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Convergence of Weak*-Scalarly Integrable Functions

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Abstract: Let $(\Omega, \mathcal{F}, \mu)$ be a complete probability space, E a separable Banach space and E' the topological dual vector space of E. We present some compactness results in $L^1_{E'}[E]$, the Banach space of weak*-scalarly integrable E'-valued functions. As well we extend the classical theorem of Komlós to the bounded sequences in $L^1_{E'}[E]$.

Keywords: weak*-scalarly integrable functions; compactness; komlós theorem

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1. Introduction

In their 2001 paper, Benabdellah and Castaing [1] established that the Ülger–Diestel–Ruess–Shachermayer characterization for weak compactness in $L^1_E(\mu)$ can be extended to $L^1_{E'}[E]$. In addition, they gave several results on weak compactness and conditionally weak compactness in $L^1_{E'}[E]$. These results are not standard and rely on a Talagrand decomposition type theorems for bounded sequences in $L^1_{E'}[E]$. Moreover, this paper paved the way for many researchers to exploit and establish further interesting results in this space (see [2–4]).

In this paper, we aim to present some compactness results in $L^1_{E'}[E]$ and a Komlós theorem in $L^1_{E'}[E]$. More precisely, we will give in the first part a decomposition theorem for a bounded sequence in $L^1_{E'}[E]$ (Theorem 2 (i)) and a Komlós-type result for the weak* convergence in E' (Theorem 2 (ii)). This will allow us to state a criterion for the $\sigma(L^1_{E'}[E], L^\infty_E(\mu))$ compactness in $L^1_{E'}[E]$ (Theorem 3). In the second part we give a result on weak compactness in $L^1_{E'}[E]$ (Theorem 4 (jj)) in terms of a Komlós theorem for the weak convergence in E' (Theorem 4 (j)). Corollary 1 provides a compactness criterion in $L^1_{E'}[E]$, which generalizes Proposition 5.1 in [1]. In this paper, we have established a Komlós theorem in $L^1_{E'}[E]$ and used it to give some weak convergence results. Other works have followed a similar approch in different function spaces, such as the space of Bochner integrable functions and the space of Pettis integrable functions (see [5–7]).

2. Notations and Preliminaries

Throughout this paper the triple $(\Omega, \mathcal{F}, \mu)$ is a complete probability space, E is a separable Banach space and E' is its topological dual. The weak topology $\sigma(E', E'')$ (resp. the weak* topology $\sigma(E', E)$) on E' will be referred to by the symbol "w" (resp. w*). A mapping $f : \Omega \to E'$ is w*-measurable, if for any $x \in E$, the function $\langle f, x\rangle : \omega \mapsto \langle f(\omega), x\rangle$ is $\mathcal{F}$-measurable. Two w*-measurable mappings f and g are said to be equivalent (shortly $f \equiv g(w^*)$) iff $\langle f, x\rangle = \langle g, x\rangle$ μ-*a.e* $\forall x \in E$. Let $L^1_E(\mu)$ denotes the set of all (equivalence classes of) Bochner integrable E-valued functions [8], recall that (see [9]) the dual of $L^1_E(\mu)$ is the (quotient) space $L^\infty_{E'}[E]$ of w*-measurable bounded functions from Ω into E'. Now, according to [4], the set $L^1_{E'}(\Omega, \mathcal{F}, \mu, [E])$, in short $L^1_{E'}[E]$, denotes the (quotient) space of all w*-measurable mappings $f : \Omega \to E'$, such that $\omega \mapsto \langle f(\omega), x\rangle$ is integrable $\forall x \in E$ and $\|f(.)\|_{E'}$ belongs to $L^1_{\mathbb{R}}(\mu)$, and the mapping

$$\overline{N}_1(f) = \int_\Omega \|f\|\,d\mu, \qquad f \in L^1_{E'}[E]$$

defines a norm in $L^1_{E'}[E]$. Furthermore, the set $L^\infty_E(\mu)$ of all (equivalence classes of) μ-measurable essentially bounded functions with value in E is included in the topological dual of $L^1_{E'}[E]$ and the mapping $f \mapsto \overline{N}_1(f)$ is lower semicontinuous on $L^1_{E'}[E]$ for the topology $\sigma(L^1_{E'}[E], L^\infty_E(\mu))$.

