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Star-Shapedness of $\mathcal{N}$-Structures in Euclidean Spaces

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Received: 2 September 2020; Accepted: 9 September 2020; Published: 12 September 2020

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Abstract: The notions of (quasi, pseudo) star-shaped sets are introduced, and several related properties are investigated. Characterizations of (quasi) star-shaped sets are considered. The translation of (quasi, pseudo) star-shaped sets are discussed. Unions and intersections of quasi star-shaped sets are conceived. Conditions for a quasi (or, pseudo) star-shaped set to be a star-shaped set are provided.

Keywords: (quasi, pseudo) star-shaped $\mathcal{N}$-structure; $\mathcal{N}$-support set; translation.

MSC: 52A20; 52A30

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1. Introduction

Convexity is a basic notion in geometry, and it is also widely used in other areas of mathematics (see [1]). Convexity also plays a most useful role in the theory and applications of fuzzy sets. In general, it can be seen that nature is not convex and is separated from possible applications. It is of independent interest to see how far the supposition of convexity can be weakened withour losing too much structure. Starshaped sets are a fairly natural extension which is also an important issue in classical convex analysis (see [2–4]). As a generalization of convexity, the term star-shapedness is being used in several contexts. For example, it is used to denote the geometrical property of a bounded planar set that from some reference point within the set all halflines intersect the boundary exactly once, and the star-shapedness is applied to linear spaces and algebraic structures etc., (see [2,3,5,6]). Brown introduced the notion of starshaped fuzzy sets (see [7]), and recently, the research of fuzzy starshaped set has been again attracting the deserving attention (see [8–10]). Star-shaped fuzzy sets are useful for processing positive information, but there is a limit to dealing with negative information. In order to deal with negative information, Jun et al. [11] introduced a new function which is called negative-valued function. It is applied to subtraction algebras and BCK/BCI-algebras (see [11–14]).

The main purpose of this article is to consider the star-shapedness of $\mathcal{N}$-structure in Euclidean spaces. We introduce the notions of (quasi, pseudo) star-shaped sets, and investigate several related properties. We discuss relations between star-shaped sets, quasi star-shaped sets and pseudo star-shaped sets. We show that the property of being (quasi, pseudo) star-shapedness is translation invariant in $\mathbb{R}^n$. We consider characterizations of (quasi) star-shaped sets. We provide conditions for a quasi (or, pseudo) star-shaped set to be a star-shaped set. We discuss union and intersection of quasi star-shaped sets.

2. Preliminaries

Let $\mathbb{R}^n$ denote the n-dimensional Euclidean space. For $x, y \in \mathbb{R}^n$, the line segment $\overline{xy}$ joining x and y is the set of all points of the form $\alpha x + \beta y$ where $\alpha \geq 0$, $\beta \geq 0$ and $\alpha + \beta = 1$. A set $S \subseteq \mathbb{R}^n$ is said to be starshaped at a point $x \in \mathbb{R}^n$ if $\overline{xy} \subseteq S$ for each point $y \in S$. A set $S \subseteq \mathbb{R}^n$ is simply said to be starshaped if there exists a point x in $\mathbb{R}^n$ such that S is starshaped relative to it.

For any family $\{a_i \mid i \in \Lambda\}$ of real numbers, we define

$$\vee\{a_i \mid i \in \Lambda\} := \begin{cases} \max\{a_i \mid i \in \Lambda\} & \text{if } \Lambda \text{ is finite,} \\ \sup\{a_i \mid i \in \Lambda\} & \text{otherwise.} \end{cases}$$

$$\wedge\{a_i \mid i \in \Lambda\} := \begin{cases} \min\{a_i \mid i \in \Lambda\} & \text{if } \Lambda \text{ is finite,} \\ \inf\{a_i \mid i \in \Lambda\} & \text{otherwise.} \end{cases}$$

Denote by $\mathscr{F}(X, [-1, 0])$ the collection of functions from a set X to $[-1, 0]$. We say that an element of $\mathscr{F}(X, [-1, 0])$ is a negative-valued function from X to $[-1, 0]$ (briefly, $\mathcal{N}$-function on X). By an $\mathcal{N}$-structure we mean an ordered pair (X, f) of X and an $\mathcal{N}$-function f on X.

