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Article

The Existence and Uniqueness of an Entropy Solution to Unilateral Orlicz Anisotropic Equations in an Unbounded Domain

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Abstract: The purpose of this work is to prove the existence and uniqueness of a class of nonlinear unilateral elliptic problem $(\mathcal{P})$ in an arbitrary domain, managed by a low-order term and non-polynomial growth described by an N-uplet of N-function satisfying the Δ_2-condition. The source term is merely integrable.

Keywords: anisotropic elliptic equation; obstacle problem; entropy solution; Sobolev–Orlicz anisotropic spaces; unbounded domain; existence and uniqueness

MSC: 35J47; 35J60

1. Introduction

Let Ω be an arbitrary domain of $\mathbb{R}^N$, $(N \geq 2)$. In this paper, we investigate the existence and uniqueness solution of the following problem:

$$(\mathcal{P}) \begin{cases} A(u) + \sum_{i=1}^{N} b_i(x,u,\nabla u) = f & \text{in } \Omega\,, \\ u \geq \psi & \text{a.e. in } \Omega\,, \end{cases}$$

where, $A(u) = \sum_{i=1}^{N} (a_i(x,u,\nabla u))_{x_i}$ is a Leray–Lions operator defined on $\mathring{W}^1_B(\Omega)$ (defined as the adherence space $C_0^\infty(\Omega)$) into its dual; $B(t) = (B_1(t), \cdots, B_N(t))$ are N-uplet Orlicz functions that satisfy Δ_2-condition; the obstacle ψ is a measurable function that belongs to $L^\infty(\Omega) \cap \mathring{W}^1_B(\Omega)$; and for $i = 1, \cdots, N$, $b_i(x,s,\xi) : \Omega \times \mathbb{R} \times \mathbb{R}^N \longrightarrow \mathbb{R}$ are Carathéodory functions (measurable with respect to x in Ω for every (s,ξ) in $\mathbb{R} \times \mathbb{R}^N$, and continuous with respect to (s,ξ) in $\mathbb{R} \times \mathbb{R}^N$ for almost every x in Ω) that do not satisfy any sign condition and the growth described by the vector N-function $B(t)$. Take $f \in L^1(\Omega)$ too.

Statement of the problems: Suppose they have non-negative measurable functions ϕ, $\varphi \in L^1(\Omega)$; and $\bar{a}$, $\tilde{a}$ are two constants, positive, such that for $\xi = (\xi_1, \cdots, \xi_N) \in \mathbb{R}^N$ and $\xi' = (\xi'_1, \cdots, \xi'_N) \in \mathbb{R}^N$, we have

$$\sum_{i=1}^{N} \left[a_i(x,s,\xi) - a_i(x,s,\xi') \right] . (\xi_i - \xi'_i) > 0, \tag{1}$$

$$\sum_{i=1}^{N} a_i(x,s,\xi)\,.\,\xi_i \geq \bar{a}\sum_{i=1}^{N} B_i(|\,\xi_i\,|) - \phi(x), \tag{2}$$

$$\sum_{i=1}^{N} |\,a_i(x,s,\xi)\,| \leq \tilde{a}\sum_{i=1}^{N} \bar{B}_i^{-1}B_i(|\,\xi\,|) + \varphi(x), \tag{3}$$

and

$$\sum_{i=1}^{N} |\,b_i(x,s,\xi)\,| \leq h(x) + l(s)\,.\sum_{i=1}^{N} B_i(|\,\xi\,|), \tag{4}$$

with $\bar{B}(t)$ being the complementary function of $B(t)$, $h \in L^1(\Omega)$ and $l : \mathbb{R} \longrightarrow \mathbb{R}^+$ being a positive continuous function such that $l \in L^1(\Omega) \cap L^\infty(\Omega)$.

We recall that in the last few decades, tremendous popularity has been achieved by the investigation of a class of nonlinear unilateral elliptic problem due to their fundamental role in describing several phenomena, such as the study of fluid filtration in porous media, constrained heating, elastoplasticity, optimal control, financial mathematics and others; for those studies, there are large numbers of mathematical articles; see [1–4] for more details.

When Ω is a bounded open set of $\mathbb{R}^N$, we refer to the celebrated paper by Bénilan [5], who presented the idea of entropy solutions adjusted to Boltzmann conditions. For more outcomes concerning the existence of solutions of this class in the Lebesgue Sobolev spaces (to be specific $B(t) = |t|^p$), we cite [6,7]. We cite [4,8,9] for the Sobolev space with variable exponent. In the case of Orlicz spaces, we have some difficulties due to the non-homogeneity of the N-functions $B(t)$ and a rather indirect definition of the norm. It is generally difficult to move essentially L^p techniques to Orlicz spaces. For more work within this framework, we quote [10–13].

On the other hand, when Ω is an unbounded domain, namely, without expecting any assumptions on the behavior when $|\,x\,| \longrightarrow +\infty$, Domanska in [14] investigated the well-posedness of nonlinear elliptic systems of equations generalizing the model equation

$$-\sum_{i=1}^{N}\left(\,|\,u_{x_i}(x)\,|^{p_i-2}u_{x_i}(x)\,\right)_{x_i} + |\,u(x)\,|^{p_0-2}u(x) = f(x),$$

with corresponding indices of nonlinearity $p_i > 1$ $(\,i = \overline{0,\,n}\,)$. In [15] Bendahmann et al. the problem $(\,\mathcal{P}\,)$ with $b(x,u,\nabla u) = div(g(u))$ and $g(u)$ a polynomial growth like u^q in L^p-spaces was solved. For more results we refer the reader to the work [16]. We mention [17–19], for the Sobolev space with variable exponent, and [20–26] for the classical anisotropic space.

The oddity of our present paper is to continue in this direction and to show the existence and uniqueness of entropy solution for equations $(\mathcal{P})$ governed with growth and described by an N-uplet of N-functions satisfying the Δ_2-condition, within the fulfilling of anisotropic Orlicz spaces. Besides, we address the challenges that come about due to the absence of some topological properties, such as the densities of bounded or smooth functions.

The outline of this work is as follows. In Section 2, we recall some definitions and properties of N-functions and the space of Sobolev–Orlicz anisotropic solutions. In Section 3, we prove the Theorem of the existence of the solutions in an unbounded domain with the help of some propositions; to be demonstrated later. In Section 4, we show the uniqueness of the solution to this problem, which is expected for strictly monotonic operators at least for a broad class of lower-order terms. Finally, there is Appendix A.

2. Mathematical Background and Auxiliary Results

In this section, we introduce the notation, recall some standard definitions and collect necessary propositions and facts that are used to establish our main result.

A comprehensive presentation of Sobolev–Orlicz anisotropic space can be found in the books of M.A Krasnoselskii and Ja. B. Rutickii [23] and in [20,25].

Definition 1. *We say that $B : \mathbb{R}^+ \longrightarrow \mathbb{R}^+$ is a N-function if B is continuous, convex, with $B(z) > 0$ for $z > 0$, $\frac{B(z)}{z} \to 0$ when $z \to 0$ and $\frac{B(z)}{z} \to \infty$ when $z \to \infty$.*

This N-function B admits the following representation: $B(z) = \int_0^z b(t)\, dt$, with $b : \mathbb{R}^+ \longrightarrow \mathbb{R}^+$ which is an increasing function on the right, with $b(0) = 0$ in the case $z > 0$ and $b(z) \longrightarrow \infty$ when $z \longrightarrow \infty$.

Its conjugate is noted by $\bar{B}(z) = \int_0^{|z|} q(t)\, dt$ with q also satisfying all the properties already quoted from b, with

$$\bar{B}(z) = \sup_{y \geq 0} (\, y\,|\,z\,| - B(y)\,), \qquad z \in \mathbb{R}. \tag{5}$$

The Young's inequality is given as follows:

$$\forall z,\, y \in \mathbb{R} \qquad |\,z\,y\,| \leq B(y) + \bar{B}(z). \tag{6}$$

Definition 2. *The N-function $B(z)$ satisfies the Δ_2-condition if $\exists\, c > 0,\ z_0 \geq 0$ such that*

$$B(2\,z) \leq c\, B(z) \qquad |\,z\,| \geq z_0. \tag{7}$$

This definition is equivalent to, $\forall k > 1,\ \exists\ c(k) > 0$ such that

$$B(k\,z) \leq c(k)\, B(z) \quad \text{for} \quad |\,z\,| \geq z_0. \tag{8}$$

Definition 3. *The N-function $B(z)$ satisfies the Δ_2-condition as long as there exist positive numbers $c > 1$ and $z_0 \geq 0$ such that for $|\,z\,| \geq z_0$ we have*

$$z\, b(z) \leq c\, B(z). \tag{9}$$

Additionally, each N-function $B(z)$satisfies the inequality

$$B(y + z) \leq c\, B(z) + c\, B(y) \qquad z,\ y \in \mathbb{R}. \tag{10}$$

Definition 4. *The N-function $B(z)$ satisfies the ∇_2-condition if $\exists\, c > 2,\ z_0 \geq 0$ such that*

$$B(2\,z) \leq c\, B(z) \qquad |\,z\,| \geq z_0. \tag{11}$$

We consider the Orlicz space $L_B(\Omega)$ provided with the norm of Luxemburg given by

$$||\,u\,||_{B,\Omega} = \inf\{\, k > 0 \,/\, \int_\Omega B\left(\frac{u(x)}{k}\right) dx \leq 1\,\}. \tag{12}$$

According with [23] we obtain the inequalities

$$\int_\Omega B\left(\frac{u(x)}{||\,u\,||_{B,\Omega}}\right) dx \leq 1, \tag{13}$$

and

$$||\,u\,||_{B,\Omega} \leq \int_\Omega B(u)\, dx + 1. \tag{14}$$

Moreover, the Hölder's inequality holds and we have for all $u \in L_B(\Omega)$ and $v \in L_{\bar{B}}(\Omega)$

$$|\int_\Omega u(x)\, v(x)\, dx\,| \leq 2\,||\,u\,||_{B,\Omega}\,.\,||\,v\,||_{\bar{B},\Omega}. \tag{15}$$

In [23,25], if $P(z)$ and $B(z)$ are two N-functions such that $P(z) \ll B(z)$ and $\text{meas}\,\Omega < \infty$, then $L_B(\Omega) \subset L_P(\Omega)$; furthermore,

$$||\,u\,||_{P,\Omega} \le A_0\,(\,\text{meas}\,\Omega\,)\,||\,u\,||_{B,\Omega} \qquad u \in L_B(\Omega). \tag{16}$$

Additionally, for all N-functions $B(z)$, if $\text{meas}\,\Omega < \infty$, then $L_\infty(\Omega) \subset L_B(\Omega)$ with

$$||\,u\,||_{B,\Omega} \le A_1\,(\,\text{meas}\,\Omega\,)\,||\,u\,||_{\infty,\Omega} \qquad u \in L_B(\Omega). \tag{17}$$

Additionally, for all N-functions $B(z)$, if $\text{meas}\,\Omega < \infty$, then $L_B(\Omega) \subset L^1(\Omega)$ with

$$||\,u\,||_{1,\Omega} \le A_2\,||\,u\,||_{B,\Omega} \qquad u \in L_B(\Omega). \tag{18}$$

We define for all N-functions $B_1(z), \cdots, B_N(z)$ the space of Sobolev–Orlicz anisotropic $\mathring{W}^1_B(\Omega)$ as the adherence space $C_0^\infty(\Omega)$ under the norm

$$||\,u\,||_{\mathring{W}^1_B(\Omega)} = \sum_{i=1}^{N} ||\,u_{x_i}\,||_{B_i,\Omega}. \tag{19}$$

Definition 5. *A sequence $\{\,u_m\,\}$ is said to converge modularly to u in $\mathring{W}^1_B(\Omega)$ if for some $k > 0$ we have*

$$\int_\Omega B\left(\frac{u_m - u}{k}\right)\,dx \longrightarrow 0 \quad \text{as} \quad m \longrightarrow \infty. \tag{20}$$

Remark 1. *Since B satisfies the Δ_2-condition, the modular convergence coincides with the norm convergence.*

Remark 2. *If the doubling condition is imposed on the modular function, but not on the conjugate, then the space for the solutions to exist is non-reflexive in general. For this reason we will assume in the remainder of this article that B satisfies the both conditions; the Δ_2-condition and ∇_2-condition, so the Propositions 1 and 2 will remain true.*

Proposition 1 ([23]). *The Sobolev–Orlicz anisotropic space $\mathring{W}^1_B(\Omega)$ is complete and reflexive.*

Proposition 2 ([23]). *The Sobolev–Orlicz anisotropic $\mathring{W}^1_B(\Omega)$ is separable.*

Proposition 3.

$$z\,B'(z) \;=\; \bar{B}(\,B'(z)\,) \,+\, B(z)\,, \qquad z \,>\, 0, \tag{21}$$

with B' being the right derivative of the N-function $B(z)$.