In addition, recall that $(L^1_{E'}[E], \overline{N}_1)$ is a Banach space ([1], Proposition 3.4) and that a subset $\mathcal{K}$ of $L^1_{E'}[E]$ is uniformly integrable (briefly UI) if the set $\{\|f\|; f \in \mathcal{K}\}$ is UI in $L^1_{\mathbb{R}}(\mu)$ ([1], Definition 4.2). A subset $\mathcal{K}$ of $L^1_{\mathbb{R}}(\mu)$ is UI if

$$\lim_{t\to\infty} \sup_{f\in\mathcal{K}} \int_{\{|f|\geq t\}} |f|\,d\mu = 0.$$

Note that every UI subset of $L^1_{E'}[E]$ is $\overline{N}_1$-bounded.

Finally let us recall the notion of the K-convergence [5]. Let $(f_n)_{n\in\mathbb{N}}$ a sequence from Ω to E' and F be a subset of E''. We say that $(f_n)_{n\in\mathbb{N}}$ is $\sigma(E', F)$-K converge almost everywhere on Ω to a function f if for every subsequence $(f'_n)_{n\in\mathbb{N}}$ of $(f_n)_{n\in\mathbb{N}}$ there exists a null set $\mathcal{N} \in \mathcal{F}$, such that for every $\omega \in \Omega\backslash\mathcal{N}$

$$\forall\, x \in F, \qquad \langle x, \frac{1}{n}\sum_{i=1}^n f'_i(\omega)\rangle \to \langle x, f(\omega)\rangle.$$

A well-known theorem of Komlós is as follows:

Theorem 1 ([10])**.** *Every bounded sequence in $L^1_{\mathbb{R}}(\mu)$ has a subsequence which K-converges a.e. to a real integrable function.*

For some K-convergence results in infinite dimension we can see [11–15], and for more details and results on $L^1_{E'}[E]$, we refer to [1–4,9].

3. Main Results

We begin by recalling the following result ([16], Lemma 4.1) which is important for the development of the work.

Lemma 1. *Let $(f_n)_{n\in\mathbb{N}}$ be a bounded sequence in $L^1_E(\mu)$. Then, there exists a subsequence $(g_n)_{n\in\mathbb{N}}$ of $(f_n)_{n\in\mathbb{N}}$, such that for every subsequence $(h_n)_{n\in\mathbb{N}}$ of $(g_n)_{n\in\mathbb{N}}$*

- ***(a)*** *The sequence $(1_{\{\|h_n\|<n\}}h_n)_{n\in\mathbb{N}}$ is uniformly integrable;*
- ***(b)*** *The sequence $(h_n - 1_{\{\|h_n\|<n\}}h_n)_{n\in\mathbb{N}}$ converges a.e to 0 in E.*

Let $(f_n)_{n\in\mathbb{N}}$ be a bounded sequence in $L^1_{E'}[E]$, as the sequence $(\|f_n\|)_{n\in\mathbb{N}}$ is bounded in $L^1_{\mathbb{R}}(\mu)$, by Lemma 1 there exists a subsequence $(\|g_n\|)_{n\in\mathbb{N}}$ of $(\|f_n\|)_{n\in\mathbb{N}}$, such that $(1_{\{\|h_n\|<n\}}\|h_n\|)_{n\in\mathbb{N}}$ is UI and $(\|h_n\| - 1_{\{\|h_n\|<n\}}\|h_n\|)_{n\in\mathbb{N}}$ converges *a.e* to 0 in $\mathbb{R}$ for each subsequence $(\|h_n\|)_{n\in\mathbb{N}}$. Then, we can see that the sequences $(g_n)_{n\in\mathbb{N}}$ and $(h_n)_{n\in\mathbb{N}}$ have the required properties of the next lemma.