For any $\mathcal{N}$-structure (X, f) and $\alpha \in [-1, 0)$, the set

$$C(f; \alpha) := \{x \in X \mid f(x) \leq \alpha\}$$

is called the closed support of (X, f) related to α.

3. Star-Shapedness of $\mathcal{N}$-Structures

Definition 1. *An $\mathcal{N}$-structure $(\mathbb{R}^n, f)$ is said to be star-shaped at $y \in \mathbb{R}^n$ if*

$$(\forall x \in \mathbb{R}^n)(\forall \lambda \in [0, 1]) \left(f(\lambda(x - y) + y) \leq f(x) \right). \tag{1}$$

Example 1. *Let $(\mathbb{R}, f)$ be an $\mathcal{N}$-structure in which f is given by*

$$f(x) = \begin{cases} -e^{x} & \text{if } \; x < 0, \\ -e^{-x} & \text{if } \; x \geq 0. \end{cases}$$

It is easy to verify that $(\mathbb{R}, f)$ is a star-shaped $\mathcal{N}$-structure at $y = 0$.

Proposition 1. *If $(\mathbb{R}^n, f)$ is a star-shaped $\mathcal{N}$-structure at $y \in \mathbb{R}^n$, then $f(y) = \inf\limits_{x \in \mathbb{R}^n} f(x)$ and $f(\lambda x + y) \leq f(x + y)$ for all $x \in \mathbb{R}^n$ and $\lambda \in [0, 1]$.*

Proof. By Equation (1), we have $f(\lambda(x - y) + y) \leq f(x)$ for all $x \in \mathbb{R}^n$ and $\lambda \in [0, 1]$. Taking $\lambda = 0$ induces $f(y) \leq f(x)$ for all $x \in \mathbb{R}^n$. Hence $f(y) = \inf\limits_{x \in \mathbb{R}^n} f(x)$. Replacing x by $x + y$ in Equation (1) induces $f(\lambda x + y) \leq f(x + y)$ for all $x \in \mathbb{R}^n$ and $\lambda \in [0, 1]$. □

Theorem 1. *For an $\mathcal{N}$-structure $(\mathbb{R}^n, f)$, the following assertions are equivalent:*

(1) $(\mathbb{R}^n, f)$ *is star-shaped at* $y \in \mathbb{R}^n$.
(2) *The nonempty closed support $C(f; \alpha)$ of $(\mathbb{R}^n, f)$ is star-shaped at $y \in \mathbb{R}^n$ for all $\alpha \in [-1, 0)$.*

Proof. Assume that $(\mathbb{R}^n, f)$ is star-shaped at $y \in \mathbb{R}^n$. Let $\alpha \in [-1, 0)$ be such that $C(f; \alpha) \neq \emptyset$. Let $x \in C(f; \alpha)$. Then $f(x) \leq \alpha$, and so

$$f(\lambda(x - y) + y) \leq f(x) \leq \alpha,$$

that is, $\lambda(x - y) + y \in C(f; \alpha)$. Hence $\overline{xy} \subseteq C(f; \alpha)$, and therefore $C(f; \alpha)$ is star-shaped at $y \in \mathbb{R}^n$ for all $\alpha \in [-1, 0)$.

Conversely, suppose that the nonempty closed support $C(f;\alpha)$ of $(\mathbb{R}^n, f)$ is star-shaped at $y \in \mathbb{R}^n$ for all $\alpha \in [-1, 0)$. For any $x \in \mathbb{R}^n$, let $f(x) = \alpha$. Then $\overline{xy} \subseteq C(f;\alpha)$, and thus $f(\lambda(x-y)+y) \leq \alpha = f(x)$ for all $\lambda \in [0,1]$. Therefore $(\mathbb{R}^n, f)$ is star-shaped at $y \in \mathbb{R}^n$. □

