-separable then it is non-metrizable. So we obtain the following result as a consequence.

Theorem 4. *Every non-separable connected locally compact abelian group has the tubby torus as a quotient group.*

As mentioned earlier, Problem 1 has been aswered for dual-like groups. In particular there is the following powerful and beautiful theorem.

Theorem 5. *[8] If B is the Banach space dual of any infinite-dimensional Banach space, then B has a separable infinite-dimensional quotient Banach space.*

Corollary 2. *If B is the Banach space dual of any infinite-dimensional Banach space, then B has the tubby torus as a quotient group.*

Recall that if G is a (Hausdorff) abelian topological group, then we denote by $\widehat{G}$ the group of all continuous homomorphisms of G into the circle group $\mathbb{T}$, where $\widehat{G}$ has the compact-open topology.

There is a natural homomorphism $\alpha : G \to \widehat{\widehat{G}}$. The Pontryagin–van Kampen duality theorem is stated below and a discussion and proof appear in [9,10].

Theorem 6. *[9,10] If G is any locally compact abelian group then the map α is an isomorphism of topological groups of G onto $\widehat{\widehat{G}}$. Also, if H is a closed subgroup of the locally compact abelian group G, then $\widehat{H}$ is a quotient group of $\widehat{G}$, and if A is a quotient group of G, then $\widehat{A}$ is isomorphic as a topological group to a closed subgroup of $\widehat{G}$. Further, the map α restricted to H is an isomorphism of topological groups of H onto the subgroup $\alpha(H)$ of $\widehat{\widehat{G}}$.*

The following is less well-known.

Let E be a locally convex space. As E is a topological group, the topological group $\widehat{E}$ consisting of all continuous group homomorphisms of E into $\mathbb{T}$ with the compact-open topology is a topological group, as is $\widehat{\widehat{E}}$. As mentioned above, there is a natural homomorphism of E into $\widehat{\widehat{E}}$.

Theorem 7. *[11] Proposition 15.2. Let E be a complete metrizable locally convex space (that is a Fréchet space). Then α is an isomorphism of topological groups of E onto $\widehat{\widehat{E}}$.*

We note that Theorem 7 does not tell us whether, for example α restricted to a closed subgroup H of E is an isomorphism of topological groups of H onto the subgroup $\alpha(H)$ of $\widehat{\widehat{E}}$. In fact this is not always true. §11 of [12] gives an example of a closed subgroup H of a Fréchet space E such that α restricted to H is not an isomorphism of topological groups of H onto its image in $\widehat{\widehat{E}}$. To see how badly things can go "wrong", we note Theorem 6.1 of [11]: Let E be a metrizable locally convex space. If E is not a nuclear space, then it has a discrete subgroup H such that there are no non-trivial continuous homomorphisms from $\mathrm{span}(H)/H$ into $\mathbb{T}$, where $\mathrm{span}(H)$ denotes the linear span in E of H.

Theorem 5 leads us then to the natural question:

Problem 3. *If E is any infinite-dimensional Fréchet space which does not have the weak topology and $\widehat{E}$ is its dual topological group, does $\widehat{E}$ have the tubby torus as a quotient group? In particular, is this the case for E a Banach space or a Schwartz space?*

This question is open, however a positive answer is given for nuclear spaces in the next section.

2. The Main Result

Definition 1. *A topological group G is said to be reflexive if the natural mapping α from G to $\widehat{\widehat{G}}$ is an isomorphism of topological groups. The topological group G is said to be strongly reflexive if every closed subgroup and every Hausdorff quotient group of G is reflexive.*

Theorem 8. *[12] (Theorem 20.35) Every complete metrizable nuclear locally convex space is strongly reflexive.*

Proposition 1. *[11] (Proposition 17.1(c)) Let H be a closed subgroup of a strongly reflexive topological group G. Then $\widehat{H}$ is isomorphic as a topological group to a quotient group of $\widehat{G}$.*

Theorem 9. *Let E be a metrizable nuclear locally convex space. Then $\widehat{E}$ has the tubby torus $\mathbb{T}^\omega$ as a quotient group if and only if E does not have the weak topology.*

Proof. By Theorem 2 of [13], if H is a dense subgroup of the metrizable topogical group G, then $\widehat{G}$ is isomorphic as a topological group to $\widehat{H}$. So the dual group $\widehat{E}$ of E is isomorphic as a topological group to the dual group of the completion of E. So there is no loss of generality in assuming that E is complete. Further, the completion of a metrizable nuclear locally convex space is a metrizable nuclear locally convex space by Theorems 20.34 and 20.20 of [12].

The theorem in [14] says that a locally convex space E has the weak topology if and only if every discrete subgroup of E is finitely generated. However, its proof there gives rather more. Namely, the locally convex space E does not have the weak topology if and only if E contains a discrete free abelian subgroup S which is not finitely generated.

So if the metrizable nuclear locally convex space E does not have the weak topology, then it has a subgroup S isomorphic as a topological group to a restricted direct product of $\mathbb{Z}_i, i = 1, 2, \ldots, n, \ldots$, where each $\mathbb{Z}_i$ is isomorphic as a topological group to the discrete $\mathbb{Z}$ of integers. Noting §3 of [15], we see that the dual group of this restricted direct product of $\mathbb{Z}_i$ is the tubby torus $\mathbb{T}^\omega$, and it then follows from Theorem 8 and Proposition 1 that $\widehat{E}$ has the tubby torus as a quotient group, as required.

On the other hand if the complete metrizable locally convex space E has the weak topology, then it is isomorphic as a locally convex space to $\mathbb{R}^\omega$. So its dual group $\widehat{E}$ is isomorphic as a topological group to the locally convex space φ. However, as mentioned earlier, it is proved in [5] (and generalized in [7]), that φ does not have the tubby torus as a quotient group, which completes the proof. □

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