Proof. By (6), we take $y \,=\, B'(z)$; then we obtain

$$B'(z)\,z \,\le B(z) \,+\, \bar{B}(B'(z)),$$

and by Ch. I [23], we get the result. □

In the following we will assume that for each N-function $B_i(z) \;=\; \displaystyle\int_0^{|\,z\,|} b_i(t)\;dt$ obeys the further condition

$$\lim_{\alpha\to\infty}\,\inf_{\theta>0}\,\frac{b_i(\alpha\,\theta\,)}{b_i(\alpha)} \;=\, \infty\,, \quad i = 1, \cdots, N. \tag{22}$$

Example 1. *The function*

$$B(z) \;=\; |\,z\,|^b\,(\,|\,ln|z|\,|\,+\,1\,),$$

with $b > 1$ checks the Δ_2-condition and (22).

Lemma 1. *Suppose that $(X, \mathcal{T}, meas)$ is a measurable set such that $mes(X) < \infty$. Let $\theta : X \longrightarrow [0, +\infty]$ be a measurable function such that $meas\{\, x \in X : \theta(x) = 0 \,\} = 0$. Then, for any $\epsilon > 0$, there exists $\delta > 0$ such that for any bounded domain Q*

$$\int_Q \theta(x)\ dx \leq \delta$$

implies that

$$meas(Q) \leq \epsilon.$$

Proof. See [27] (Lemma 2). □

3. The Existence of an Entropy Solution

This section is devoted to the proofs of our main results which will be split into different steps. For $m \in \mathbb{N}^*$, we define the truncation at height m, $T_m(u) : \mathbb{R} \longrightarrow \mathbb{R}$ by

$$T_m(u) = \begin{cases} u & \text{if } \ |\, u \,| \leq m\,, \\ m & \text{if } \ |\, u \,| > m\,. \end{cases}$$

Definition 6. *A measurable function u is said to be an entropy solution for the problem $(\mathcal{P})$, if $u \in \mathring{W}^1_B(\Omega)$ such that $u \geq \psi$ a.e. in Ω and*

$$\begin{aligned} &\sum_{i=1}^{N} \int_\Omega a_i(x,u,\nabla u)\,.\,\nabla(u-v)\ dx + \sum_{i=1}^{N} \int_\Omega b_i(x,u,\nabla u)\,.\,(u-v)\ dx \\ &\quad + \int_\Omega m\,.\,T_m(u-\psi)^-\,.\,sg_{\frac{1}{m}}(u)\,.\,(u-v)\ dx \\ &\quad \leq \int_\Omega f(x)\,.\,(u-v)\ dx \qquad \forall v \in K_\psi \cap L^\infty(\Omega), \end{aligned}$$

where, $K_\psi = \{\, u \in \mathring{W}^1_B(\Omega) / u \geq \psi$ a.e. in $\Omega \,\}$, and $sg_m(s) = \frac{T_m(s)}{m}$.

We have $f^m \longrightarrow f$ in $L^1(\Omega)$, $m \to \infty$, $|\, f^m(x) \,| \leq |\, f(x) \,|$ and for $i = 1, \cdots, N$, $a_i^m(x, u_m, \nabla u_m) : (\,\mathring{W}^1_B(\Omega)\,)^N \longrightarrow (\,\mathring{W}^{-1}_{\bar{B}}(\Omega)\,)^N$ being Carathéodory functions with

$$a_i^m(x,u,\nabla u) = a_i(x, T_m(u), \nabla u),$$

and $b_i^m(x, u_m, \nabla u_m) : \Omega \times \mathbb{R} \times \mathbb{R}^N \longrightarrow \mathbb{R}$ again being Carathéodory functions not satisfying any sign condition, with

$$b^m(x,u,\nabla u) = \frac{b(x,u,\nabla u)}{1 + \frac{1}{m}\,|\, b(x,u,\nabla u) \,|},$$

and

$$|\, b^m(x,u,\nabla u) \,| = |\, b(x, T_m(u), \nabla u) \,| \leq m \text{ for all } \ m \in \mathbb{N}^*, \tag{23}$$

and for all $v \in \mathring{W}^1_B(\Omega)$, we consider the following approximate problem:

$$\begin{aligned} (\mathcal{P}_m) : &\sum_{i=1}^{N} \int_\Omega a_i^m(x,u_m,\nabla u_m)\,.\,\nabla(u_m - v)\ dx + \sum_{i=1}^{N} \int_\Omega b_i^m(x,u_m,\nabla u_m)\,.\,(u_m - v)\ dx \\ &+ \int_\Omega m\,.\,T_m(u_m-\psi)^-\,.\,sg_{\frac{1}{m}}(u_m)\,.\,(u_m - v)\ dx = \int_\Omega f^m(x)\,.\,(u_m - v)\ dx. \end{aligned}$$

Theorem 1. *Assume that conditions (1)–(4) and (22) hold true, then there exists at least one solution of the approximate problem* $(\mathcal{P}_m)$.

Proof. See Appendix A. □

Theorem 2. *Under assumptions (1)–(4), the problem* $(\mathcal{P})$ *has at least one entropy solution.*

Proof. Let $R > 0$ and $\Omega(R) = \{ x \in \Omega : |x| \leq R \}$. Note by $h(t) = \left(\prod_{i=1}^{N} \frac{B_i^{-1}(t)}{t} \right)^{\frac{1}{N}}$ and we assume that $\int_0^1 \frac{h(t)}{t}\, dt$ converge, so we consider the N-functions $B^*(z)$ defined by $(B^*)^{-1}(z) = \int_0^{|z|} \frac{h(t)}{t}\, dt$. □

Lemma 2 ([20]). *Let* $u \in \mathring{W}_B^1(\Omega(R))$. *If*

$$\int_1^{\infty} \frac{h(t)}{t}\, dt = \infty, \tag{24}$$

then $\mathring{W}_B^1(\Omega(R)) \subset L_{B^*}(\Omega(R))$ *and* $||\, u\, ||_{B^*,\Omega(R)} \leq \frac{N-1}{N} ||\, u\, ||_{\mathring{W}_B^1(\Omega(R))}$. *If*

$$\int_1^{\infty} \frac{h(t)}{t}\, dt \leq \infty, \tag{25}$$

then $\mathring{W}_B^1(\Omega(R)) \subset L_\infty(\Omega(R))$ *and* $||\, u\, ||_{\infty,\Omega(R)} \leq \beta\, ||\, u\, ||_{\mathring{W}_B^1(\Omega(R))}$, *with* $\beta = \int_0^{\infty} \frac{h(t)}{t}\, dt$.

Step 1. A priori estimate of $\{ u_m \}$:

Let $v = u_m - \eta \exp(G(u_m))\,.\,T_k(u_m - v_0)^+$ where $G(s) = \int_0^s \frac{l(t)}{\bar{a}}\, dt$, $k > 0$ and $\eta \geq 0$; we have $v \in \mathring{W}_B^1(\Omega)$ and for a small enough η we deduce that $v \geq \psi$. Thus v is an admissible test function in $(\mathcal{P}_m)$ and we get for all $v_0 \in K_\psi \cap L^\infty(\Omega)$ that

$$\begin{aligned}
&\sum_{i=1}^{N} \int_\Omega a_i^m(x, u_m, \nabla u_m)\,.\,\nabla(\exp(G(u_m))\,.\,T_k(u_m - v_0)^+)\, dx \\
&\quad + \sum_{i=1}^{N} \int_\Omega b_i^m(x, u_m, \nabla u_m)\,.\,\exp(G(u_m))\,.\,T_k(u_m - v_0)^+\, dx \\
&\quad + \int_\Omega m\,.\,T_m(u_m - \psi)^-\,.\,sg_{\frac{1}{m}}(u_m)\,.\,\exp(G(u_m))\,.\,T_k(u_m - v_0)^+\, dx \\
&\quad \leq \int_\Omega f^m(x)\,.\,\exp(G(u_m))\,.\,T_k(u_m - v_0)^+\, dx;
\end{aligned}$$

then,

$$\sum_{i=1}^{N}\int_{\Omega} a_i^m(x,u_m,\nabla u_m)\,.\,\nabla u_m\,.\,\frac{l(u_m)}{\bar{a}}\,.\,\exp(G(u_m))\,.\,T_k(u_m-v_0)^+\;dx$$
$$+\sum_{i=1}^{N}\int_{\Omega} a_i^m(x,u_m,\nabla u_m)\,.\,\exp(G(u_m))\,.\,\nabla T_k(u_m-v_0)^+\;dx$$
$$+\int_{\Omega} m\,.\,T_m(u_m-\psi)^-\,.\,sg_{\frac{1}{m}}(u_m)\,.\,\exp(G(u_m))\,.\,T_k(u_m-v_0)^+\;dx$$
$$\leq \sum_{i=1}^{N}\int_{\Omega} |\,b_i^m(x,u_m,\nabla u_m)\,|\,.\,\exp(G(u_m))\,.\,T_k(u_m-v_0)^+\;dx$$
$$+\int_{\Omega} f^m(x)\,.\,\exp(G(u_m))\,.\,T_k(u_m-v_0)^+\;dx,$$

by (2) and (4), we obtain

$$\sum_{i=1}^{N}\int_{\Omega} a_i^m(x,u_m,\nabla u_m)\,.\,\exp(G(u_m))\,.\,\nabla T_k(u_m-v_0)^+\;dx$$
$$+\int_{\Omega} m\,.\,T_m(u_m-\psi)^-\,.\,sg_{\frac{1}{m}}(u_m)\,.\,\exp(G(u_m))\,.\,T_k(u_m-v_0)^+\;dx$$
$$\leq \int_{\Omega}\Big[\,h(x)+f^m(x)+\phi(x)\,.\,\frac{l(u_m)}{\bar{a}}\,\Big]\,.\,\exp(G(u_m))\,.\,T_k(u_m-v_0)^+\;dx,$$

so

$$\sum_{i=1}^{N}\int_{\{|\,u_m-v_0\,|\leq k\}} a_i^m(x,u_m,\nabla u_m)\,.\,\nabla u_m\,.\,\exp(G(u_m))\;dx$$
$$-\,c\sum_{i=1}^{N}\int_{\{|\,u_m-v_0\,|\leq k\}} a_i^m(x,u_m,\nabla u_m)\,.\,\frac{\nabla v_0}{c}\,.\,\exp(G(u_m))\;dx$$
$$+\int_{\Omega} m\,.\,T_m(u_m-\psi)^-\,.\,sg_{\frac{1}{m}}(u_m)\,.\,\exp(G(u_m))\,.\,T_k(u_m-v_0)^+\;dx$$
$$\leq \int_{\Omega}\Big[\,h(x)+f^m(x)+\phi(x)\,.\,\frac{l(u_m)}{\bar{a}}\,\Big]\,.\,\exp(G(u_m))\,.\,T_k(u_m-v_0)^+\;dx,$$

where c is a constant such that $0<c<1$, and since $h,\ f^m,\ \phi\in L^1(\Omega)$ we deduce that

$$\sum_{i=1}^{N}\int_{\{|\,u_m-v_0\,|\leq k\}} a_i^m(x,u_m,\nabla u_m)\,.\,\nabla u_m\,.\,\exp(G(u_m)\;dx$$
$$+\int_{\Omega} m\,.\,T_m(u_m-\psi)^-\,.\,sg_{\frac{1}{m}}(u_m)\,.\,\exp(G(u_m))\,.\,T_k(u_m-v_0)^+\;dx$$
$$\leq -\,c\sum_{i=1}^{N}\int_{\{|\,u_m-v_0\,|\leq k\}}\left[a_i^m(x,u_m,\nabla u_m)\;-\;a_i^m(x,u_m,\frac{\nabla v_0}{c})\right].\,\nabla(u_m-\frac{\nabla v_0}{c})\,.\,\exp(G(u_m))\;dx$$
$$+\,c\sum_{i=1}^{N}\int_{\{|\,u_m-v_0\,|\leq k\}} a_i^m(x,u_m,\nabla u_m)\,.\,\nabla u_m\,.\,\exp(G(u_m))\;dx$$
$$+\,c\sum_{i=1}^{N}\int_{\{|\,u_m-v_0\,|\leq k\}} |\,a_i^m(x,u_m,\frac{\nabla v_0}{c})\,|\,.\,|\nabla(u_m-\frac{\nabla v_0}{c})\,|\,.\,\exp(G(u_m))\;dx+c_1,$$

by (1)

$$
\begin{aligned}
(1-c)\sum_{i=1}^{N}\int_{\{|u_m-v_0|\le k\}} & a_i^m(x,u_m,\nabla u_m)\,.\,\nabla u_m\,.\,\exp(G(u_m))\;dx \\
&+\int_\Omega m\,.\,T_m(u_m-\psi)^-\,.\,sg_{\frac{1}{m}}(u_m)\,.\,\exp(G(u_m))\,.\,T_k(u_m-v_0)^+\;dx \\
&\le c\sum_{i=1}^{N}\int_{\{|u_m-v_0|\le k\}}|\,a_i^m(x,u_m,\frac{\nabla v_0}{c})\,|\,.\,|\nabla u_m\,|\,.\,\exp(G(u_m))\;dx \\
&+c\sum_{i=1}^{N}\int_{\{|u_m-v_0|\le k\}}|\,a_i^m(x,u_m,\frac{\nabla v_0}{c})\,|\,.\left|\frac{\nabla v_0}{c}\right|.\,\exp(G(u_m))\;dx+c_1,
\end{aligned}
$$

since $\frac{\nabla v_0}{c}\in \mathring{W}^1_B(\Omega)$, and by (8), (3) and (6) and the fact that $\exp(G(\pm\infty))\le \exp\left(\frac{||l||_{L^1(\mathbb{R})}}{\tilde{a}}\right)$ we have

$$
\begin{aligned}
(1-c)\sum_{i=1}^{N}\int_{\{|u_m-v_0|\le k\}} a_i^m(x,u_m,\nabla u_m)\,.\,\nabla u_m\;dx &+\int_{\{|u_m-v_0|\ge 0\}} m\,.\,T_m(u_m-\psi)^-\,.\,sg_{\frac{1}{m}}(u_m)\;dx \\
&\le \frac{\tilde{a}\,(1-c)}{2}\sum_{i=1}^{N}\int_{\{|u_m-v_0|\le k\}} B_i(\nabla u_m)\;dx+c_2(k)\,.\,c_1,
\end{aligned}
$$

where $c_2(k)$ is a positive constant which depends only on k.