Definition 2. *An $\mathcal{N}$-structure $(\mathbb{R}^n, f)$ is said to be*

- *quasi star-shaped at $y \in \mathbb{R}^n$ if*

$$(\forall x \in \mathbb{R}^n)(\forall \lambda \in [0,1])\,(f(\lambda x + (1-\lambda)y) \leq \vee\{f(x), f(y)\})\,. \tag{2}$$

- *pseudo star-shaped at $y \in \mathbb{R}^n$ if*

$$(\forall x \in \mathcal{N}_{\mathrm{spt}}(f))(\forall \lambda \in [0,1])\,(f(\lambda x + (1-\lambda)y) \leq \lambda f(x) + (1-\lambda) f(y))\,. \tag{3}$$

where $\mathcal{N}_{\mathrm{spt}}(f)$ is the closure of the set $\{x \in \mathbb{R}^n \mid f(x) < 0\}$ and is called the $\mathcal{N}$-support set of f.

Example 2. *(1) Let $(\mathbb{R}, f)$ be an $\mathcal{N}$-structure in which f is given by*

$$f(x) = \begin{cases} -x-2 & \text{if } x \in [-2,-1), \\ -x^2 & \text{if } x \in [-1,1], \\ -x+2 & \text{if } x \in (1,2], \\ 0 & \text{otherwise} \end{cases}$$

It is routine to verify that $(\mathbb{R}, f)$ is a quasi star-shaped $\mathcal{N}$-structure at $y = 0$.
(2) Let $(\mathbb{R}, g)$ be an $\mathcal{N}$-structure in which g is given by

$$g(x) = \begin{cases} -x-1.5 & \text{if } x \in [-1.5,-0.5), \\ 0.4x-0.8 & \text{if } x \in [-0.5,0], \\ -0.4x-0.8 & \text{if } x \in (0,0.5], \\ x-1.5 & \text{if } x \in (0.5,1.5], \\ 0 & \text{otherwise} \end{cases}$$

It is routine to verify that $(\mathbb{R}, g)$ is a pseudo star-shaped $\mathcal{N}$-structure at $y = 0$.

Theorem 2. *If $(\mathbb{R}^n, f)$ is a star-shaped $\mathcal{N}$-structure at $y \in \mathbb{R}^n$, then it is a quasi star-shaped $\mathcal{N}$-structure at $y \in \mathbb{R}^n$.*

Proof. Straightforward. □

Theorem 3. *If $(\mathbb{R}^n, f)$ is a pseudo star-shaped $\mathcal{N}$-structure at $y \in \mathbb{R}^n$, then it is a quasi star-shaped $\mathcal{N}$-structure at $y \in \mathbb{R}^n$.*

Proof. For any $x \in \mathbb{R}^n$ and $\lambda \in [0,1]$, we have

$$f(\lambda x + (1-\lambda)y) \leq \lambda f(x) + (1-\lambda) f(y) \leq \vee\{f(x), f(y)\}.$$

Therefore $(\mathbb{R}^n, f)$ is a quasi star-shaped $\mathcal{N}$-structure at $y \in \mathbb{R}^n$. □

The converse of Theorem 3 is not true. In fact, the quasi star-shaped $\mathcal{N}$-structure $(\mathbb{R}, f)$ at $y = 0$ in Example 2(1) is not a pseudo star-shaped $\mathcal{N}$-structure at $y = 0$ because if we take $x = 0.8 \in \mathcal{N}_{\mathrm{spt}}(f)$ and $\lambda = 0.5$ then

$$f(\lambda x + (1-\lambda)0) = f(\lambda x) > \lambda f(x) = \lambda f(x) + (1-\lambda) f(0).$$

The following example shows that any quasi (resp., pseudo) star-shaped $\mathcal{N}$-structure may not be a star-shaped $\mathcal{N}$-structure.

Example 3. *Let $(\mathbb{R}, f)$ be an $\mathcal{N}$-structure in which f is given by*

$$f(x) = \begin{cases} \frac{1}{2}\sin x & \text{if } x \in [-\pi, 0), \\ -\frac{1}{2}\sin x & \text{if } x \in [0, \pi], \\ 0 & \text{otherwise} \end{cases}$$

It is routine to verify that $(\mathbb{R}, f)$ is both a quasi and a pseudo star-shaped $\mathcal{N}$-structure at $y = 0$. But it is not a star-shaped $\mathcal{N}$-structure at $y = 0$ since $f(\lambda x) > f(x)$ by taking $x = -\frac{\pi}{5}$ and $\lambda = \frac{1}{2}$.