Lemma 2. *Let $(f_n)_{n\in\mathbb{N}}$ be a bounded sequence in $L^1_{E'}[E]$. Then there exists a subsequence $(g_n)_{n\in\mathbb{N}}$ of $(f_n)_{n\in\mathbb{N}}$, such that for every subsequence $(h_n)_{n\in\mathbb{N}}$ of $(g_n)_{n\in\mathbb{N}}$*

- ***(a′)*** *The sequence $(1_{\{\|h_n\|<n\}}h_n)_{n\in\mathbb{N}}$ is uniformly integrable;*
- ***(b′)*** *The sequence $(h_n - 1_{\{\|h_n\|<n\}}h_n)_{n\in\mathbb{N}}$ converges a.e to 0 in E'.*

The following simple result is useful.

Lemma 3. *Every bounded set in $L^\infty_{E'}[E]$ is sequentially relatively compact for the topology $\sigma(L^\infty_{E'}[E], L^\infty_E(\mu))$.*

Proof. Let H be a bounded set in $L^{\infty}_{E'}[E] = (L^1_E(\mu))'$, by the Banach–Alaoglu theorem H is relatively compact for the topology $\sigma(L^{\infty}_{E'}[E], L^1_E(\mu))$. As $L^1_E(\mu)$ is separable because E it is, H is sequentially relatively compact for the topology $\sigma(L^{\infty}_{E'}[E], L^1_E(\mu))$, and since $L^{\infty}_E(\mu)$ is a subspace of $L^1_E(\mu)$, we deduce that H is $\sigma(L^{\infty}_{E'}[E], L^{\infty}_E(\mu))$-sequentially relatively compact. □

Lemma 4. *Let $(f_n)_{n\in\mathbb{N}}$ be a sequence of $L^1_{E'}[E]$ which converges $\sigma(L^1_{E'}[E], L^{\infty}_E(\mu))$ to a function $f \in L^1_{E'}[E]$. Then, there exists an integer m such that*

$$\overline{N}_1(f) \leq 2 \inf_{n\geq m} \overline{N}_1(f_n).$$

Proof. As the mapping $\overline{N}_1$ is lower semicontinuous on $L^1_{E'}[E]$ for the topology $\sigma(L^1_{E'}[E], L^{\infty}_E(\mu))$, we have $\overline{N}_1(f) \leq \liminf_n \overline{N}_1(f_n)$. If $\liminf_n \overline{N}_1(f_n) = 0$, then the result is obvious. Now, if $\liminf_n \overline{N}_1(f_n) > 0$, we have

$$\overline{N}_1(f) < 2 \liminf_n \overline{N}_1(f_n) = \sup_{m\geq 1} 2 \inf_{n\geq m} \overline{N}_1(f_n).$$

Hence there exists $m \in \mathbb{N}^*$, satisfying the inequality. □

Now we are able to state our first main result of this paper.

Theorem 2. *Let $(f_n)_{n\in\mathbb{N}}$ be a bounded sequence in $L^1_{E'}[E]$. Then, there exists a function f in $L^1_{E'}[E]$ and a subsequence $(g_n)_{n\in\mathbb{N}}$ of $(f_n)_{n\in\mathbb{N}}$ such that for every subsequence $(h_n)_{n\in\mathbb{N}}$ of $(g_n)_{n\in\mathbb{N}}$ the following holds*

- ***(i)*** *$(1_{\{\|h_n\|<n\}}h)_{n\in\mathbb{N}}$ converges $\sigma(L^1_{E'}[E], L^{\infty}_E(\mu))$ to f in $L^1_{E'}[E]$ and $(h_n - 1_{\{\|h_n\|<n\}}h)_{n\in\mathbb{N}}$ converges a.e. to 0 in E';*
- ***(ii)*** *$(\frac{1}{n}\sum_{i=1}^{n} h_i)_{n\in\mathbb{N}}$ w^*-converges a.e. to f.*

