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A Proof of Komlós Theorem for Super-Reflexive Valued Random Variables

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Abstract: We give a geometrical proof of Komlós' theorem for sequences of random variables with values in super-reflexive Banach space. Our approach is inspired by the elementary proof given by Guessous in 1996 for the Hilbert case and uses some geometric properties of smooth spaces.

Keywords: Bochner; convergence; Komlós; super-reflexive space; truncation technique; uniform smoothness

1. Introduction

In the Scottish Book [1], H. Steinhaus raised the following problem: Is there a family F of measurable functions defined on a measure space (Ω, Σ, μ) such that $|f(x)| = 1$ for all $x \in X$ and $f \in F$, and for each sequence $(f_n)_{n\geq 1}$ in F the sequence of averages:

$$\frac{1}{m}\sum_{k=1}^{m} f_k(x)$$

is divergent for almost all x? In [2], Révész showed that if $(f_n)_{n\geq 1}$ is a bounded sequence in $L^2_{\mathbb{R}}$, then there is a subsequence $(g_n)_{n\geq 1}$ of $(f_n)_{n\geq 1}$ and $f \in L^2_{\mathbb{R}}$ such that

$$\sum_n a_n(g_n - f) \text{ converges } \mu\text{-}a.e.,$$

whenever $\sum_n |c_n|^2 < +\infty$. In Particular, if we take $c_n = \frac{1}{n}$, then we may conclude via Kronecker's lemma, that every L^2-bounded sequence of random variables, has a subsequence which is Cesàro-convergent to an square integrable function. In 1967 Komlós [3] has shown that we can extract a subsequence from every L^1-bounded sequence of random variables, such that every further subsequence converges Cesàro a.e. to the same limit. In 1978, Garling [4] has generalized the Komlós theorem for sequences of random variables with values in super-reflexive Banach spaces.

Other extensions of Komlós theorem have been studied with respect to the weak convergence, for example, by Balder [5] in L^1_X and by Chakraborty and Choudhury [6] for Pettis integrable functions. Other papers have considered cases in which the functions f_n take their values in some Banach lattices (see, for example, [7–9]), Cassese [10] replaced the functions f_n by additive set functions, Lennard [11] proved a converse to Komlós' theorem for convex subsets of $L^1_{\mathbb{R}}$ and recently, the authors of this paper have proved that Komlós theorem for Hilbert valued random variables, remains valid after any permutation of the terms of the subsequence [12].

In this paper, we aim to develop Guessous' proof of Komlós' theorem for Hilbert space-valued functions [13] to super-reflexive Banach space-valued functions. As in [13] we do not appeal to martingale technique as it was done in [4], all we need are truncation technique, weak compactness in L_X^q and some geometric properties of Banach spaces, more precisely the characteristic inequalities of uniformly smooth Banach spaces [14]. Furthermore, it is possible that the idea used here may be useful for related problems in probability theory.

2. Notations and Preliminaries

Let $(\Omega,\ \Sigma,\ \mu)$ be a probability space, X a real Banach space and X^* the dual space of X. If $1 \le p < \infty$, $L_X^p = L_X^p\,(\Omega,\ \Sigma,\ \mu)$ denotes the Banach space of (equivalence classes of) all strongly Σ-measurable functions $f : \Omega \to X$, such that $\|f\,(.)\|^p$ is integrable. For any $a \ge 0$ and $f : \Omega \to X$, we set

$$F_a(f)\,(w) = \begin{cases} f(w) & \text{if} \quad \|f(w)\| < a, \\ 0 & \text{if not.} \end{cases}$$

For any pair $x \in X$ and $x^* \in X^*$, $x^*(x)$ is denoted by $< x, x^* >$.