Finally, by (2) we obtain

$$
\sum_{i=1}^{N}\int_{\{|u_m-v_0|\le k\}} B_i(\nabla u_m)\;dx\le c_3\,.\,k, \tag{26}
$$

and

$$
0\le\int_{\{|u_m-v_0|\ge 0\}} m\,.\,T_m(u_m-\psi)^-\,.\,sg_{\frac{1}{m}}(u_m)\;dx\le c_1. \tag{27}
$$

Similarly, taking $v=u_m-\eta\,.\,\exp(G(u_m))\,.\,T_k(u_m-v_0)^-$ as a test function in $(\mathcal{P}_m)$, we obtain

$$
\begin{aligned}
\sum_{i=1}^{N}\int_\Omega & a_i^m(x,u_m,\nabla u_m)\,.\,\nabla(\,\exp(G(u_m))\,.\,T_k(u_m-v_0)^-)\;dx \\
&+\sum_{i=1}^{N}\int_\Omega b_i^m(x,u_m,\nabla u_m)\,.\,\exp(-G(u_m))\,.\,T_k(u_m-v_0)^-\;dx \\
&+\int_\Omega m\,.\,T_m(u_m-\psi)^-\,.\,sg_{\frac{1}{m}}(u_m)\,.\,\exp(-G(u_m))\,.\,T_k(u_m-v_0)^-\;dx \\
&\le\int_\Omega f^m(x)\,.\,\exp(-G(u_m))\,.\,T_k(u_m-v_0)^-\;dx,
\end{aligned}
$$

and using same techniques, we obtain also

$$
\sum_{i=1}^{N}\int_{\{|u_m-v_0|\ge k\}} B_i(\nabla u_m)\;dx\le c_4\,.\,k, \tag{28}
$$

and

$$
0\le\int_{\{|u_m-v_0|\le 0\}} m\,.\,T_m(u_m-\psi)^-\,.\,sg_{\frac{1}{m}}(u_m)\;dx\le c_5. \tag{29}
$$

Additionally, by (26), (27), (28) and (29) we conclude that

$$
\int_\Omega B(\nabla T_k(u_m))\;dx\le c\,.\,k, \tag{30}
$$

with c_3, c_4, c_5, c_6 being positive constants.
Step 2. Almost everywhere convergence of $\{ u_m \}$:

Firstly, we prove that meas$\{ x \in \Omega : |u_m| \geq k \} \to 0$.
According to Lemma 2, we have

$$\begin{aligned} ||\, T_k(u_m) \,||_{B^*} &\leq c \,||\, \nabla T_k(u_m) \,||_B \\ &\leq c \,.\, \epsilon(k) \int_\Omega B(\nabla T_k(u_m))\, dx \\ &\leq c \,.\, \epsilon(k) \,.\, k \quad \text{for } k > 1, \end{aligned} \tag{31}$$

with c being a positive constant and $\epsilon(k) \to 0$ when $k \to \infty$. By (31) we obtain

$$\begin{aligned} B^*\left(\frac{k}{||\, T_k(u_m) \,||_{B^*}} \right) \text{meas}\, \{ x \in \Omega : |u_m| \geq k \} &\leq \int_\Omega B^*\left(\frac{T_k(u_m)}{||\, T_k(u_m) \,||_{B^*}} \right) dx \\ &\leq \int_\Omega B^*\left(\frac{k}{||\, T_k(u_m) \,||_{B^*}} \right) dx. \end{aligned} \tag{32}$$

Thus, we deduce that

$$B^*\left(\frac{k}{||\, T_k(u_m) \,||_{B^*}} \right) \longrightarrow \infty \text{ when } k \to \infty.$$

Hence

$$\text{meas}\{ x \in \Omega : |u_m| \geq k \} \to 0 \text{ as } k \to \infty \text{ for all } m \in \mathbb{N}.$$

Secondly we show that for all $\{u_m\}$ measurable function on Ω such that

$$T_k(u_m) \in \mathring{W}^1_B(\Omega) \qquad \forall k \geq 1,$$

we have

$$\text{meas}\{ x \in \Omega : B(\nabla u_m) \geq \alpha \} \to 0 \text{ as } \alpha \to \infty. \tag{33}$$

In the beginning

$$\begin{aligned} \text{meas}\{ x \in \Omega : B(\nabla u_m) \geq 0 \} = &\text{meas}\{ \{ x \in \Omega : |u_m| \geq k,\, B(\nabla u_m) \geq \alpha \} \\ &\cup \{ x \in \Omega : |u_m| < k,\, B(\nabla u_m) \geq \alpha \} \}, \end{aligned}$$

if we denote

$$g(\alpha, k) = \text{meas}\{ x \in \Omega : |u_m| \geq k,\, B(\nabla u_m) \geq \alpha \},$$

we have

$$\text{meas}\{ x \in \Omega : |u_m| < k,\, B(\nabla u_m) \geq \alpha \} = g(\alpha, 0) - g(\alpha, k),$$

then

$$\int_{\{ x \in \Omega : |u_m| < k \}} B(\nabla u_m)\, dx = \int_0^\infty \big(g(\alpha, 0) - g(\alpha, k) \big)\, d\alpha \leq c \,.\, k, \tag{34}$$

with $\alpha \to g(\alpha, k)$ is a decreasing map; then

$$\begin{aligned} g(\alpha, 0) &\leq \frac{1}{\alpha} \int_0^\alpha g(s, 0)\, ds \\ &\leq \frac{1}{\alpha} \int_0^\alpha \big(g(s, 0) - g(s, k) \big)\, ds + \frac{1}{\alpha} \int_0^\alpha g(s, k)\, ds \\ &\leq \frac{1}{\alpha} \int_0^\alpha \big(g(s, 0) - g(s, k) \big)\, ds + g(0, k), \end{aligned} \tag{35}$$

and according to (34) and (35) we have

$$g(\alpha,0) \leq \frac{c\,.\,k}{\alpha} + g(0,k),$$

like [28] we obtain

$$\lim_{k\to\infty} g(0,k) = 0.$$

Hence

$$g(\alpha,0) \to 0 \text{ as } \alpha \to \infty.$$

We have now to demonstrate that the almost everywhere convergence of $\{\, u_m : \}$

$$u_m \longrightarrow u \text{ almost everywhere in } \Omega.$$

Let $g(k) = \sup_{m\in\mathbb{N}} \text{meas}\{\, x \in \Omega : |\, u_m \,| > k \,\} \to 0$ as $k \to \infty$. Since Ω is unbounded domain in $\mathbb{R}^N$, we define

$$\eta_R(x) = \begin{cases} 1 & \text{if } \ x < R\,, \\ R+1-\alpha & \text{if } \ R \leq x < R+1\,, \\ 0 & \text{if } \ x \geq R+1\,. \end{cases}$$

For R, $k > 0$, we have by (10)

$$\begin{aligned} \int_\Omega B(\nabla \eta_R(|\,x\,|)\,.\,T_k(u_m))\; dx \leq c \int_{\{\,x\in\Omega:\,|\,u_m\,|<k\,\}} B(\nabla u_m)\; dx \\ + c \int_\Omega B(T_k(u_m)\,.\,\nabla \eta_R(|\,x\,|)\; dx \\ \leq c(k,\, R), \end{aligned}$$

which implies that the sequence $\{\, \eta_R(|\,x\,|)\, T_k(u_m) \,\}$ is bounded in $\mathring{W}^1_B(\Omega(R+1))$, and by embedding Theorem, for an N-function P with $P \ll B$ we have

$$\mathring{W}^1_B(\Omega(R+1)) \hookrightarrow L_P(\Omega(R+1))$$

and since $\eta_R = 1$ in $\Omega(R)$, we have:

$$\eta_R\, T_k(u_m) \longrightarrow v_k \text{ in } L_P(\Omega(R+1)) \text{ as } m \longrightarrow \infty.$$

For $k \in \mathbb{N}^*$,

$$T_k(u_m) \longrightarrow v_k \text{ in } L_P(\Omega(R+1)) \text{ as } m \longrightarrow \infty$$

by a diagonal process, we prove that there is measurable $u : \Omega \longrightarrow \mathbb{R}$ such that $u_m \longrightarrow u$ a.e. in Ω.

Lemma 3 ([29]). *Let an N-function $\bar{B}(t)$ satisfy the Δ_2-condition and $u_m,\ m \geq 1$ and u be two functions of $L_B(\Omega)$ such that*

$$||\, u_m \,||_B \leq c \qquad m = 1,2,\cdots.$$

$$u_m \longrightarrow u \textit{ almost everywhere in } \Omega,\ m \to \infty.$$

Then,

$$u_m \rightharpoonup u \textit{ weakly in } L_B(\Omega) \textit{ as } m \to \infty.$$

Hence,

$$\textit{meas}\,\{\, x \in \Omega :\ |\, u_m \,| \geq k \,\} \longrightarrow 0 \textit{ when } k \to \infty \textit{ for all } m \in \mathbb{N}.$$

Step 3. Weak convergence of the gradient:

Since $\mathring{W}^1_B(\Omega)$ reflexive, there exists a subsequence

$$T_k(u_m) \rightharpoonup v \text{ weakly in } \mathring{W}^1_B(\Omega), \quad m \to \infty,$$

and since

$$\mathring{W}^1_B(\Omega) \hookrightarrow L_B(\Omega),$$

we have

$$\nabla T_k(u_m) \rightharpoonup \nabla u_m \text{ in } L_B(\Omega) \text{ as } m \to \infty,$$

since

$$u_m \longrightarrow u \text{ almost everywhere in } \Omega \text{ as } m \to \infty, \tag{36}$$

implies the local convergence in measure and, therefore, the local Cauchy property of u_m in measure

$$meas\{\,\Omega(R) : |\,u_m - u_n\,| \geq k\,\} \to 0 \text{ as } m,\, n \to \infty \text{ for any } k > 0. \tag{37}$$

Proving that

$$\nabla u_m \longrightarrow \nabla u \text{ locally in measure as } m \to \infty. \tag{38}$$

For that, we borrow ideas from Evans [13], Demangel-Hebey [12] and Koznikova L. M. [21,22]. Let $\delta > 0$ be given. By Egoroff's Theorem, there exists $E_{\delta,k,\alpha} \subset\subset \Omega$ such that

$$\begin{aligned} E_{\delta,k,\alpha}(R) =\{\,&\Omega(R) : |\,u_m - u_n\,| < k,\; B(\nabla u_m) < \alpha,\; B(\nabla u_n) < \alpha,\; |\,u_m\,| \leq \alpha,\\ &|\,u_n\,| \leq \alpha,\; |\,\nabla(u_m - u_n)\,| \geq \delta\,\} \end{aligned}$$

$$\begin{aligned} \{\,\Omega(R) : |\,\nabla(u_m - u_n)\,| \geq \delta\,\} \subset &\{\,\Omega : B(\nabla u_m) > \alpha\,\} \cup \{\,\Omega : B(\nabla u_n) > \alpha\,\}\\ &\cup \{\,\Omega(R) : |\,u_m - u_n\,| \geq k\,\} \cup \{\,\Omega : |\,u_m\,| > \alpha\,\}\\ &\cup \{\,\Omega :\; |\,u_n\,| > \alpha\,\} \cup E_{\delta,k,\alpha}(R). \end{aligned}$$