We provide a condition for a quasi (or, pseudo) star-shaped $\mathcal{N}$-structure to be a star-shaped $\mathcal{N}$-structure.

Theorem 4. *For an $\mathcal{N}$-structure $(\mathbb{R}^n, f)$, let $y \in \mathbb{R}^n$ be such that $f(y) = \inf_{x \in \mathbb{R}^n} f(x)$. If $(\mathbb{R}^n, f)$ is a quasi star-shaped $\mathcal{N}$-structure at y, then it is a star-shaped $\mathcal{N}$-structure at y.*

Proof. Assume that $(\mathbb{R}^n, f)$ is a quasi star-shaped $\mathcal{N}$-structure at y with $f(y) = \inf_{x \in \mathbb{R}^n} f(x)$. Then

$$f(\lambda x + (1-\lambda)y) \leq \vee\{f(x), f(y)\} = f(x)$$

for all $x \in \mathbb{R}^n$ and $\lambda \in [0, 1]$. Therefore $(\mathbb{R}^n, f)$ is a star-shaped $\mathcal{N}$-structure at y. □

Combining Theorems 3 and 4, we have the following corollary.

Corollary 1. *For an $\mathcal{N}$-structure $(\mathbb{R}^n, f)$, let $y \in \mathbb{R}^n$ be such that $f(y) = \inf_{x \in \mathbb{R}^n} f(x)$. If $(\mathbb{R}^n, f)$ is a pseudo star-shaped $\mathcal{N}$-structure at y, then it is a star-shaped $\mathcal{N}$-structure at y.*

Theorem 5. *Given an $\mathcal{N}$-structure $(\mathbb{R}^n, f)$ and $y \in \mathcal{N}_{\mathrm{spt}}(f)$ with $f(y) \neq 0$, the following assertions are equivalent:*

(1) $(\mathbb{R}^n, f)$ is a quasi star-shaped $\mathcal{N}$-structure at y.
(2) The closed support $C(f; \alpha)$ of $(\mathbb{R}^n, f)$ is star-shaped at y for all $\alpha \in [f(y), 0)$.

Proof. Assume that $(\mathbb{R}^n, f)$ is a quasi star-shaped $\mathcal{N}$-structure at y. Let $x \in C(f; \alpha)$ for $\alpha \in [f(y), 0)$. Then $f(x) \leq \alpha$ and $f(y) \leq \alpha$, that is, $x, y \in C(f; \alpha)$. It follows from Equation (2) that

$$f(\lambda x + (1-\lambda)y) \leq \vee\{f(x), f(y)\} \leq \alpha$$

for all $x \in \mathbb{R}^n$ and $\lambda \in [0, 1]$, that is, $\lambda x + (1-\lambda)y \in C(f; \alpha)$. Thus $\overline{xy} \subseteq C(f; \alpha)$, and so $C(f; \alpha)$ is star-shaped at y.

Conversely, suppose that the closed support $C(f; \alpha)$ of $(\mathbb{R}^n, f)$ is star-shaped at y for all $\alpha \in [f(y), 0)$. For any $x \in \mathbb{R}^n$, if $f(x) \in [f(y), 0)$ then $x, y \in C(f; \beta)$ by putting $f(x) = \beta$. Thus $\overline{xy} \subseteq C(f; \beta)$, and hence

$$f(\lambda x + (1-\lambda)y) \leq \beta = \vee\{f(x), f(y)\}$$

for all $\lambda \in [0, 1]$. If $f(x) \notin [f(y), 0)$ and $f(x) \neq 0$, then $\beta := f(y) > f(x)$ which implies that $x, y \in C(f; \beta)$. Hence $\overline{xy} \subseteq C(f; \beta)$, and so

$$f(\lambda x + (1-\lambda)y) \leq \beta = \vee\{f(x), f(y)\}$$

for all $\lambda \in [0,1]$. If $f(x) = 0$, then clearly

$$f(\lambda x + (1-\lambda)y) \leq 0 = \vee\{f(x), f(y)\}$$

for all $\lambda \in [0,1]$. Therefore $(\mathbb{R}^n, f)$ is a quasi star-shaped $\mathcal{N}$-structure at y. $\square$

Since $\mathcal{N}_{\text{spt}}(f) = \overline{\bigcup_{\alpha \in [-1,0)} C(f;\alpha)}$, we have the following corollary.