3. About the Geometry of Banach Spaces

In this section, we recall some basic concepts and results of Banach spaces geometry. For the definitions and further properties of uniform smoothness, duality mappings and super-reflexivity please refer to [15–17]. If X is a normed space, its modulus of smoothness $\rho_X\,(t)$ is defined by:

$$\forall t \in [0, +\infty[\, : \rho_X\,(t) = \sup\left\{ \frac{\|x+ty\| + \|x-ty\|}{2} - 1,\ \ \|x\| = \|y\| = 1 \right\}.$$

We denote by $J_q\ (q > 1)$ the generalized duality mapping from X into 2^{X^*} given by:

$$J_q\,(x) := \left\{ j_q\,(x) \in X^* : \left\langle x, j_q\,(x) \right\rangle = \left\| j_q\,(x) \right\| \|x\| \ \text{ and } \ \left\| j_q\,(x) \right\| = \|x\|^{q-1} \right\}.$$

The following geometric lemma gives an elementary inequality in a real normed general space:

Lemma 1. *Let $q > 1$ and X be a real normed space. For any $x, y \in X$ and for any $j_q(x+y) \in J_q(x+y)$, we have:*

$$\|x+y\|^q \le \|x\|^q + q\left\langle y, j_q(x+y) \right\rangle. \tag{1}$$

In particular for any $j_q(x) \in J_q(x)$,

$$||x+y||^q \ge ||x||^q + q\left\langle y, j_q(x) \right\rangle. \tag{2}$$

Proof. By Corollary 2.5.19 in [15], J_q is the subdifferential of the functional $\frac{1}{q}||.||^q$. Hence, by the sub-differential inequality, for all $x, y \in X$ and $j_q(x+y) \in J_q(x+y)$ we have:

$$\frac{1}{q}\|x\|^q - \frac{1}{q}\|x+y\|^q \ge \left\langle x - (x+y)\,, j_q(x+y) \right\rangle,$$

so that

$$\|x+y\|^q \le \|x\|^q + q\left\langle y, j_q(x+y) \right\rangle.$$

Applying (1) to $u = x + y$ and $v = -y$ we get (2). □

A Banach space X is called smooth if for every $x \in X$, $\|x\| = 1$, there exists a unique $j\,(x)$ in X^* such that $\|j\,(x)\| = 1$ and $\langle x, j\,(x) \rangle = 1$ ($J_q\,(x)$ is single-valued, $\forall q > 1$).

In order to shorten the terminology, we shall say that X is q-uniformly smooth, if

$$\forall t > 0,\ \rho_X\,(t) \le Ct^q \text{ for some constant } C > 0. \tag{3}$$

A Banach space X is called q-smoothable if it's q-uniformly smooth for some equivalent norm.

A Banach space X is said to be finitely representable in the Banach space Y if for every $\varepsilon > 0$ and every finite-dimensional subspace X_0 of X there exists a finite-dimensional subspace Y_0 of Y and a linear isomorphism $T : X_0 \to Y_0$ such that $\|T\| \left\|T^{-1}\right\| \leq 1 + \varepsilon$.

A super-reflexive Banach space is defined to be a Banach space X which has the property that no non-reflexive Banach space is finitely representable in X.

The following theorem is due to Pisier [17].

Theorem 1. *A Banach space X is super-reflexive if and only if X is q-uniformly smoothable for some $1 < q \leq 2$.*

The following lemma is due to Xu ([14], p. 1129, Corollary 1) and gives a characteristic inequality of q-uniformly smooth Banach spaces in terms of the generalized duality map.