Then, by Lemma 3 and (33) we obtain that

$$\begin{aligned} meas\,\{\,\Omega(R) : |\,\nabla(u_m - u_n)\,| \geq \delta\,\} \leq &4\epsilon + meas\, E_{\delta,k,\alpha}(R)\\ &+ meas\,\{\,\Omega(R) : |\,u_m - u_n\,| \geq k\,\} \quad \forall n,\, m \in \mathbb{N}^*. \end{aligned} \tag{39}$$

According to (1) and the fact that a continuous function on a compact set achieves the lowest value, there exists a function $\theta(x) > 0$ almost everywhere in Ω, such that, for $B(\xi) \leq \alpha$, $B(\xi') \leq \alpha$, $|\,s\,| \leq \alpha$ and for $i = 1, \cdots, N$, $|\,\xi_i - \xi_i'\,| \geq k$, we have that

$$\sum_{i=1}^{N} \left[\,a_i^m(x,s,\xi) - a_i^m(x,s,\xi')\,\right].(\xi_i - \xi_i') \geq \theta(x), \tag{40}$$

holds. Writing $(\mathcal{P}_m)$ twice for $\{\,u_m\,\}$ and $\{\,u_n\,\}$, and by subtracting the second relation from the first and according to (23), (27), (29) and (36) we obtain

$$\sum_{i=1}^{N} \int_\Omega \left[\,a_i^m(x,u_m,\nabla u_m) - a_i^m(x,u_n,\nabla u_n)\,\right].\nabla(u_m - u_n - v)\; dx = 0.$$

Consider the following test function:

$$v = u_m - u_n - \eta_R(|\,x\,|)\,\eta_\alpha(|\,u_n\,|)\,\eta_\alpha(|\,u_m\,|)\;\exp(G(|\,u_m - u_n\,|))\,T_\delta(u_m - u_n).$$

Further on, by applying (40), we get

$$\begin{aligned}\int_{E_{\delta,k,\alpha}(R)} \theta(x)\, dx \leq \sum_{i=1}^{N} \int_{E_{\delta,k,\alpha}(R)} & \left[\, a_i^m(x, u_m, \nabla u_m) - a_i^m(x, u_m, \nabla u_n) \,\right] \\ & \times \nabla(\, \eta_R(|\, x\,|)\, \eta_\alpha(|\, u_n\,|)\, \eta_\alpha(|\, u_m\,|)\, \exp(G(|\, u_m - u_n\,|))\, T_\delta(u_m - u_n)\,)\, dx \\ & \leq \sum_{i=1}^{N} \int_{\{\,\Omega:|\, u_m - u_n\,| < k\,\}} \eta_R(|\, x\,|)\, \eta_\alpha(|\, u_n\,|)\, \eta_\alpha(|\, u_m\,|)\, \exp(G(|\, u_m - u_n\,|)) \\ & \times \left[\, a_i^m(x, u_m, \nabla u_m) - a_i^m(x, u_m, \nabla u_n) \,\right] . \nabla(u_m - u_n)\, dx \\ & = A_1^m(x) + A_2^m(x),\end{aligned}$$

with

$$\begin{aligned}A_1^m(x) = \sum_{i=1}^{N} \int_{\{\,|\, u_m - u_n\,| < k\,\}} & \eta_R(|\, x\,|)\, \eta_\alpha(|\, u_n\,|)\, \eta_\alpha(|\, u_m\,|)\, \exp(G(|\, u_m - u_n\,|)) \\ & \times \left[\, a_i^m(x, u_m, \nabla u_m) - a_i^m(x, u_n, \nabla u_n) \,\right] . \nabla(u_m - u_n)\, dx,\end{aligned}$$

and

$$\begin{aligned}A_2^m(x) = \sum_{i=1}^{N} \int_{\{\,|\, u_m - u_n\,| < k\,\}} & \eta_R(|\, x\,|)\, \eta_\alpha(|\, u_n\,|)\, \eta_\alpha(|\, u_m\,|)\, \exp(G(|\, u_m - u_n\,|)) \\ & \times \left[\, a_i^m(x, u_n, \nabla u_n) - a_i^m(x, u_m, \nabla u_n) \,\right] . \nabla(u_m - u_n)\, dx.\end{aligned}$$

Since $B(u)$ satisfies the Δ_2-condition, by (14) we have

$$\int_{\Omega} B(u)\, dx \leq c_0\, ||\, u\,||_{B,\Omega}. \tag{41}$$

According to Lemma 3, we get

$$||\, u_m\,||_{\mathring{W}_B^1(\Omega)} \leq c_1 \qquad m \in \mathbb{N}^*, \tag{42}$$

and

$$||\, B(\nabla u_m)\,||_1 \leq c_2 \qquad m \in \mathbb{N}^*. \tag{43}$$

Additionally, using (14) and (3) we have

$$\begin{aligned}||\, a(x, u, \nabla u)\,||_{L_{\bar{B}}(\Omega)} &= \sum_{i=1}^{N} ||\, a_i(x, u, \nabla u)\,||_{L_{\bar{B}_i}(\Omega)} \\ &\leq \sum_{i=1}^{N} \int_{\Omega} \bar{B}_i(\, a_i(x, u, \nabla u)\,)\, dx + N \\ &\leq c_3\, ||\, B(u)\,||_{1,\Omega} + ||\, \varphi\,||_{1,\Omega} + N \\ &\leq c_4.\end{aligned} \tag{44}$$

Hence,

$$
\begin{aligned}
A_1^m(x) = &\sum_{i=1}^{N} \int_{\{\Omega: |u_m-u_n|<k, |x|<R, |u_m|<\alpha, |u_n|<\alpha\}} \exp(G(|u_m-u_n|)) \\
&\times \left[a_i^m(x,u_m,\nabla u_m) - a_i^m(x,u_n,\nabla u_n) \right] . \nabla(u_m-u_n)\, dx \\
&+ \sum_{i=1}^{N} \int_{\{\Omega: |u_m-u_n|<k, R\le |x| \le R+1, \alpha \le |u_m| \le \alpha+1, \alpha \le |u_n| \le \alpha+1\}} (R+1-|x|) \\
&\times (\alpha+1-|u_n|) . (\alpha+1-|u_m|) . \exp(G(|u_m-u_n|)) \\
&\times \left[a_i^m(x,u_m,\nabla u_m) - a_i^m(x,u_n,\nabla u_n) \right] . \nabla(u_m-u_n)\, dx,
\end{aligned}
$$

since $\exp(G(\pm\infty)) \le \exp\left(\frac{||l||_{L^1(\mathbb{R}}}{\bar{a}}\right)$, and according to (42), (43), (44) and (15) we obtain that

$$A_1^m(x) \le c_1(R,\alpha).k;$$

the same for $A_2^m(x)$. We get

$$A_2^m(x) \le c_2(R,\alpha).k.$$

Then,

$$\int_{E_{\delta,k,\alpha}(R)} \theta(x)\, dx \le c_3(R,\alpha).k. \tag{45}$$

For any arbitrary $\delta > 0$ for fixed m and α, by choosing k from (45) we establish the following inequality

$$\int_{E_{\delta,k,\alpha}(R)} \theta(x)\, dx < \delta.$$

By applying Lemma 1, for any $\epsilon > 0$, we find

$$meas\, E_{\delta,k,\alpha}(R) < \epsilon. \tag{46}$$

In addition, according to (37), we have

$$meas\, \{\Omega(R) : |u_m - u_n| \ge k\} < \epsilon, \quad m, n > 0. \tag{47}$$

By combining (39), (46) and (47) we deduce the inequality

$$meas\, \{\Omega(R) : |\nabla(u_m-u_n)| \ge \delta\} < 6\epsilon, \quad n, m > 0.$$

Hence, the sequence $\{\nabla u_m\}$ is fundamental in measure on the set $\Omega(R)$ for any $R > 0$. This implies (38) and the selective convergence,

$$\nabla u_m \longrightarrow \nabla u \text{ almost everywhere in } \Omega, \quad m \to \infty. \tag{48}$$

Then, we obtain for any fixed $k > 0$

$$\nabla T_k(u_m) \longrightarrow \nabla T_k(u) \text{ almost everywhere in } \Omega \text{ as } m \to \infty.$$

Applying Lemma 3, we have the following weak convergence

$$\nabla T_k(u_m) \rightharpoonup \nabla T_k(u) \text{ in } L_B(\Omega) \text{ as } m \to \infty.$$

Proposition 4. *Suppose that Conditions (1)–(4) are satisfied and let $(u_m)_{m\in\mathbb{N}}$ be a sequence in $\mathring{W}_B^1(\Omega(R))$ such that*

(a) $u_m \rightharpoonup u$ *in* $\mathring{W}^1_B(\Omega(R))$
(b) $a^m(x, u_m, \nabla u_m)$ *is bounded in* $L_{\bar{B}}(\Omega(R))$
(c) $\sum_{i=1}^{N} \int_{\Omega(R)} [\, a_i^m(x, u_m, \nabla u_m) - a_i^m(x, u_m, \nabla u\chi_\epsilon)\,.\nabla(u_m - u\chi_\epsilon)\; dx \to 0$ *as* $\epsilon \to +\infty$ *(*χ_ϵ *the characteristic function of* $\Omega_\epsilon(R) = \{\, x \in \Omega;\ |\nabla u| \le \epsilon\,\}$*). Then*

$$B(|\nabla u_m|) \longrightarrow B(|\nabla u|) \text{ in } L^1(\Omega(R))$$

Proof. Let $\epsilon > 0$ fixed, and $\eta > \epsilon$; then from (1) we have

$$\begin{aligned}
0 &\le \sum_{i=1}^{N} \int_{\Omega_\eta(R)} [\, a_i^m(x, u_m, \nabla u_m) - a_i^m(x, u_m, \nabla u)\,]\,.\nabla(u_m - u)\; dx \\
&\le \sum_{i=1}^{N} \int_{\Omega_\epsilon(R)} [\, a_i^m(x, u_m, \nabla u_m) - a_i^m(x, u_m, \nabla u)\,]\,.\nabla(u_m - u)\; dx \\
&= \sum_{i=1}^{N} \int_{\Omega_\epsilon(R)} [\, a_i^m(x, u_m, \nabla u_m) - a_i^m(x, u_m, \nabla u\chi_\epsilon)\,]\,.\nabla(u_m - u\chi_\epsilon)\; dx
\end{aligned}$$

$$\le \sum_{i=1}^{N} \int_{\Omega(R)} [\, a_i^m(x, u_m, \nabla u_m) - a_i^m(x, u_m, \nabla u\chi_\epsilon)\,]\,.\nabla(u_m - u\chi_\epsilon)\; dx,$$

using the condition (c) we get

$$\lim_{m\to\infty} \sum_{i=1}^{N} \int_{\Omega(R)} [\, a_i^m(x, u_m, \nabla u_m) - a_i^m(x, u_m, \nabla u\chi_\epsilon)\,]\,.\nabla(u_m - u\chi_\epsilon)\; dx = 0$$

proceeding as in [28], and we obtain $\nabla u_m \longrightarrow \nabla u$; by letting $\epsilon \longrightarrow \infty$ we get

$$\nabla u_m \chi_\epsilon \longrightarrow \nabla u,$$

Thus, since

$$\begin{aligned}
&\sum_{i=1}^{N} \int_{\Omega(R)} a_i^m(x, u_m, \nabla u_m)\,.\nabla u_m\; dx \\
&\quad = \sum_{i=1}^{N} \int_{\Omega(R)} [\, a_i^m(x, u_m, \nabla u_m) - a_i^m(x, u_m, \nabla u\chi_\epsilon)\,]\,.\nabla(u_m - u\chi_\epsilon)\; dx \\
&\quad + \sum_{i=1}^{N} \int_{\Omega(R)} a_i^m(x, u_m, \nabla\chi_\epsilon)\,.\nabla(u_m - u\chi_\epsilon)\; dx + \sum_{i=1}^{N} \int_{\Omega(R)} a_i^m(x, u_m, \nabla u_m)\,.\nabla u\chi_\epsilon\; dx,
\end{aligned}$$

using (b), we have

$$\sum_{i=1}^{N} a_i^m(x, u_m, \nabla u_m) \rightharpoonup \sum_{i=1}^{N} a_i^m(x, u, \nabla u) \text{ weakly in } (L_{\bar{B}}(\Omega(R)))^N.$$

Therefore

$$\sum_{i=1}^{N} \int_{\Omega(R)} a_i^m(x, u_m, \nabla u_m)\,.\nabla u\chi_\epsilon\; dx \longrightarrow \sum_{i=1}^{N} \int_{\Omega(R)} a_i^m(x, u, \nabla u)\,.\nabla u\; dx \text{ as } m \to \infty,\ \epsilon \to \infty.$$