Proof. **(i)** We have $\|1_{\{\|f_n\|<k\}}f_n\|_{L^{\infty}_{E'}[E]} \leq k$ for all $(k,n) \in (\mathbb{N}^*)^2$. For $k = 1$, there exists by Lemma 3 a subsequence $(f^1_n)_{n\in\mathbb{N}}$ of $(f_n)_{n\in\mathbb{N}}$, such that the sequence $(1_{\{\|f^1_n\|<1\}}f^1_n)_{n\in\mathbb{N}}$ converges $\sigma(L^{\infty}_{E'}[E], L^{\infty}_E(\mu))$ to $v_1 \in L^{\infty}_{E'}[E]$ and there exists for all $k \geq 1$ a subsequence $(f^{k+1}_n)_{n\in\mathbb{N}}$ of $(f^k_n)_{n\in\mathbb{N}}$, such that the sequence $(1_{\{\|f^{k+1}_n\|<k+1\}}f^{k+1}_n)_{n\in\mathbb{N}}$ converges $\sigma(L^{\infty}_{E'}[E], L^{\infty}_E(\mu))$ to v_{k+1} in $L^{\infty}_{E'}[E]$. Let $f'_n = f^n_n$ $(n \geq 1)$, then, for every $k \geq 1$, the sequence $(1_{\{\|f'_n\|<k\}}f'_n)_{n\in\mathbb{N}}$ converges $\sigma(L^{\infty}_{E'}[E], L^{\infty}_E(\mu))$ to v_k in $L^{\infty}_{E'}[E]$.

Claim: $(v_k)_{k\in\mathbb{N}}$ converges to a function f in $L^1_{E'}[E]$.

Put $v_0 = 0$, as $(L^1_{E'}[E], \overline{N}_1)$ is a Banach space, it is enough to prove that the series $\sum_{k\geq 1} \overline{N}_1(v_k - v_{k-1})$ converges. For every $k \geq 1$, the sequence $(1_{\{\|f'_n\|<k\}}f'_n - 1_{\{\|f'_n\|<k-1\}}f'_n)_{n\in\mathbb{N}}$ converges $\sigma(L^{\infty}_{E'}[E], L^{\infty}_E(\mu))$ to $(v_k - v_{k-1})$ in $L^{\infty}_{E'}[E]$ and, therefore, also in $L^1_{E'}[E]$ for the topology $\sigma(L^1_{E'}[E], L^{\infty}_E(\mu))$. By Lemma 4, there exists $m_k \in \mathbb{N}^*$, such that

$$\overline{N}_1(v_k - v_{k-1}) \leq 2 \inf_{n\geq m_k} \overline{N}_1(1_{\{\|f'_n\|<k\}}f'_n - 1_{\{\|f'_n\|<k-1\}}f'_n).$$

Let $N \in \mathbb{N}^*$ and $n \geq \max(m_1, .., m_N)$. Then we have

$$\begin{aligned}
\sum_{k=1}^{N} \overline{N}_1(v_k - v_{k-1}) &\leq 2\sum_{k=1}^{N} \overline{N}_1(1_{\{\|f'_n\|<k\}} f'_n - 1_{\{\|f'_n\|<k-1\}} f'_n) \\
&= 2\sum_{k=1}^{N} \int \|1_{\{\|f'_n\|<k\}} f'_n - 1_{\{\|f'_n\|<k-1\}} f'_n)\| \, d\mu \\
&\leq 2\int \|f'_n\| \, d\mu \\
&\leq 2 \sup_{p\geq 1} \overline{N}_1(f'_p) < +\infty
\end{aligned}$$

and therefore $\sum_{k=1}^{+\infty} \overline{N}_1(v_k - v_{k-1}) < +\infty$. This proves the *Claim*.

Now applying Lemma 2 to $(f'_n)_{n\in\mathbb{N}}$ we get a subsequence $(f''_n)_{n\in\mathbb{N}}$ of $(f'_n)_{n\in\mathbb{N}}$ such that for every subsequence $(h_n)_{n\in\mathbb{N}}$ of $(f''_n)_{n\in\mathbb{N}}$

$$(1_{\{\|h_n\|<n\}} h_n)_{n\in\mathbb{N}} \text{ is UI,} \tag{1}$$

$$(h_n - 1_{\{\|h_n\|<n\}} h_n)_{n\in\mathbb{N}} \text{ converges } a.e. \text{ to } 0 \text{ in } E'. \tag{2}$$