Lemma 2. *Let X be a real smooth Banach space and let $1 < q \leq 2$. The following statements are equivalent:*
(i) X is q-uniformly smooth.
(ii) There is a constant $a > 0$ such that for all $x, y \in X$

$$\|x + y\|^q \leq \|x\|^q + q \left\langle y, j_q(x) \right\rangle + a \|y\|^q .$$

Remark 1. *Suppose X is a q-uniformly smooth real Banach space and let $L = \max(a, 1)$ (where a is the constant mentioned in Lemma 2). Then $L > 0$ and for all $x_1, x_2 ... x_n \in X$*

$$\| \sum_{k=1}^{n} x_k \|^q \leq L \sum_{k=1}^{n} \|x_k\|^q + q \sum_{2 \leq j \leq n} < x_j, \; j_q(\sum_{1 \leq i < j} x_i) > . \tag{4}$$

We will use the following lemma,

Lemma 3. *Let (x_n) be a weakly convergent sequence in a normed space X and x its limit. Then there exists an integer N such that:*

$$\|x\| \leq 2 \inf_{n \geq N} \|x_n\| .$$

4. Proof of Komlós Theorem in Super-Reflexive Banach Spaces

We recall the following lemma, which was used in [4,13,18,19] as a generalization of basic results of Komlós [3]. For the proof, see, for example, Theorem 1 in [4].

Lemma 4. *Suppose that X is reflexive. Let $1 < q \leq 2$ and $(f_n)_{n \geq 1}$ a bounded sequence in L^1_X. Then there exists a subsequence $(g_n)_{n \geq 1}$ of $(f_n)_{n \geq 1}$ and a sequence $(u_k)_{k \geq 1}$ in L^q_X such that for any subsequence $(h_n)_{n \geq 1}$ of $(g_n)_{n \geq 1}$ we have the following:*

1. *$\lim_n F_k(h_n) = u_k$ weakly in L^q_X, for each $k \geq 1$,*
2. *$(u_k)_{k \geq 1}$ converges μ-a.e. and strongly in L^1_X,*
3. *$\sum_{n \geq 1} \frac{1}{n^q} \|F_n(h_n)\|_q^q < +\infty$,*
4. *$\sum_{n \geq 1} \mu(\|h_n\| \geq n) < \infty$.*

Theorem 2. *Let $(f_n)_{n \geq 1}$ be a bounded sequence in L^1_X, X a super-reflexive Banach space. Then, there exists a subsequence $(g_n)_{n \geq 1}$ of $(f_n)_{n \geq 1}$ and f in L^1_X such that for any subsequence $(h_n)_{n \geq 1}$ of $(g_n)_{n \geq 1}$*

$$\frac{1}{n} \sum_{k=1}^{n} h_k \text{ converges } \mu\text{-a.e. to } f \text{ as } n \to \infty.$$

Proof. We may assume without loss of generality that $(f_n)_{n\geq 1}$ is a sequence of simple functions. Indeed, consider a sequence $(f_n^{'})_{n\geq 1}$ of simple functions such that

$$\left\| f_n - f_n^{'} \right\|_1 \leq \frac{1}{2^n}.$$

Then $(f_n')_{n\geq 1}$ is bounded in L^1_X and $\left(f_n - f_n^{'}\right)_{n\geq 1}$ converges μ-a.e to zero, since $\sum\limits_n \left(f_n - f_n^{'}\right)$ is absolutely convergent μ-a.e. Then $\frac{1}{n}\sum\limits_{k=1}^{n} h_k$ converges μ-a.e. to f if and only if $\frac{1}{n}\sum\limits_{k=1}^{n} h_k'$ converges μ-a.e. to f where $h_i^{'} = f_{n_i}^{'}$ if $h_i = f_{n_i}$, because $\frac{1}{n}\sum\limits_{k=1}^{n}\left(h_k - h_k^{'}\right)$ converges μ-a.e. to 0. In the sequel, we assume that $(f_n)_{n\geq 1}$ is a sequence of simple functions. Considering an equivalent norm on X if necessary, we may assume that X is q- uniformly smooth $(1 < q \leq 2)$. This will enable us to denote the single-valued generalized duality map on X by j_q. By passing to a further subsequence if necessary, we can suppose that, for each $k \geq 1$, the sequence $(F_k(f_n))_{n\geq 1}$ converges weakly to some limit u_k in L^q_X. Then by Lemma 3, there exist subsequences $(f_n^1)_{n\geq 1}, (f_n^2)_{n\geq 1}, ..., (f_n^k)_{n\geq 1}, ...$ of $(f_n)_{n\geq 1}$, where $(f_n^{k+1})_{n\geq 1}$ is a subsequence of $(f_n^k)_{n\geq 1}$ and $N_1 < N_2 < ...N_k < N_{k+1} < ...$ such that