Thus,

$$\sum_{i=1}^{N}\int_{\Omega(R)}\left[\,a_i^m(x,u_m,\nabla u_m)-a_i^m(x,u_m,\nabla u\chi_\epsilon)\,\right].\nabla(u_m-\nabla u\chi_\epsilon)\;dx\longrightarrow 0 \text{ as } m\to\infty,\,\epsilon\to\infty.$$

and

$$\sum_{i=1}^{N}\int_{\Omega(R)} a_i^m(x,u_m,\nabla u\chi_\epsilon)\,.\,\nabla(u_m-u\chi_\epsilon)\;dx\longrightarrow 0 \text{ as } m\to\infty,\,\epsilon\to\infty.$$

Thus,

$$\lim_{m\to\infty}\sum_{i=1}^{N}\int_{\Omega(R)} a_i^m(x,u_m,\nabla u_m)\,.\,\nabla u_m\;dx=\sum_{i=1}^{N}\int_{\Omega(R)} a_i^m(x,u,\nabla u)\,.\,\nabla u\;dx,$$

from (2), and the vitali's Theorem, we get

$$\bar{a}\sum_{i=1}^{N}\int_{\Omega(R)} B_i(|\,\nabla u_m\,|)\;dx-\int_{\Omega(R)}\phi(x)\;dx\geq\bar{a}\sum_{i=1}^{N}\int_{\Omega(R)} B_i(|\,\nabla u\,|)\;dx-\int_{\Omega(R)}\phi(x)\;dx,$$

Consequently, by Lemma 2.6 in [11] and (48), we get

$$B(|\,\nabla u_m\,|)\longrightarrow B(|\,\nabla u\,|) \text{ in } \mathring{W}^1_B(\Omega(R)),$$

thanks to lemma 1 (see [20]) and (48), we have

$$B(|\,\nabla u_m\,|)\longrightarrow B(|\,\nabla u\,|) \text{ in } L^1(\Omega(R)).$$

□

Step 4. Strong convergence of the gradient:

In this step we consider again the following test function:

$$v=u_m+\eta\;\exp(G(|u_m|))\,(\,T_k(u_m)-T_k(u))\,\eta_j(|u_m|)\,h_j(u_m),$$

$$\text{with, } h_j(u_m))=1-|\,T_1(u_m-T_j(u_m))\,|=\begin{cases}1 & \text{if } \{|u_m|\geq j\}\\ 0 & \text{if } \{|u_m|\geq j+1\}\\ j+1-|u_m| & \text{if } \{j\leq|u_m|\leq j+1\}\end{cases}$$

and, $|T_k(u_m)-T_k(u)|$ at the same sign when $u_m\in\{\,|u_m|>k\}$, where $j\geq k>0$ and η is small enough, we obtain

$$\begin{aligned}&\sum_{i=1}^{N}\int_{\Omega} a_i^m(x,u_m,\nabla u_m)\,.\,\nabla(\exp(G(|u_m|))\,(\,T_k(u_m)-T_k(u))\,\eta_j(|u_m|)\,h_j(u_m))\;dx\\&+\int_{\Omega} m\,.\,T_m(u_m-\psi)^-\,.\,sg_{\frac{1}{m}}(u_m)\,.\,\exp(G(|u_m|))\,(\,T_k(u_m)-T_k(u))\,\eta_j(|u_m|)\,h_j(u_m)\;dx\\&+\sum_{i=1}^{N}\int_{\Omega} b_i^m(x,u_m,\nabla u_m)\,.\,\exp(G(|u_m|))\,(\,T_k(u_m)-T_k(u))\,\eta_j(|u_m|)\,h_j(u_m)\;dx\\&\leq\int_{\Omega} f^m(x)\,.\,\exp(G(|u_m|))\,(\,T_k(u_m)-T_k(u))\,\eta_j(|u_m|)\,h_j(u_m)\;dx,\end{aligned}$$

which implies,

$$
\begin{aligned}
&\sum_{i=1}^{N}\int_{\Omega} a_i^m(x,u_m,\nabla u_m)\,.\,\nabla(\exp(G(|u_m|)))\,(\,T_k(u_m)-T_k(u))\,\eta_j(|u_m|)\,h_j(u_m)\;dx\\
&\quad+\sum_{i=1}^{N}\int_{\Omega} a_i^m(x,u_m,\nabla u_m)\,.\,\exp(G(|u_m|))\,\nabla((\,T_k(u_m)-T_k(u)))\,\eta_j(|u_m|)\,h_j(u_m)\;dx\\
&\quad+\sum_{i=1}^{N}\int_{\Omega} a_i^m(x,u_m,\nabla u_m)\,.\,\exp(G(|u_m|))\,(\,T_k(u_m)-T_k(u))\,\nabla\eta_j(|u_m|)\,h_j(u_m)\;dx\\
&\quad+\sum_{i=1}^{N}\int_{\Omega} a_i^m(x,u_m,\nabla u_m)\,.\,\exp(G(|u_m|)))\,(\,T_k(u_m)-T_k(u))\,\eta_j(|u_m|)\,\nabla h_j(u_m)\;dx\\
&\quad+\sum_{i=1}^{N}\int_{\Omega} b_i^m(x,u_m,\nabla u_m)\,.\,\exp(G(|u_m|))\,(\,T_k(u_m)-T_k(u))\,\eta_j(|u_m|)\,h_j(u_m)\;dx\\
&\quad+\int_{\Omega} m\,.\,T_m(u_m-\psi)^-\,.\,sg_{\frac{1}{m}}(u_m)\,.\,\exp(G(|u_m|))\,(\,T_k(u_m)-T_k(u))\,\eta_j(|u_m|)\,h_j(u_m)\;dx\\
&\quad\le\int_{\Omega} f^m(x)\,.\,\exp(G(|u_m|))\,(\,T_k(u_m)-T_k(u))\,\eta_j(|u_m|)\,h_j(u_m)\;dx;
\end{aligned}
$$

then,

$$
\begin{aligned}
&\sum_{i=1}^{N}\int_{\Omega} a_i^m(x,u_m,\nabla u_m)\,.\,\nabla u_m\,.\,\frac{l(|u_m|)}{\bar{a}}\,.\,\exp(G(|u_m|))\,(\,T_k(u_m)-T_k(u)\,)\,\eta_j(|u_m|)\,h_j(u_m)\;dx\\
&\quad+\sum_{i=1}^{N}\int_{\Omega} a_i^m(x,u_m,\nabla u_m)\,.\,\exp(G(|u_m|))\,\nabla((\,T_k(u_m)-T_k(u)))\,\eta_j(|u_m|)\,h_j(u_m)\;dx\\
&\quad+\sum_{i=1}^{N}\int_{\Omega} a_i^m(x,u_m,\nabla u_m)\,.\,\exp(G(|u_m|))\,(\,T_k(u_m)-T_k(u))\,\nabla\eta_j(|u_m|)\,h_j(u_m)\;dx\\
&\quad+\sum_{i=1}^{N}\int_{\Omega} a_i^m(x,u_m,\nabla u_m)\,.\,\exp(G(|u_m|)))\,(\,T_k(u_m)-T_k(u))\,\eta_j(|u_m|)\,\nabla h_j(u_m)\;dx\\
&\quad+\int_{\Omega} m\,.\,T_m(u_m-\psi)^-\,.\,sg_{\frac{1}{m}}(u_m)\,.\,\exp(G(|u_m|))\,(\,T_k(u_m)-T_k(u))\,\eta_j(|u_m|)\,h_j(u_m)\;dx\\
&\quad\le\sum_{i=1}^{N}\int_{\Omega} |\,b_i^m(x,u_m,\nabla u_m)\,|\,.\,\exp(G(|u_m|))\,(\,T_k(u_m)-T_k(u))\,\eta_j(|u_m|)\,h_j(u_m)\;dx\\
&\quad+\int_{\Omega} f^m(x)\,.\,\exp(G(|u_m|))\,(\,T_k(u_m)-T_k(u))\,\eta_j(|u_m|)\,h_j(u_m)\;dx,
\end{aligned}
$$

by (2) and (4) we get

$$
\begin{aligned}
&\bar{a}\sum_{i=1}^{N}\int_{\Omega} B_i(|\nabla u_m|)\,.\,\frac{l(|u_m|)}{\bar{a}}\,.\,\exp(G(|u_m|))\,(\,T_k(u_m)-T_k(u)\,)\,\eta_j(|u_m|)\,h_j(u_m)\;dx\\
&\quad+\sum_{i=1}^{N}\int_{\Omega} a_i^m(x,u_m,\nabla u_m)\,.\,\exp(G(|u_m|))\,\nabla((\,T_k(u_m)-T_k(u)))\,\eta_j(|u_m|)\,h_j(u_m)\;dx\\
&\quad+\sum_{i=1}^{N}\int_{\Omega} a_i^m(x,u_m,\nabla u_m)\,.\,\exp(G(|u_m|))\,(\,T_k(u_m)-T_k(u))\,\nabla\eta_j(|u_m|)\,h_j(u_m)\;dx\\
&\quad+\sum_{i=1}^{N}\int_{\Omega} a_i^m(x,u_m,\nabla u_m)\,.\,\exp(G(|u_m|)))\,(\,T_k(u_m)-T_k(u))\,\eta_j(|u_m|)\,\nabla h_j(u_m)\;dx\\
&\quad+\int_{\Omega} m\,.\,T_m(u_m-\psi)^-\,.\,sg_{\frac{1}{m}}(u_m)\,.\,\exp(G(|u_m|))\,(\,T_k(u_m)-T_k(u))\,\eta_j(|u_m|)\,h_j(u_m)\;dx\\
&\quad\le\sum_{i=1}^{N}\int_{\Omega} B_i(|\nabla u_m|)\,l(|u_m|)\,.\,\exp(G(|u_m|))\,(\,T_k(u_m)-T_k(u)\,)\,\eta_j(|u_m|)\,h_j(u_m)\;dx\\
&\qquad+\int_{\Omega}\big(\,h(x)+f^m(x)+\phi(x)\,.\,\frac{l(|u_m|)}{\bar{a}}\,\big)\,.\,\exp(G(|u_m|))\,(\,T_k(u_m)-T_k(u)\,)\,\eta_j(|u_m|)\,h_j(u_m)\;dx;
\end{aligned}
$$

we then obtain

$$
\begin{aligned}
&\sum_{i=1}^{N}\int_{\{\,|u_m|\le j\,\}} a_i^m(x,u_m,\nabla u_m)\,.\,\exp(G(|u_m|))\,(\,\nabla T_k(u_m)-\nabla T_k(u))\,\eta_j(|u_m|)\;dx\\
&\quad-\sum_{i=1}^{N}\int_{\{\,j\le|u_m|\le j+1\,\}} a_i^m(x,u_m,\nabla u_m)\,.\,\nabla u_m\,.\,(\,j+1+|u_m|\,)\,.\,\nabla T_k(u)\,\eta_j(|u_m|)\;\exp(G(|u_m|))\;dx\\
&\quad+\sum_{i=1}^{N}\int_{\Omega} a_i^m(x,u_m,\nabla u_m)\,.\,\exp(G(|u_m|))\,(\,T_k(u_m)-T_k(u))\,\nabla\eta_j(|u_m|)\,h_j(u_m)\;dx\\
&\quad+\sum_{i=1}^{N}\int_{\{\,j\le|u_m|\le j+1\,\}} a_i^m(x,u_m,\nabla u_m)\,.\,\nabla u_m\,.\,\exp(G(|u_m|))\,(\,T_k(u_m)-T_k(u))\,\eta_j(|u_m|)\;dx\\
&\quad+\int_{\Omega} m\,.\,T_m(u_m-\psi)^-\,.\,sg_{\frac{1}{m}}(u_m)\,.\,\exp(G(|u_m|))\,(\,T_k(u_m)-T_k(u))\,\eta_j(|u_m|)\,h_j(u_m)\;dx\\
&\quad\le\int_{\Omega}\big(\,h(x)+f^m(x)+\phi(x)\,.\,\frac{l(|u_m|)}{\bar{a}}\,\big)\,.\,\exp(G(|u_m|))\,(\,T_k(u_m)-T_k(u)\,)\,\eta_j(|u_m|)\,h_j(u_m)\;dx.
\end{aligned}
$$