Corollary 2. *If $(\mathbb{R}^n, f)$ is a quasi star-shaped $\mathcal{N}$-structure at $y \in \mathcal{N}_{\text{spt}}(f)$ with $f(y) \neq 0$, then $\mathcal{N}_{\text{spt}}(f)$ is star-shaped at y.*

Given $x_0 \in \mathbb{R}^n$, the $\mathcal{N}$-structure $(\mathbb{R}^n, x_0 + f)$ in which $(x_0 + f)(x) = f(x - x_0)$ for all $x \in \mathbb{R}^n$ is called the translation with respect to x_0 (briefly, x_0-translation) of $(\mathbb{R}^n, f)$.

Theorem 6. *Given $x_0 \in \mathbb{R}^n$, if $(\mathbb{R}^n, f)$ is a (quasi, pseudo) star-shaped $\mathcal{N}$-structure at $y \in \mathbb{R}^n$, then its x_0-translation $(\mathbb{R}^n, x_0 + f)$ is a (quasi, pseudo) star-shaped $\mathcal{N}$-structure at $x_0 + y \in \mathbb{R}^n$.*

Proof. Assume that $(\mathbb{R}^n, f)$ is a star-shaped $\mathcal{N}$-structure at $y \in \mathbb{R}^n$. Then

$$\begin{aligned}
&(x_0 + f)(\lambda(x - (x_0 + y)) + x_0 + y) = (x_0 + f)(\lambda(x - x_0 - y)) + x_0 + y) \\
&= f(\lambda(x - x_0 - y) + y) \leq f(x - x_0) = (x_0 + f)(x)
\end{aligned}$$

for all $x \in \mathbb{R}^n$ and $\lambda \in [0,1]$. Therefore $(\mathbb{R}^n, x_0 + f)$ is a star-shaped $\mathcal{N}$-structure at $x_0 + y \in \mathbb{R}^n$. Similarly, we can prove that if $(\mathbb{R}^n, f)$ is a quasi (resp., pseudo) star-shaped $\mathcal{N}$-structure at $y \in \mathbb{R}^n$, then its x_0-translation $(\mathbb{R}^n, x_0 + f)$ is also a quasi (resp, pseudo) star-shaped $\mathcal{N}$-structure at $x_0 + y \in \mathbb{R}^n$. $\square$

Theorem 6 shows that the property of being (quasi, pseudo) star-shapedness is translation invariant in $\mathbb{R}^n$.

Theorem 7. *Let $T : \mathbb{R}^n \to \mathbb{R}^n$ be an invertible linear transformation. If $(\mathbb{R}^n, f)$ is a (quasi, pseudo) star-shaped $\mathcal{N}$-structure at $y \in \mathbb{R}^n$, then $(\mathbb{R}^n, T(f))$ is a (quasi, pseudo) star-shaped $\mathcal{N}$-structure at $T(y) \in \mathbb{R}^n$.*

Proof. Suppose that $(\mathbb{R}^n, f)$ is a star-shaped $\mathcal{N}$-structure at $y \in \mathbb{R}^n$ and let $x \in \mathbb{R}^n$ and $\lambda \in [0,1]$. Then

$$\begin{aligned}
T(f)(\lambda(x - T(y)) + T(y)) &= f(T^{-1}(\lambda(x - T(y)) + T(y))) \\
&= f(\lambda(T^{-1}(x) - y) + y) \\
&\leq f(T^{-1}(x)) = T(f)(x).
\end{aligned}$$

Hence $(\mathbb{R}^n, T(f))$ is a star-shaped $\mathcal{N}$-structure at $T(y) \in \mathbb{R}^n$.