$$\|u_k\|_q \leq 2\left\|F_k(f_n^k)\right\|_q, \forall n \geq N_k.$$

Put $f_n' = f_{N_n}^n$. Then$(f_n')_{n\geq k}$ is a subsequence of $(f_n^k)_{n\geq N_k}$ and

$$\|u_k\|_q \leq 2\left\|F_k(f_n')\right\|_q, \forall n \geq k \geq 1.$$

So, we suppose that

$$\|u_k\|_q \leq 2\left\|F_k(f_n)\right\|_q, \forall n \geq k \geq 1.$$

Applying the last inequality, Lemma 4 and a passage to a further subsequence if necessary, we can assume that, for any subsequence $(h_n)_{n\geq 1}$ of $(f_n)_{n\geq 1}$ we have the following:

1. The sequence $(u_k)_{k\geq 1}$ converges μ-a.e. to some limit $f \in L^1_X$.
2. $$\sum_{n\geq 1}\frac{1}{n^q}\left\|F_n\left(h_n\right)\right\|_q^q < +\infty. \tag{5}$$
3. $$\sum_{n\geq 1}\mu\left(\left\|h_n - F_n\left(h_n\right)\right\| > 0\right) < \infty. \tag{6}$$

Put $\varepsilon_k = \frac{1}{2^k}\left(k \geq 1\right)$. Since $\|u_k\|_q \leq \underline{\lim}_n \left\|F_k\left(f_n\right)\right\|_q \leq k$, there is a measurable simple functions v_k such that $\|v_k\|_q \leq k$ and

$$\|u_k - v_k\|_q \leq \min\left(\inf_{n\geq k}\left\|F_k\left(f_n\right)\right\|_q, \frac{\varepsilon_k}{(2k)^{q-1}}, \frac{1}{k}\right).$$

Remark that if $\inf_{n\geq k}\left\|F_k\left(f_n\right)\right\|_q = 0$, then $u_k = 0$ and therefore we can take $v_k = 0$. For every $n \geq k \geq 1$ we have

$$\begin{aligned}
\|v_k\|_q &\leq \|u_k - v_k\|_q + \|u_k\|_q \\
&\leq \left\|F_k\left(f_n\right)\right\|_q + 2\left\|F_k\left(f_n\right)\right\|_q \\
&= 3\left\|F_k\left(f_n\right)\right\|_q
\end{aligned}$$

then,

$$\|F_k(f_n) - v_k\|_q \leq \|F_k(f_n)\|_q + \|v_k\|_q \leq 4\|F_k(f_n)\|_q$$

consequently, for any subsequence $(h_n)_{n\geq 1}$ of $(f_n)_{n\geq 1}$ we have

$$\sum_{k\geq 1} \frac{1}{k^q} \|F_k(h_k) - v_k\|_q^q \leq 4^q \sum_{k\geq 1} \frac{1}{k^q} \|F_k(h_k)\|_q^q < +\infty.$$

Let $\mathcal{F}_n$ be the smallest sub-σ-algebra of Σ with respect to which all f_m and v_m for $m \leq n$ are $\mathcal{F}_n$-measurable. Then $\mathcal{F}_n$ contains only finitely many sets, because all these functions are simple.