By (2) we get

$$\begin{aligned}
&\sum_{i=1}^{N}\int_{\{\,|u_m|\le j\,\}} a_i^m(x,u_m,\nabla u_m)\,.\,\exp(G(|u_m|))\,(\,\nabla T_k(u_m)-\nabla T_k(u))\,\eta_j(|u_m|)\;dx\\
&\quad+\sum_{i=1}^{N}\int_\Omega a_i^m(x,u_m,\nabla u_m)\,.\,\exp(G(|u_m|))\,(\,T_k(u_m)-T_k(u))\,\nabla\eta_j(|u_m|)\,h_j(u_m)\;dx\\
&\quad+\int_\Omega m\,.\,T_m(u_m-\psi)^-\,.\,sg_{\frac{1}{m}}(u_m)\,.\,\exp(G(|u_m|))\,(\,T_k(u_m)-T_k(u))\,\eta_j(|u_m|)\,h_j(u_m)\;dx\\
&\quad\le\int_\Omega\Big(\,h(x)+f^m(x)+\phi(x)\,.\,\frac{l(|u_m|)}{\bar a}\,\Big)\,.\,\exp(G(|u_m|))\,(\,T_k(u_m)-T_k(u)\,)\,\eta_j(|u_m|)\,h_j(u_m)\;dx\\
&\quad\;-\sum_{i=1}^{N}\int_{\{\,j\le|u_m|\le j+1\,\}}\phi(x)\,.\,(\,j+1+|u_m|\,)\,.\,\nabla T_k(u)\;\exp(G(|u_m|))\,\eta_j(|u_m|)\;dx\\
&\quad\;+\sum_{i=1}^{N}\int_{\{\,j\le|u_m|\le j+1\,\}}\phi(x)\,.\,\exp(G(|u_m|))\,(\,T_k(u_m)-T_k(u))\,\eta_j(|u_m|)\;dx\\
&\quad\;+\bar a\sum_{i=1}^{N}\int_{\{\,j\le|u_m|\le j+1\,\}}B_i(|\nabla u_m|)\,(\,j+1+|u_m|\,)\,.\,\nabla T_k(u)\;\exp(G(|u_m|))\,\eta_j(|u_m|)\;dx\\
&\quad\;-\bar a\sum_{i=1}^{N}\int_{\{\,j\le|u_m|\le j+1\,\}}B_i(|\nabla u_m|)\;\exp(G(|u_m|))\,(\,T_k(u_m)-T_k(u)\,)\,\eta_j(|u_m|)\;dx.
\end{aligned}$$

According to (27), (29); and $T_k(u_m)\rightharpoonup T_k(u)$ weakly in $\mathring{W}_B^1(\Omega)$, $h_j\ge 0$, $\eta_j(|u_m|)\ge 0$, and $u_m\,(\,T_k(u_m)-T_k(u)\,)\ge 0$ and $\exp(G(\pm\infty))\le\exp\left(\frac{||\,l\,||_{L^1(\mathbb{R})}}{\bar a}\right)$, we deduce that

$$\sum_{i=1}^{N}\int_{\{\,|u_m|\le j\,\}} a_i^m(x,u_m,\nabla u_m)\,.\,\exp(G(|u_m|))\,(\,\nabla T_k(u_m)-\nabla T_k(u))\,\eta_j(|u_m|)\;dx\le C(\,k,j,m\,). \tag{49}$$

Then,

$$\begin{aligned}
&\sum_{i=1}^{N}\int_\Omega\big[\,a_i(x,T_k(u_m),\nabla T_k(u_m))\;-\;a_i(x,T_k(u_m),\nabla T_k(u))\,\big]\\
&\qquad\times\exp(G(|u_m|))\,(\,\nabla T_k(u_m)-\nabla T_k(u)\,)\,\eta_j(|u_m|)\;dx\\
&\quad\le-\sum_{i=1}^{N}\int_\Omega a_i(x,T_k(u_m),\nabla T_k(u))\,.\,\exp(G(|u_m|))\,|\,\nabla T_k(u_m)-\nabla T_k(u)\,|\,\eta_j(|u_m|)\;dx\\
&\quad\;-\sum_{i=1}^{N}\int_{\{\,|u_m|\le k\,\}}a_i(x,T_k(u_m),\nabla T_k(u))\,.\,\exp(G(|u_m|))\,\nabla T_k(u)\,\eta_j(|u_m|)\;dx\;+\;C(\,k,j,m\,).
\end{aligned}\tag{50}$$

By Lebesgue dominated convergence theorem, we have $T_k(u_m)\longrightarrow T_k(u)$ strongly in $\mathring{W}_{B,loc}^1(\Omega)$ and $\nabla T_k(u_m)\rightharpoonup\nabla T_k(u)$ weakly in $\mathring{W}_B^1(\Omega)$; then the terms on the right hand side of (50) go to zeros as k,j,m tend to infinity, which gives

$$\begin{aligned}
&\sum_{i=1}^{N}\int_\Omega\big[\,a_i(x,T_k(u_m),\nabla T_k(u_m))\;-\;a_i(x,T_k(u_m),\nabla T_k(u))\,\big]\\
&\qquad\times(\,\nabla T_k(u_m)-\nabla T_k(u)\,)\;\exp(G(|u_m|))\,\eta_j(|u_m|)\;dx\;\longrightarrow 0.
\end{aligned}\tag{51}$$

By Proposition 4 and the diagonal process, we deduce for $k\longrightarrow\infty$ that

$$B(|\nabla u_m|)\longrightarrow B(|\nabla u|)\;\text{ in }\;L^1(\Omega). \tag{52}$$

Hence, we obtain for a subsequence

$$\nabla u_m \longrightarrow \nabla u \text{ a.e. in } \Omega. \tag{53}$$

Step 5. The equi-integrability of $b_i^m(x, u_m, \nabla u_m)$:

In this step we will show that

$$b_i^m(x, u_m, \nabla u_m) \longrightarrow b_i(x, u, \nabla u). \tag{54}$$

Therefore, it is enough to show that $b_i^m(x, u_m, \nabla u_m)$ is uniformly equi-integrable. We take the following test function

$$v = u_m - \eta \exp(2\,G(|u_m|))\,\eta_j(|u_m|)\,T_1(u_m - T_j(u_m)).$$

We have

$$\begin{aligned}
&\sum_{i=1}^{N}\int_\Omega a_i^m(x,u_m,\nabla u_m)\,.\,\nabla(\exp(2\,G(|u_m|))\,\eta_j(|u_m|)\,T_1(u_m - T_j(u_m)))\,dx\\
&\quad+\sum_{i=1}^{N}\int_\Omega b_i^m(x,u_m,\nabla u_m)\,.\,\exp(2\,G(|u_m|))\,\eta_j(|u_m|)\,T_1(u_m - T_j(u_m))\,dx\\
&\quad+\int_\Omega m\,.\,T_m(u_m-\psi)^-\,.\,sg_{\frac{1}{m}}(u_m)\,.\,\exp(2\,G(|u_m|))\,\eta_j(|u_m|)\,T_1(u_m - T_j(u_m))\,dx\\
&\quad\le\int_\Omega f^m(x)\,.\,\exp(2\,G(|u_m|))\,\eta_j(|u_m|)\,T_1(u_m - T_j(u_m))\,dx;
\end{aligned}$$

then,

$$\begin{aligned}
&\sum_{i=1}^{N}\int_\Omega a_i^m(x,u_m,\nabla u_m)\,.\,\nabla u_m\,.\,\frac{l(|u_m|)}{\bar a}\,\exp(2\,G(|u_m|))\,\eta_j(|u_m|)\,T_1(u_m - T_j(u_m))\,dx\\
&\quad+\sum_{i=1}^{N}\int_\Omega a_i^m(x,u_m,\nabla u_m)\,.\,\exp(2\,G(|u_m|))\,\nabla(\eta_j(|u_m|))\,T_1(u_m - T_j(u_m))\,dx\\
&\quad+\sum_{i=1}^{N}\int_\Omega a_i^m(x,u_m,\nabla u_m)\,.\,\exp(2\,G(|u_m|))\,\eta_j(|u_m|)\,\nabla T_1(u_m - T_j(u_m))\,dx\\
&\quad+\int_\Omega m\,.\,T_m(u_m-\psi)^-\,.\,sg_{\frac{1}{m}}(u_m)\,.\,\exp(2\,G(|u_m|))\,\eta_j(|u_m|)\,T_1(u_m - T_j(u_m))\,dx\\
&\quad\le\sum_{i=1}^{N}\int_\Omega |\,b_i^m(x,u_m,\nabla u_m)\,|\,.\,\exp(2\,G(|u_m|))\,\eta_j(|u_m|)\,T_1(u_m - T_j(u_m))\,dx\\
&\qquad+\int_\Omega f^m(x)\,.\,\exp(2\,G(|u_m|))\,\eta_j(|u_m|)\,T_1(u_m - T_j(u_m))\,dx.
\end{aligned}$$

By (2) and (4) we get

$$\begin{aligned}
&\sum_{i=1}^{N}\int_\Omega a_i^m(x,u_m,\nabla u_m)\,.\,\exp(2\,G(|u_m|))\,\nabla(\eta_j(|u_m|))\,T_1(u_m - T_j(u_m))\,dx\\
&\quad+\sum_{i=1}^{N}\int_\Omega a_i^m(x,u_m,\nabla u_m)\,.\,\exp(2\,G(|u_m|))\,\eta_j(|u_m|)\,\nabla T_1(u_m - T_j(u_m))\,dx\\
&\quad+\int_\Omega m\,.\,T_m(u_m-\psi)^-\,.\,sg_{\frac{1}{m}}(u_m)\,.\,\exp(2\,G(|u_m|))\,\eta_j(|u_m|)\,T_1(u_m - T_j(u_m))\,dx\\
&\quad\le\int_\Omega \left(h(x)+f^m(x)+\phi(x)\,.\,\frac{l(|u_m|)}{\bar a}\right).\,\exp(2\,G(|u_m|))\,\eta_j(|u_m|)\,T_1(u_m - T_j(u_m))\,dx;
\end{aligned}$$

we deduce that

$$\begin{aligned}&\sum_{i=1}^{N}\int_{\Omega} a_i^m(x,u_m,\nabla u_m)\,.\,\exp(2\,G(|u_m|))\,\nabla(\eta_j(|u_m|))\,T_1(u_m-T_j(u_m))\;dx\\&\quad+\sum_{i=1}^{N}\int_{\{j\le|u_m|\le j+1\}} a_i^m(x,u_m,\nabla u_m)\,.\,\nabla u_m\,\exp(2\,G(|u_m|))\,\eta_j(|u_m|)\;dx\\&\quad+\int_{\Omega} m\,.\,T_m(u_m-\psi)^-\,.\,sg_{\frac{1}{m}}(u_m)\,.\,\exp(2\,G(|u_m|))\,\eta_j(|u_m|)\,T_1(u_m-T_j(u_m))\;dx\\&\quad\le\int_{\Omega}\big(h(x)+f^m(x)+\phi(x)\,.\,\frac{l(|u_m|)}{\bar a}\,\big)\,.\,\exp(2\,G(|u_m|))\,\eta_j(|u_m|)\,T_1(u_m-T_j(u_m))\;dx.\end{aligned}$$

Since $a_i^m(x,u_m,\nabla u_m)$ is bounded in $\mathring{W}_B^1(\Omega)$, and $\eta_j(|u_m|)\ge 0$ then by (27), (29) we obtain

$$\begin{aligned}&\sum_{i=1}^{N}\int_{\{j+1<|u_m|\}} B_i(|\nabla u_m|)\;dx\\&\quad\le\exp\left(2\,\frac{||\,l\,||_{L^1(\mathbb{R})}}{\bar a}\right).\int_{\{j<|u_m|\}}\big[\,|C(x)|+\phi(x)+h(x)+f^m(x)+\phi(x)\,.\,\frac{l(|u_m|)}{\bar a}\,\big]\;dx.\end{aligned}$$

Thus, $\forall\epsilon>0\;\exists\, j(\epsilon)>0$ such that

$$\sum_{i=1}^{N}\int_{\{j+1<|u_m|\}} B_i(|\nabla u_m|)\;dx\;\le\;\frac{\epsilon}{2}\qquad\forall j\;>\;j(\epsilon). \tag{55}$$

Let $\mathring{V}(\Omega(R))$ be an arbitrary bounded subset for Ω; then, for any measurable set $E\subset\mathring{V}(\Omega(R))$ we have

$$\begin{aligned}&\sum_{i=1}^{N}\int_{E} B_i(|\nabla u_m|)\;dx\\&\quad\le\sum_{i=1}^{N}\int_{E} B_i(|\nabla T_k(u_m)|)\;dx\;+\;\sum_{i=1}^{N}\int_{\{j+1<|u_m|\}} B_i(|\nabla u_m|)\;dx.\end{aligned}\tag{56}$$

We conclude that $\forall E\subset\mathring{V}(\Omega(R))$ with $\text{meas}(E)<\beta(\epsilon)$, and $T_k(u_m)\longrightarrow T_k(u)$ in $\mathring{W}_B^1(\Omega)$,

$$\sum_{i=1}^{N}\int_{E} B_i(|\nabla T_k(u_m)|)\;dx\;\le\;\frac{\epsilon}{2}. \tag{57}$$

Finally, by combining the last formulas we obtain

$$\sum_{i=1}^{N}\int_{E} B_i(|\nabla u_m|)\;dx\;\le\;\epsilon\qquad\forall E\subset\mathring{V}(\Omega(R))\text{ such that }\text{meas}(E)<\beta(\epsilon),$$

giving the assumed results.