Now assume that $(\mathbb{R}^n, f)$ is a quasi star-shaped $\mathcal{N}$-structure at $y \in \mathbb{R}^n$. For any $x \in \mathbb{R}^n$ and $\lambda \in [0,1]$, we have

$$\begin{aligned}
T(f)(\lambda x + (1-\lambda)T(y)) &= f(T^{-1}(\lambda x + (1-\lambda)T(y))) \\
&= f(\lambda T^{-1}(x) + (1-\lambda)T^{-1}(T(y))) \\
&= f(\lambda T^{-1}(x) + (1-\lambda)y) \\
&\leq \vee\{f(T^{-1}(x)), f(y)\} \\
&= \vee\{T(f)(x), T(f)(T(y))\}.
\end{aligned}$$

Therefore $(\mathbb{R}^n, T(f))$ is a quasi star-shaped $\mathcal{N}$-structure at $T(y) \in \mathbb{R}^n$.

Similar way shows that if $(\mathbb{R}^n, f)$ is a pseudo star-shaped $\mathcal{N}$-structure at $y \in \mathbb{R}^n$, then $(\mathbb{R}^n, T(f))$ is a pseudo star-shaped $\mathcal{N}$-structure at $T(y) \in \mathbb{R}^n$. □

Theorem 8. *If $(\mathbb{R}^n, f)$ and $(\mathbb{R}^n, g)$ are star-shaped at $y \in \mathbb{R}^n$, then $(\mathbb{R}^n, f \cup g)$ is star-shaped at $y \in \mathbb{R}^n$ where $(f \cup g)(x) = \vee\{f(x), g(x)\}$ for all $x \in \mathbb{R}^n$.*

Proof. Let $(\mathbb{R}^n, f)$ and $(\mathbb{R}^n, g)$ be star-shaped at $y \in \mathbb{R}^n$. Then

$$f(\lambda x + (1-\lambda)y) \leq f(x) \text{ and } g(\lambda x + (1-\lambda)y) \leq g(x)$$

for all $x \in \mathbb{R}^n$ and $\lambda \in [0,1]$. it follows that

$$\begin{aligned}(f \cup g)(\lambda x + (1-\lambda)y) &= \vee\{f(\lambda x + (1-\lambda)y), g(\lambda x + (1-\lambda)y)\} \\ &\leq \vee\{f(x), g(x)\} = (f \cup g)(x)\end{aligned}$$

for all $x \in \mathbb{R}^n$ and $\lambda \in [0,1]$. Therefore $(\mathbb{R}^n, f \cup g)$ is star-shaped at $y \in \mathbb{R}^n$. □

Corollary 3. *If $(\mathbb{R}^n, f)$ and $(\mathbb{R}^n, g)$ are star-shaped at $y \in \mathbb{R}^n$, then $(\mathbb{R}^n, f \cup g)$ is quasi star-shaped at $y \in \mathbb{R}^n$.*

Theorem 9. *If $(\mathbb{R}^n, f)$ and $(\mathbb{R}^n, g)$ are quasi star-shaped at $y \in \mathbb{R}^n$, then $(\mathbb{R}^n, f \cup g)$ is quasi star-shaped at $y \in \mathbb{R}^n$.*

Proof. Assume that $(\mathbb{R}^n, f)$ and $(\mathbb{R}^n, g)$ are quasi star-shaped at $y \in \mathbb{R}^n$. Then

$$f(\lambda x + (1-\lambda)y) \leq \vee\{f(x), f(y)\}$$

and

$$g(\lambda x + (1-\lambda)y) \leq \vee\{g(x), g(y)\}$$

for all $x \in \mathbb{R}^n$ and $\lambda \in [0,1]$. Hence

$$\begin{aligned}(f \cup g)(\lambda x + (1-\lambda)y) &= \vee\{f(\lambda x + (1-\lambda)y), g(\lambda x + (1-\lambda)y)\} \\ &\leq \vee\{f(x), f(y), g(x), g(y)\} \\ &= \vee\{(f \cup g)(x), (f \cup g)(y)\}\end{aligned}$$

for all $x \in \mathbb{R}^n$ and $\lambda \in [0,1]$. Therefore $(\mathbb{R}^n, f \cup g)$ is quasi star-shaped at $y \in \mathbb{R}^n$. □