Take $n_1 = 1$ and let us prove that there exist integers $1 < n_2 < \ldots < n_r < n_{r+1} < \ldots$ such that, whenever $r \geq 2$ and $2 \leq k \leq r$ we have

$$\sup_{1\leq w\leq p\leq k-1} \; \sup_{1\leq s_w<s_{w+1}<\ldots<s_p\leq r-1} \; \sup_{B\in\mathcal{F}_{n_{r-1}}} \left| \left\langle F_k(f_{n_r}) - u_k,\ 1_B\, j_q\Big(\sum_{i=w}^{p} \frac{F_i\left(f_{n_{s_i}}\right) - v_i}{i}\Big) \right\rangle \right| \leq \varepsilon_k. \qquad (*)$$

To prove $(*)$ we proceed by recurrence on r. By the weak convergence of $(F_2(f_n) - u_2)_{n\geq 1}$ to 0 in L^q_X, there exist $n_2 > 1$ such that

$$\sup_{B\in\mathcal{F}_2} \left|\left\langle F_2(f_{n_2}) - u_2,\ 1_B\, j_q\left(F_1(f_{n_1}) - v_1\right)\right\rangle\right| \leq \varepsilon_2.$$

Let us suppose $r \geq 3$ and that $n_1 < n_2 < ... < n_{r-1}$ have been obtained. Then, for each $2 \leq k \leq r$, the sequence $(F_k(f_n) - u_k)_{n\geq 1}$ converges weakly to 0 in L^q_X and the set

$$\left\{ 1_B\, j_q\left(\sum_{i=w}^{p} \frac{F_i\left(f_{n_{s_i}}\right) - v_i}{i}\right) : 1 \leq w \leq p \leq k-1,\ 1 \leq s_w < s_{w+1} < ... < s_p \leq r-1 \text{ and } B \in \mathcal{F}_{n_{r-1}} \right\}$$

is a finite set of $L^{q^*}_{X^*}$ $\left(\frac{1}{q} + \frac{1}{q^*} = 1\right)$. Then there exists $n_r > n_{r-1}$ such that, for each $k \in \{2, ..., r\}$ we have

$$\left| \left\langle F_k(f_{n_r}) - u_k,\ 1_B\, j_q\left(\sum_{i=w}^{p} \frac{F_i\left(f_{n_{s_i}}\right) - v_i}{i}\right) \right\rangle \right| \leq \varepsilon_k$$

uniformly on $1 \leq w \leq p \leq k-1$, $1 \leq s_w < s_{w+1} < ... < s_p \leq r-1$ and $B \in \mathcal{F}_{n_{r-1}}$. This completes the proof of $(*)$.

Now, for each $2 \leq k \leq r$, $1 \leq w \leq p \leq k-1$, $1 \leq s_w < s_{w+1} < ... < s_p \leq r-1$ and $B \in \mathcal{F}_{n_{r-1}}$ we have

$$\begin{aligned}
\left|\left\langle F_k\left(f_{n_r}\right)-v_k,\ 1_B\, j_q\left(\sum_{i=w}^{p}\frac{F_i\left(f_{n_{s_i}}\right)-v_i}{i}\right)\right\rangle\right| &\leq \left|\left\langle F_k\left(f_{n_r}\right)-u_k,\ 1_B\, j_q\left(\sum_{i=w}^{p}\frac{F_i\left(f_{n_{s_i}}\right)-v_i}{i}\right)\right\rangle\right|+\\
&\quad \left|\left\langle u_k-v_k,\ 1_B\, j_q\left(\sum_{i=w}^{p}\frac{F_i\left(f_{n_{s_i}}\right)-v_i}{i}\right)\right\rangle\right|\\
&\leq \varepsilon_k+\left\|\sum_{i=w}^{p}\frac{F_i\left(f_{n_{s_i}}\right)-v_i}{i}\right\|_q^{q-1}\left\|u_k-v_k\right\|_q\\
&\leq \varepsilon_k+\left(\sum_{i=1}^{k}\frac{\left\|F_i\left(f_{n_{s_i}}\right)\right\|_\infty+\left\|v_i\right\|_\infty}{i}\right)^{q-1}\left\|u_k-v_k\right\|_q\\
&\leq \varepsilon_k+(2k)^{q-1}\frac{\varepsilon_k}{(2k)^{q-1}}.
\end{aligned}$$