Step 6. Passing to the limit:

Let $\varphi\in\mathring{W}_B^1(\Omega)\cap L^\infty(\Omega)$; we take the following test function:

$$v=u_m-\psi_k\,T_k(u_m-\varphi),\;\;\psi_K\in\mathcal{D}(\Omega),$$

such that

$$\psi_k(x) = \begin{cases} 1 & \text{for } \Omega(R) \\ 0 & \text{for } \Omega(R+1)\backslash\Omega(R) \end{cases}$$

and $|u_m| - ||\varphi||_\infty < |u_m - \varphi| \leq j$. Then, by $\{|u_m - \varphi| \leq j\} \subset \{|u_m| \leq j + ||\varphi||_\infty\}$ we obtain

$$\begin{aligned} &\sum_{i=1}^{N} \int_{\Omega(R+1)} a_i(x, T_m(u_m), \nabla u_m)\, \psi_k\, \nabla T_k(u_m - \varphi)\; dx \\ &\quad + \sum_{i=1}^{N} \int_{\Omega(R+1)} a_i(x, T_m(u_m), \nabla u_m)\, T_k(u_m - \varphi)\, \nabla \psi_k\; dx \\ &\quad + \sum_{i=1}^{N} \int_{\Omega(R+1)} b_i^m(x, u_m, \nabla u_m)\, \psi_k\, T_k(u_m - \varphi)\; dx \\ &\quad + \int_{\Omega(R+1)} m\,.\, T_m(u_m - \psi)^-\,.\, sg_{\frac{1}{m}}(u_m)\,.\, \psi_k\, T_k(u_m - \varphi)\; dx \\ &\quad \leq \int_{\Omega(R+1)} f^m(x)\, \psi_k\, T_k(u_m - \varphi)\; dx, \end{aligned}$$

which implies that

$$\begin{aligned} &\sum_{i=1}^{N} \int_{\Omega(R+1)} a_i(x, T_m(u_m), \nabla u_m)\, \psi_k\, \nabla T_k(u_m - \varphi)\; dx \\ &\quad = \sum_{i=1}^{N} \int_{\Omega(R+1)} a_i(x, T_{j+||\varphi||_\infty}(u_m), \nabla T_{j+||\varphi||_\infty} u_m)\, \psi_k\, \nabla T_k(u_m - \varphi)\; dx \\ &\quad = \sum_{i=1}^{N} \int_{\Omega(R+1)} \left[a_i(x, T_{j+||\varphi||_\infty}(u_m), \nabla T_{j+||\varphi||_\infty}(u_m)) - a_i(x, T_{j+||\varphi||_\infty}(u_m), \nabla \varphi) \right] \\ &\qquad\qquad \times\; \nabla T_{j+||\varphi||_\infty}(u_m - \varphi)\,.\,\chi_{\{\,|u_m - \varphi| < j\,\}}\; dx \\ &\quad + \sum_{i=1}^{N} \int_{\Omega(R+1)} a_i(x, T_{j+||\varphi||_\infty}(u_m), \nabla \varphi)\, \nabla T_{j+||\varphi||_\infty}(u_m - \varphi)\,.\,\chi_{\{\,|u_m - \varphi| < j\,\}}\; dx. \end{aligned}$$

By Fatou's Lemma we get

$$\begin{aligned} &\liminf_{m\to\infty} \sum_{i=1}^{N} \int_{\Omega(R+1)} a_i(x, T_{j+||\varphi||_\infty}(u_m), \nabla \varphi)\, \nabla T_{j+||\varphi||_\infty}(u_m - \varphi)\,.\,\chi_{\{\,|u_m - \varphi| < j\,\}}\; dx \\ &\qquad = \sum_{i=1}^{N} \int_{\Omega(R+1)} a_i(x, T_{j+||\varphi||_\infty}(u), \nabla \varphi)\, \nabla T_{j+||\varphi||_\infty}(u - \varphi)\,.\,\chi_{\{\,|u - \varphi| < j\,\}}\; dx, \end{aligned}$$

and the fact that

$$a_i(x, T_{j+||\varphi||_\infty}(u_m), \nabla T_{j+||\varphi||_\infty}(u_m)) \rightharpoonup a_i(x, T_{j+||\varphi||_\infty}(u), \nabla T_{j+||\varphi||_\infty}(u)) \tag{58}$$

weakly in $\mathring{W}^1_B(\Omega)$. Additionally, since $\psi_k\, T_k(u_m - \varphi) \rightharpoonup \psi_k\, T_k(u - \varphi)$ weakly in $\mathring{W}^1_B(\Omega)$, and by (53) we obtain

$$\sum_{i=1}^{N} \int_{\Omega(R+1)} b_i^m(x, u_m, \nabla u_m)\, \psi_k\, T_k(u_m - \varphi)\; dx \longrightarrow \sum_{i=1}^{N} \int_{\Omega(R+1)} b_i(x, u, \nabla u)\, \psi_k\, T_k(u - \varphi)\; dx,$$

and

$$\int_{\Omega(R+1)} f^m(x)\, \psi_k\, T_k(u_m - \varphi)\ dx \longrightarrow \int_{\Omega(R+1)} f(x)\, \psi_k\, T_k(u - \varphi)\ dx,$$

and

$$\int_{\Omega(R+1)} m\,.\,T_m(u_m - \psi)^-\,.\,sg_{\frac{1}{m}}(u_m)\,.\,\psi_k\, T_k(u_m - \varphi)\ dx \longrightarrow \int_{\Omega(R+1)} m\,.\,T_m(u - \psi)^-\,.\,sg_{\frac{1}{m}}(u)\,.\,\psi_k\, T_k(u - \varphi)\ dx, \tag{59}$$

so we get

$$\begin{aligned}
&\sum_{i=1}^{N} \int_{\Omega(R+1)} a_i(x,u,\nabla u)\, \psi_k\, \nabla T_k(u - \varphi)\ dx + \sum_{i=1}^{N} \int_{\Omega(R+1)} a_i(x,u,\nabla u)\, T_k(u - \varphi)\, \nabla \psi_k\ dx \\
&\quad + \sum_{i=1}^{N} \int_{\Omega(R+1)} b_i(x,u,\nabla u)\, \psi_k\, T_k(u - \varphi)\ dx \\
&\quad + \int_{\Omega(R+1)} m\,.\,T_m(u - \psi)^-\,.\,sg_{\frac{1}{m}}(u)\,.\,\psi_k\, T_k(u - \varphi)\ dx \\
&\quad \le \int_{\Omega(R+1)} f(x)\, \psi_k\, T_k(u - \varphi)\ dx,
\end{aligned}$$

now passing to the limit to infinity in k, we obtain the entropy solution of the problem.

4. Uniqueness of the Entropy Solution

Theorem 3. *Suppose that conditions (1)–(3) are true, and $b_i(x,u,\nabla u) : \Omega \times \mathbb{R} \times \mathbb{R}^N \longrightarrow \mathbb{R}$ are strictly monotonic operators, at least for a broad class of lower order terms. Then, the problem $(\mathcal{P})$ has a unique solution.*

Proof. Let u and $\bar{u}$ belong to $K_\psi \cap L^\infty(\Omega)$ being two solutions of problem $(\mathcal{P})$ with $u \neq \bar{u}$. In accordance with Definition 6, we obtain

$$\begin{aligned}
&\sum_{i=1}^{N} \int_\Omega a_i(x,u,\nabla u)\,.\,\nabla(u - v)\ dx + \sum_{i=1}^{N} b_i(x,u,\nabla u)\,.\,(u - v)\ dx \\
&\quad + \int_\Omega m\,.\,T_m(u - \psi)^-\,.\,sg_m(u)\,.\,(u - v)\ dx \\
&\quad \le \int_\Omega f(x)\,.\,(u - v)\ dx
\end{aligned} \tag{60}$$

and

$$\begin{aligned}
&\sum_{i=1}^{N} \int_\Omega a_i(x,\bar{u},\nabla \bar{u})\,.\,\nabla(\bar{u} - v)\ dx + \sum_{i=1}^{N} b_i(x,\bar{u},\nabla \bar{u})\,.\,(\bar{u} - v)\ dx \\
&\quad + \int_\Omega m\,.\,T_m(\bar{u} - \psi)^-\,.\,sg_m(\bar{u})\,.\,(\bar{u} - v)\ dx \\
&\quad \le \int_\Omega f(x)\,.\,(\bar{u} - v)\ dx
\end{aligned} \tag{61}$$

Denote $v = u - \mu(x)(u - \bar{u})(x)$ and $v = \bar{u} - \mu(x)(u - \bar{u})(x)$ with

$$\mu(x) = \begin{cases} 0 & \text{if } \ x \ge k\,, \\ k - \frac{|\,x\,|^2}{k} & \text{if } \ |\,x\,| < k\,, \\ 0 & \text{if } \ x \le -k\,, \end{cases}$$

as test functions in (60) and (61) respectively. Using (1), (27), (29) and the condition of a strictly monotonic for the operator $b_i(x, u, \nabla u)$, we substract the equations to obtain

$$\sum_{i=1}^{N} \int_{\Omega} \left[\, a_i(x,u,\nabla u) - a_i(x,\bar{u},\nabla \bar{u}) \,\right] . (u-\bar{u}) \nabla \mu(x)\ dx \leq 0,$$

According to (6), we obtain

$$\begin{aligned} &\sum_{i=1}^{N} \int_{\Omega} \bar{B}_i\big(\, a_i(x,u,\nabla u) - a_i(x,\bar{u},\nabla \bar{u}) \,\big)\ dx + \sum_{i=1}^{N} \int_{\Omega} B_i\big(\, (u-\bar{u}) \,. \nabla \mu(x) \,\big)\ dx \\ &\quad \leq \sum_{i=1}^{N} \int_{\Omega} \bar{B}_i\big(\, a_i(x,u,\nabla u) - a_i(x,\bar{u},\nabla \bar{u}) \,\big)\ dx + 2 \sum_{i=1}^{N} \int_{\Omega} B_i(u-\bar{u})\ dx \\ &\quad \leq 0. \end{aligned} \tag{62}$$

Since the N-functions $\bar{B}_i$ verified the same conditions and properties of B_i; then by (10), we have

$$\begin{aligned} &\sum_{i=1}^{N} \int_{\Omega} \bar{B}_i\big(\, a_i(x,u,\nabla u) - a_i(x,\bar{u},\nabla \bar{u}) \,\big)\ dx \\ &\quad \leq c \sum_{i=1}^{N} \int_{\Omega} |\, \bar{B}_i(\, a_i(x,u,\nabla u) \,) \,|\ dx - c \sum_{i=1}^{N} \int_{\Omega} |\, \bar{B}_i(\, a_i(x,\bar{u},\nabla \bar{u}) \,) \,|\ dx; \end{aligned}$$

according to (3), we obtain

$$\begin{aligned} &\sum_{i=1}^{N} \int_{\Omega} \bar{B}_i\big(\, a_i(x,u,\nabla u) - a_i(x,\bar{u},\nabla \bar{u}) \,\big)\ dx \\ &\quad \leq \tilde{a}c \sum_{i=1}^{N} \int_{\Omega} B_i(\nabla(u-\bar{u})\,)\ dx \\ &\quad \leq \tilde{a}c \,||\, B(u-\bar{u}) \,||_{1,\Omega}. \end{aligned} \tag{63}$$

Combined with (62) and (63) we get

$$0 \leq (\, \tilde{a}c + 2 \,) \,.\, ||\, B(u-\bar{u}) \,||_{1,\Omega} \leq 0.$$

Finally, $||\, B(u-\bar{u}) \,||_{1,\Omega} = 0$; therefore, $u = \bar{u}$ a.e. in Ω. □

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Appendix A

Let S be the operator defined by

$$\begin{aligned} S(u) = &\sum_{i=1}^{N} \int_{\Omega} a_i^m(x,u,\nabla u)\ dx + \sum_{i=1}^{N} \int_{\Omega} b_i^m(x,u,\nabla u)\ dx + \int_{\Omega} m \,.\, T_m(u-\psi)^- \,.\, sg_{\frac{1}{m}}(u)\ dx \\ &- \int_{\Omega} f^m(x)\ dx, \end{aligned}$$

and for any $v \in \mathring{W}^1_B(\Omega)$, $u \in \mathring{W}^1_{B,loc}(\overline{\Omega(R)})$ we have

$$\langle S(u), v \rangle = \sum_{i=1}^{N} \int_{\Omega(R)} a_i^m(x,u,\nabla u) \,.\, \nabla v \; dx + \sum_{i=1}^{N} \int_{\Omega(R)} b_i^m(x,u,\nabla u) \,.\, v \; dx \\ + \int_{\Omega(R)} m \,.\, T_m(u-\psi)^- \,.\, sg_{\frac{1}{m}}(u) \,.\, v \; dx - \int_{\Omega(R)} f^m(x) \,.\, v \; dx. \tag{A1}$$

In order to show the result of the Theorem 1, it is sufficient to show that operator S is bounded, coercive and pseudo-monotonic.