Theorem 10. *For two $\mathcal{N}$-structures $(\mathbb{R}^n, f)$ and $(\mathbb{R}^n, g)$, let $y \in \mathbb{R}^n$ be such that $f(y) = g(y)$. If $(\mathbb{R}^n, f)$ and $(\mathbb{R}^n, g)$ are quasi star-shaped at $y \in \mathbb{R}^n$, then $(\mathbb{R}^n, f \cap g)$ is quasi star-shaped at $y \in \mathbb{R}^n$ where $(f \cap g)(x) = \wedge\{f(x), g(x)\}$ for all $x \in \mathbb{R}^n$.*

Proof. Suppose that $(\mathbb{R}^n, f)$ and $(\mathbb{R}^n, g)$ are quasi star-shaped at $y \in \mathbb{R}^n$. Then

$$f(\lambda x + (1-\lambda)y) \leq \vee\{f(x), f(y)\}$$

and

$$g(\lambda x + (1-\lambda)y) \leq \vee\{g(x), g(y)\}$$

for all $x \in \mathbb{R}^n$ and $\lambda \in [0,1]$. Since $f(y) = g(y)$, it follows that

$$\begin{aligned}(f \cap g)(\lambda x + (1-\lambda)y) &= \wedge\{f(\lambda x + (1-\lambda)y), g(\lambda x + (1-\lambda)y)\} \\ &\leq \wedge\{\vee\{f(x), f(y)\}, \vee\{g(x), g(y)\}\} \\ &\leq \vee\{\wedge\{f(x), g(x)\}, f(y)\}\} \\ &= \vee\{(f \cap g)(x), (f \cap g)(y)\}\end{aligned}$$

for all $x \in \mathbb{R}^n$ and $\lambda \in [0,1]$. Thus $(\mathbb{R}^n, f \cap g)$ is quasi star-shaped at $y \in \mathbb{R}^n$. $\square$

Before ending our discussion, we pose a question.

Question. If $(\mathbb{R}^n, f)$ and $(\mathbb{R}^n, g)$ are pseudo star-shaped at $y \in \mathbb{R}^n$, then are $(\mathbb{R}^n, f \cup g)$ and $(\mathbb{R}^n, f \cap g)$ pseudo star-shaped at $y \in \mathbb{R}^n$?

4. Conclusions and Future Works

Star-shaped fuzzy sets are useful for processing positive information, but there is a limit to dealing with negative information. So we came to think about how to use star-shapedness to deal with negative information. We discussed star-shapedness for $\mathcal{N}$-structure to handle negative information, and applied it to Euclidean spaces. We introduced the notions of (quasi, pseudo) star-shaped sets, and investigated several properties. We considered the relationship between the star-shaped set, the quasi-star-shaped set, and the pseudo-star-shaped set. We shown that the property of being (quasi, pseudo) star-shapedness is translation invariant in the n-dimensional Euclidean space. We considered characterizations of (quasi) star-shaped sets. We found and arranged the conditions for a quasi (or, pseudo) star-shaped set to be a star-shaped set. We discussed the union and intersection of quasi star-shaped sets. In future work, we will consider how to apply the ideas or results of this paper to the algebraic structure. We also want to study star-shapedness on bipolar $\mathcal{N}$-structure.

Author Contributions: Create and conceptualize ideas, Y.B.J. and S.-Z.S.; writing—original draft preparation, Y.B.J.; writing—review and editing, K.J.L.; funding acquisition, S.-Z.S. All authors have read and agreed to the published version of the manuscript.

Funding: This work was supported by the 2020 education, research and student guidance grant funded by Jeju National University.

Acknowledgments: We would like to thank the anonymous reviewers for their very careful reading and valuable comments/suggestions.

Conflicts of Interest: The authors declare no conflict of interest.

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