Therefore, whenever $r \geq 2$ and $2 \leq k \leq r$ we have

$$\sup_{1\leq w\leq p\leq k-1}\ \sup_{1\leq s_w<s_{w+1}<...<s_p\leq r-1}\ \sup_{B\in\mathcal{F}_{n_{r-1}}}\left|\left\langle F_k\left(f_{n_r}\right)-v_k,\ 1_B\, j_q\Big(\sum_{i=w}^{p}\frac{F_i\left(f_{n_{s_i}}\right)-v_i}{i}\Big)\right\rangle\right| \leq \varepsilon_{k-1}. \qquad (**)$$

Put $g_r = f_{n_r}$ and let $(h_n)_{n\geq 1}$ be a subsequence of $(g_n)_{n\geq 1}$. We note

$$S_k = \sum_{n=1}^{k}\frac{1}{n}\left(F_n\left(h_n\right)-v_n\right).$$

We will use the Cauchy criterion to prove that $(S_k)_{k\geq 1}$ converges μ-a.e. For this, if $\varepsilon > 0$ and $m \in \mathbb{N}^*$ we prove that $\lim_m \mu\left(A_m\right) = 1$, where

$$A_m = \left\{\sup_{j\geq 1}\left\|S_{m+j}-S_m\right\| \leq \varepsilon\right\}.$$

Denote:

$$A_{m,0} = \Omega,\ A_{m,k} = \left\{\sup_{1\leq j\leq k}\left\|S_{m+j}-S_m\right\| \leq \varepsilon\right\}$$

and

$$B_{m,k} = A_{m,k-1} - A_{m,k} = \left\{\sup_{1\leq j\leq k-1}\left\|S_{m+j}-S_m\right\| \leq \varepsilon \text{ and } \left\|S_{m+k}-S_m\right\| \geq \varepsilon\right\}.$$

Then $\left(A_{m,n}^c\right)_{n\geq 1}$ is an increasing sequence of measurable sets, $\underset{n\geq 1}{\cup} A_{m,n}^c = A_m^c$ and $(B_{m,k})_{1\leq k\leq n}$ is a Σ-partition of $A_{m,n}^c$. Using (2) in L_X^q we obtain the following estimation

$$\begin{aligned}
\left\|1_{B_{m,k}}\left(S_{m+n}-S_m\right)\right\|_q^q &= \left\|1_{B_{m,k}}\left(S_{m+k}-S_m\right)+1_{B_{m,k}}\left(S_{m+n}-S_{m+k}\right)\right\|_q^q \\
&\geq \left\|1_{B_{m,k}}\left(S_{m+k}-S_m\right)\right\|_q^q \\
&\quad +q\left\langle 1_{B_{m,k}}\left(S_{m+n}-S_{m+k}\right),\, j_q\left(1_{B_{m,k}}\left(S_{m+k}-S_m\right)\right)\right\rangle \\
&\geq \varepsilon^q\mu\left(B_{m,k}\right)+q\left\langle 1_{B_{m,k}}\left(S_{m+n}-S_{m+k}\right),\, j_q\left(1_{B_{m,k}}\left(S_{m+k}-S_m\right)\right)\right\rangle \\
&= \varepsilon^q\mu\left(B_{m,k}\right)+q\left\langle S_{m+n}-S_{m+k},\, 1_{B_{m,k}}\, j_q\left(S_{m+k}-S_m\right)\right\rangle.
\end{aligned}$$

Consequently

$$\begin{aligned}
\left\|S_{m+n}-S_m\right\|_q^q &\geq \sum_{k=1}^{n}\left\|1_{B_{m,k}}\left(S_{m+n}-S_m\right)\right\|_q^q \\
&\geq \varepsilon^q\mu\left(A_{m,n}^c\right)+q\sum_{k=1}^{n}\left\langle S_{m+n}-S_{m+k},\, 1_{B_{m,k}}\, j_q\left(S_{m+k}-S_m\right)\right\rangle.
\end{aligned}$$