Let us start by demonstrating that S is bounded. Additionally, according to (A1), (15), (27) and (29) we obtain

$$|\langle S(u), v \rangle| \leq 2 \,||\, a^m(x,u,\nabla u) \,||_{\bar{B},\Omega(R)} \,.\, ||\, v \,||_{\mathring{W}^1_B(\Omega)} + 2 \,||\, b^m(x,u,\nabla u) \,||_{B,\Omega(R)} \,.\, ||\, v \,||_{\mathring{W}^1_B(\Omega)} \\ - c_0 \,.\, ||\, v \,||_{\mathring{W}^1_B(\Omega)}.$$

or

$$||\, a^m(x,u,\nabla u) \,||_{\bar{B},\Omega(R)} \leq \sum_{i=1}^{N} \int_{\Omega(R)} \bar{B}_i(\, a_i^m(x,u,\nabla u)\,) \; dx + N$$

and by diagonal process we obtain

$$||\, a^m(x,u,\nabla u) \,||_{\bar{B},\Omega(R)} \leq \bar{a} \,||\, u \,||_{\mathring{W}^1_B(\Omega)} + ||\, \varphi \,||_1 + N$$

and by (23) we deduce that

$$b^m(x,u,\nabla u) \text{ bounded in } L_B(\Omega)$$

if u with bounded support, and if support $u = \overline{\Omega(R)}$, then (A1) is bounded.

Next, we will move to proving that S is coercive. By (27), (29) for any $u \in \mathring{W}^1_B(\Omega)$

$$\langle S(u), u \rangle \geq \bar{a} \sum_{i=1}^{N} \int_{\Omega} B_i\left(\left| \frac{\partial u}{\partial x_i} \right| \right) dx - \int_{\Omega} \varphi(x) \; dx + \sum_{i=1}^{N} \int_{\Omega} b_i^m(x,u,\nabla u) \,.\, u \; dx - \int_{\Omega} f^m(x) \,.\, u \; dx,$$

so

$$\frac{\langle S(u), u \rangle}{||\, u \,||_{\mathring{W}^1_B(\Omega)}} \geq \frac{1}{||\, u \,||_{\mathring{W}^1_B(\Omega)}} \left[\bar{a} \sum_{i=1}^{N} \int_{\Omega} B_i\left(\left| \frac{\partial u}{\partial x_i} \right| \right) dx - c_1 - c_0 \right] \\ + \frac{1}{||\, u \,||_{\mathring{W}^1_B(\Omega)}} \sum_{i=1}^{N} \int_{\Omega} b_i^m(x,u,\nabla u) \,.\, u \; dx,$$

using (23), we obtain

$$\frac{1}{||\, u \,||_{\mathring{W}^1_B(\Omega)}} \sum_{i=1}^{N} \int_{\Omega} b_i^m(x,u,\nabla u) \,.\, u \; dx \geq -2\, c(m).$$

Thus,

$$\frac{\langle S(u), u \rangle}{||\, u \,||_{\mathring{W}^1_B(\Omega)}} \geq \frac{1}{||\, u \,||_{\mathring{W}^1_B(\Omega)}} \left[\bar{a} \sum_{i=1}^{N} \int_{\Omega} B_i\left(\left| \frac{\partial u}{\partial x_i} \right| \right) dx - c_1 - c_0 \right] - 2\, c(m)$$

according to (22) we have for all $k > 0$, $\exists \alpha_0 > 0$ such that

$$b_i(\,|\, u_{x_i} \,|\,) > k\, b_i\left(\frac{|\, u_{x_i} \,|}{||\, u_{x_i} \,||_{B_i,\Omega}} \right) \qquad i = 1, \cdots, N;$$

we take $||\, u_{x_i}\, ||_{B_i,\Omega} > \alpha_0,\ i = 1, \cdots, N$. Additionally, since Ω is unbounded domain, then we can assume that $||\, u^j\, ||_{\mathring{W}^1_B(\Omega)} \longrightarrow \infty$ as $j \longrightarrow \infty$. We suppose

$$||\, u^j_{x_1}\, ||_{B_1,\Omega} + \cdots + ||\, u^j_{x_N}\, ||_{B_N,\Omega} \geq N\, \alpha_0,$$

according to (9) we get

$$|\, u^j\, |\, b(\, |\, u^j\, |\,) < c\, B(\, u^j\,),$$

so,

$$\begin{aligned} \frac{\langle\, S(u^j), u^j\, \rangle}{||\, u^j\, ||_{\mathring{W}^1_B(\Omega)}} &\geq \frac{\bar{a}}{N\, \alpha_0} \sum_{i=1}^{N} \int_\Omega B_i\Big(\Big|\, \frac{\partial u^j}{\partial x_i}\, \Big| \Big)\, dx - \frac{c_2}{N\, \alpha_0} - 2\, c(m) \\ &\geq \frac{\bar{a}}{c\, N\, ||\, u^j_{x_i}\, ||_{B_i}} \sum_{i=1}^{N} \int_\Omega |\, u^j_{x_i}\, |\, .\, b_i(\, |\, u^j_{x_i}\, |\,)\, dx - \frac{c_2}{N\, \alpha_0} - 2\, c(m) \\ &\geq \frac{\bar{a}\, k}{c\, N} \sum_{i=1}^{N} \int_\Omega B_i\Big(\frac{|\, u^j_{x_i}\, |}{||\, u^j_{x_i}\, ||_{B_i,\Omega}} \Big)\, dx - \frac{c_2}{N\, \alpha_0} - 2\, c(m), \end{aligned}$$

with $c_2 = c_0 + c_1$. Now, by the Luxemburg norm, we have

$$||\, u^j\, ||_B = \inf\{\, k > 0\, /\, \int_\Omega B\Big(\frac{u^j(x)}{k} \Big)\, dx \leq 1\, \}$$

then

$$\sum_{i=1}^{N} \int_\Omega B_i\Big(\frac{|\, u^j_{x_i}\, |}{||\, u^j_{x_i}\, ||_{B_i,\Omega}} \Big)\, dx \geq \sum_{i=1}^{N} ||\, u^j_{x_i}\, ||_{B_i}.$$

Hence,

$$\frac{\langle\, S(u^j), u^j\, \rangle}{||\, u^j\, ||_{\mathring{W}^1_B(\Omega)}} \geq \frac{\bar{a}\, k}{c\, N}\, ||\, u^j\, ||_{\mathring{W}^1_B(\Omega)} - \frac{c_2}{N\, \alpha_0} - 2\, c(m) \longrightarrow \infty \text{ as } ||\, u^j\, ||_{\mathring{W}^1_B(\Omega)} \longrightarrow \infty$$

which gives the coercivity of the operator S.

Finally, we will end it by the demonstration of pseudo-monotonic of S. Following up this assumption, since the space $\mathring{W}^1_B(\Omega)$ is separable, then $\exists (u^j) \in C^\infty_0(\Omega)$ such that

$$u^j \rightharpoonup u \text{ in } \mathring{W}^1_B(\Omega), \tag{A2}$$

and

$$S(u^j) \rightharpoonup y \text{ in } (\, \mathring{W}^1_B(\Omega)\,)^{'}; \tag{A3}$$

according to (A2), we have for all subsequences denoted again by u^j,

$$||\, u^j\, ||_{\mathring{W}^1_B(\Omega)} \leq c_2,\ j \in \mathbb{N}$$

$(u^j)_{j\in\mathbb{N}}$ is bounded in $\mathring{W}^1_B(\Omega)$, and since $\mathring{W}^1_B(\Omega)$ is continuously and compactly injected into $L_B(\Omega)$

$$u^j \rightharpoonup u \text{ in } L_B(\Omega),$$

$$u^j \longrightarrow u \text{ a.e. in } \Omega,\ j \in \mathbb{N},$$

and according to (53), we have

$$a^m_i(x, u^j, \nabla u^j) \longrightarrow a^m_i(x, u, \nabla u) \text{ a.e. in } \Omega,\ j \in \mathbb{N}$$

and

$$b_i^m(x, u^j, \nabla u^j) \longrightarrow b_i^m(x, u, \nabla u) \text{ a.e. in } \Omega,\ j \in \mathbb{N}$$

and

$$m \,.\, T_m(u^j - \psi)^- \,.\, sg_{\frac{1}{m}}(u^j) \longrightarrow m \,.\, T_m(u - \psi)^- \,.\, sg_{\frac{1}{m}}(u) \text{ a.e. in } \Omega,\ j \in \mathbb{N}$$

from (A2) and (A3), $\exists \tilde{a}^m \in L_{\bar{B}}(\Omega)$ such that

$$a_i^m(x, u^j, \nabla u^j) \rightharpoonup \tilde{a}^m,\ j \in \mathbb{N} \tag{A4}$$

and $\exists \tilde{b}^m \in L_B(\Omega)$ such that

$$b_i^m(x, u^j, \nabla u^j) \rightharpoonup \tilde{b}^m,\ j \in \mathbb{N}; \tag{A5}$$

by (27) and (29) it is clear that for any $v \in \mathring{W}_B^1(\Omega)$, we get

$$\begin{aligned} \langle y, v \rangle &= \lim_{j\to\infty} \sum_{i=1}^{N} \int_\Omega a_i^m(x, u^j, \nabla u^j) \,.\, \nabla v \, dx + \lim_{j\to\infty} \sum_{i=1}^{N} \int_\Omega b_i^m(x, u^j, \nabla u^j) \,.\, v \, dx \\ &= \int_\Omega \tilde{a}^m \,.\, \nabla v \, dx + \int_\Omega \tilde{b}^m \,.\, v \, dx \end{aligned} \tag{A6}$$

whereof

$$\begin{aligned} \limsup_{j\to\infty} < S(u^j), u^j > &= \limsup_{j\to\infty} \{\sum_{i=1}^{N} \int_\Omega a_i^m(x, u^j, \nabla u^j) \nabla u^j \, dx \\ &+ \lim_{j\to\infty} \sum_{i=1}^{N} \int_\Omega b_i^m(x, u^j, \nabla u^j) \, u^j \, dx\} \le \int_\Omega \tilde{a}^m \nabla u^j \, dx + \int_\Omega \tilde{b}^m \, u^j \, dx, \end{aligned} \tag{A7}$$

by (A5), we have

$$\int_\Omega b^m(x, u^j, \nabla u^j) \, u^j \, dx \longrightarrow \int_\Omega \tilde{b}^m \, u \, dx; \tag{A8}$$

consequently,

$$\limsup_{j\to\infty} \sum_{i=1}^{N} \int_\Omega a_i^m(x, u^j, \nabla u^j) \nabla u^j \, dx \le \int_\Omega \tilde{a}^m \nabla u^j \, dx. \tag{A9}$$

On the other hand, we have by the condition of monotony,

$$\sum_{i=1}^{N} (\, a_i^m(x, u^j, \nabla u^j \,) - a_i^m(x, u^j, \nabla u \,) \,) \,.\, \nabla(u^j - u) \ge 0,$$

which implies

$$\sum_{i=1}^{N} (\, a_i(x, T_m(u^j), \nabla u^j) - a_i(x, T^j(u^j), \nabla u) \,) \,.\, \nabla(u^j - u) \ge 0, \tag{A10}$$

then;

$$\sum_{i=1}^{N} a_i(x, T_m(u^j), \nabla u^j) \,.\, \nabla u^j \ge \sum_{i=1}^{N} a_i(x, T_m(u^j), \nabla u) \,.\, \nabla(u^j - u) + \sum_{i=1}^{N} a_i(x, T_m(u^j), \nabla u^j) \,.\, \nabla u,$$

and by Step 3, we get

$$\sum_{i=1}^{N} a_i(x, T_m(u^j), \nabla u) \longrightarrow \sum_{i=1}^{N} a_i(x, T_m(u), \nabla u) \quad \text{in } L_{\bar{B}}(\Omega);$$

according to (A4), we obtain

$$\liminf_{j\to\infty}\sum_{i=1}^{N}\int_{\Omega}a_i^m(x,u^j,\nabla u^j)\,.\,\nabla u^j\;dx\;\geq\;\int_{\Omega}\tilde{a}^m\,.\,\nabla u^j\;dx; \tag{A11}$$

therefore, from (A9), we have

$$\lim_{j\to\infty}\sum_{i=1}^{N}\int_{\Omega}a_i^m(x,u^j,\nabla u^j)\,.\,\nabla u^j\;dx\;=\;\int_{\Omega}\tilde{a}^m\,.\,\nabla u^j\;dx; \tag{A12}$$

according to (A6), (A8) and (A11) we get

$$< S(u^j), u^j > \longrightarrow < y, u > \quad \text{as} \quad j\to\infty.$$

Hence, from (A12), and (53) we obtain

$$\lim_{j\to\infty}\sum_{i=1}^{N}\int_{\Omega}(\,a_i^m(x,u^j,\nabla u^j\,)\;-\;a_i^m(x,u^j,\nabla u\,)\,)\,.\,\nabla(u^j\;-\;u)\;dx\;=\;0.$$

By (A6) we can conclude that

$$<y\,,\,u>=<S(u)\,,\,u> \qquad \forall\, u\in \mathring{W}_B^1(\Omega).$$

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