On the other hand by (4), there exists a positive constant L such that for all $w \in \Omega$

$$\begin{aligned}
\left\|\left(S_{m+n}-S_m\right)(w)\right\|^q &= \left\|\sum_{k=1}^{n}\frac{\left(F_{m+k}\left(h_{m+k}\right)-v_{m+k}\right)(w)}{m+k}\right\|^q \\
&\leq L\sum_{k=1}^{n}\frac{\left\|\left(F_{m+k}\left(h_{m+k}\right)-v_{m+k}\right)(w)\right\|^q}{(m+k)^q}+ \\
&\quad q\sum_{2\leq j\leq n}\frac{1}{m+j}\left\langle\left(F_{m+j}\left(h_{m+j}\right)-v_{m+j}\right)(w),\, j_q\left(\sum_{1\leq i<j}\frac{\left(F_{m+i}\left(h_{m+i}\right)-v_{m+i}\right)(w)}{m+i}\right)\right\rangle.
\end{aligned}$$

Integrating both sides of the last inequality with respect to w, we obtain

$$\begin{aligned}
\left\|S_{m+n}-S_m\right\|_q^q &\leq L\sum_{k=1}^{n}\frac{\left\|F_{m+k}\left(h_{m+k}\right)-v_{m+k}\right\|_q^q}{(m+k)^q}+ \\
&\quad q\sum_{2\leq j\leq n}\frac{1}{m+j}\left\langle F_{m+j}\left(h_{m+j}\right)-v_{m+j},\, j_q\left(\sum_{1\leq i<j}\frac{F_{m+i}\left(h_{m+i}\right)-v_{m+i}}{m+i}\right)\right\rangle.
\end{aligned}$$

Hence

$$\begin{aligned}
\mu\left(A_{m,n}^c\right) &\leq \frac{1}{\varepsilon^q}\Big(L\sum_{k=1}^{n}\frac{\left\|F_{m+k}\left(h_{m+k}\right)-v_{m+k}\right\|_q^q}{(m+k)^q} \\
&\quad +q\sum_{2\leq j\leq n}\frac{1}{m+j}\left\langle F_{m+j}\left(h_{m+j}\right)-v_{m+j},\, j_q\left(\sum_{i=m+1}^{m+j-1}\frac{F_i\left(h_i\right)-v_i}{i}\right)\right\rangle \\
&\quad +q\sum_{k=1}^{n}\left|\left\langle S_{m+n}-S_{m+k},\, 1_{B_{m,k}} j_q\left(S_{m+k}-S_m\right)\right\rangle\right|\Big).
\end{aligned}$$

To get $\lim_m \mu\left(A_m\right)=1$, we have to prove that $\lim_m \lim_n \mu\left(A_{m,n}^c\right)=0$ using the previous majorations.

(i) As the series $\sum_{k\geq 1}\frac{1}{k^q}\left\|F_k\left(h_k\right)-v_k\right\|_q^q$ is convergent, then $\sum_{k=1}^{n}\frac{\left\|F_{m+k}\left(h_{m+k}\right)-v_{m+k}\right\|_q^q}{(m+k)^q}$ converges to 0 when $m \to +\infty$ uniformly in n.

(ii) For $m \geq 1$ and $j \geq 2$ we have $h_{m+j} = f_{n_r}$ with $m + j \leq r$ and, for $m + 1 \leq i \leq m + j - 1$ we have $h_i = f_{n_{s_i}}$ with $1 \leq s_i \leq r - 1$. By $(**)$ we get

$$\left\langle F_{m+j}\left(h_{m+j}\right) - v_{m+j}\,,\, j_q\left(\sum_{i=m+1}^{m+j-1} \frac{F_i\left(h_i\right) - v_i}{i}\right)\right\rangle \quad \leq \quad \varepsilon_{m+j-1}.$$

Consequently

$$\begin{aligned}\sum_{2\leq j\leq n} \frac{1}{m+j}\left\langle F_{m+j}\left(h_{m+j}\right) - v_{m+j},\ j_q\left(\sum_{1\leq i<j} \frac{F_{m+i}\left(h_{m+i}\right) - v_{m+i}}{m+i}\right)\right\rangle \quad &\leq \quad \frac{1}{m}\sum_{2\leq j\leq n} \varepsilon_{m+j-1} \\ &\leq \quad \frac{1}{m2^m},\end{aligned}$$

which converges to 0 when $m \to \infty$ uniformly in n.

(iii) Now observe that the sets $B_{m,k}$ belong to the smallest sub-σ-algebra $\mathcal{F}$ of Σ such that the functions $h_1, h_2, ..., h_{m+k}, v_1, v_2, ..., v_{m+k}$ are $\mathcal{F}$-measurable. Using similar arguments to those given before and taking $B = B_{m,k}$, we can apply $(**)$ to obtain the following estimation

$$\begin{aligned}\sum_{k=1}^{n}\left|\left\langle S_{m+n} - S_{m+k},\ 1_{B_{m,k}}\, j_q\left(S_{m+k} - S_m\right)\right\rangle\right| \quad &\leq \quad \sum_{k=1}^{n}\sum_{i=m+k+1}^{m+n} \tfrac{1}{i}\left|\left\langle F_i\left(h_i\right) - v_i,\ 1_{B_{m,k}}\, j_q\left(\sum_{j=m+1}^{m+k} \tfrac{F_j\left(h_j\right) - v_j}{j}\right)\right\rangle\right| \\ &\leq \quad \sum_{k=1}^{n}\ \sum_{i=m+k+1}^{m+n} \frac{1}{i}\varepsilon_{i-1} \\ &\leq \quad \frac{1}{m+1}\sum_{k=1}^{n}\ \sum_{i=m+k+1}^{m+n} \frac{1}{2^{i-1}} \\ &\leq \quad \frac{1}{m+1}\sum_{k=1}^{n} \frac{1}{2^{m+k-1}} \\ &\leq \quad \frac{1}{m+1}\frac{1}{2^{m-1}},\end{aligned}$$

which converges to 0 when $m \to \infty$ uniformly in n. Then, $(S_k)_{k\geq 1}$ converges μ-a.e.

On the other hand

$$\sum_{n\geq 1}\left\|\frac{u_n - v_n}{n}\right\|_1 \leq \sum_{n\geq 1} \frac{\|u_n - v_n\|_q}{n} \leq \sum_{n\geq 1} \frac{1}{n^2} < \infty,$$

which implies that the series $\sum_{n\geq 1} \frac{u_n - v_n}{n}$ converges μ-a.e. and as $(S_k)_{k\geq 1}$ converges μ-a.e. we deduce that $\sum_{n\geq 1} \frac{1}{n}\left(F_n\left(h_n\right) - u_n\right)$ converges μ-a.e. Hence, applying Kronecker's lemma, we obtain

$$\frac{1}{k}\sum_{n=1}^{k}\left(F_n\left(h_n\right) - u_n\right) \text{ converges to } 0\ \mu\text{-}a.e.$$

By the Borel–Cantelli lemma we deduce from (6) that the sequence $\left(h_n - F_n\left(h_n\right)\right)_{n\geq 1}$ converges μ-a.e. to 0, therefore

$$\frac{1}{k}\sum_{n=1}^{k}\left(h_n - u_n\right) \text{ converges to } 0\ \mu\text{-}a.e.,$$

as $(u_n)_{n\geq 1}$ converges to f μ-a.e., then

$$\frac{1}{k}\sum_{n=1}^{k} h_n \text{ converges to } f\ \mu\text{-}a.e.$$

This completes the proof of the theorem. □

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