It remains to show that $(1_{\{\|h_n\|<n\}} h_n)_{n\in\mathbb{N}}$ converges $\sigma(L^1_{E'}[E], L^\infty_E(\mu))$ to f in $L^1_{E'}[E]$. Let us consider $\zeta \in L^\infty_E(\mu)$ with norm ≤ 1 and $\epsilon > 0$. By (1) and the convergence of $(v_k)_{k\in\mathbb{N}}$ to f in $L^1_{E'}[E]$, there exists $n_0 \in \mathbb{N}$, such that

$$\sup_n \int_{\{\|1_{\{\|h_n\|<n\}} h_n\| \geq n_0\}} \|1_{\{\|h_n\|<n\}} h_n\| d\mu = \sup_n \overline{N}_1(1_{\{\|h_n\|<n\}} h_n - 1_{\{\|h_n\|<n_0\}} h_n) \leq \frac{\epsilon}{3}$$

and

$$\overline{N}_1(v_{n_0} - f) \leq \frac{\epsilon}{3}.$$

As $(1_{\{\|h_n\|<n_0\}} h_n)_{n\in\mathbb{N}}$ converges $\sigma(L^1_{E'}[E], L^\infty_E(\mu))$ to v_{n_0} in $L^1_{E'}[E]$, there exists $n_1 \geq n_0$ such that

$$n \geq n_1 \Rightarrow \left\langle \zeta, 1_{\{\|h_n\|<n_0\}} h_n - v_{n_0} \right\rangle \leq \frac{\epsilon}{3}.$$

Then, for $n \geq n_1$ we have

$$\begin{aligned}
\left\langle \zeta, 1_{\{\|h_n\|<n\}} h_n - f \right\rangle &\leq \left\langle \zeta, 1_{\{\|h_n\|<n\}} h_n - 1_{\{\|h_n\|<n_0\}} h_n \right\rangle \\
&+ \left\langle \zeta, 1_{\{\|h_n\|<n_0\}} h_n - v_{n_0} \right\rangle + \langle \zeta, v_{n_0} - f \rangle \\
&\leq \overline{N}_1(1_{\{\|h_n\|<n\}} h_n - 1_{\{\|h_n\|<n_0\}} h_n) \\
&+ \left\langle \zeta, 1_{\{\|h_n\|<n_0\}} h_n - v_{n_0} \right\rangle + \overline{N}_1(v_{n_0} - f) \\
&\leq \epsilon.
\end{aligned}$$

(ii) It is sufficient to show that there is a subsequence $(g_n)_{n\in\mathbb{N}}$ of $(f''_n)_{n\in\mathbb{N}}$, such that $(\frac{1}{n}\sum_{i=1}^{n} h_i)_{n\in\mathbb{N}}$ w*-converges *a.e.* to f for every subsequence $(h_n)_{n\in\mathbb{N}}$ of $(g_n)_{n\in\mathbb{N}}$. With E being separable, let $D = (x_j)_{j\in\mathbb{N}^*}$, a norm-dense sequence in E. The sequences $(\|f''_n(.)\|)_{n\in\mathbb{N}}$ and $(\langle f''_n, x_j\rangle)_{n\in\mathbb{N}}$ $j = 1, 2, ...$ are bounded in $L^1_{\mathbb{R}}(\mu)$, so we apply Komlós' theorem to suitably chosen sequences and a diagonal method to get functions φ_0,φ_1,φ_2,...,φ_j, ... in $L^1_{\mathbb{R}}(\mu)$ and a subsequence $(g_n)_{n\in\mathbb{N}}$ of $(f''_n)_{n\in\mathbb{N}}$, such that for every subsequence $(h_n)_{n\in\mathbb{N}}$ of $(g_n)_{n\in\mathbb{N}}$

$$\frac{1}{n}\sum_{i=1}^{n}\|h_i(\omega)\| \to \varphi_0(\omega) \qquad a.e., \tag{3}$$

$$\forall\, j \in \mathbb{N}^*, \qquad \frac{1}{n}\sum_{i=1}^{n}\langle h_i(\omega), x_j\rangle \to \varphi_j(\omega) \qquad a.e. \tag{4}$$

Let $(h_n)_{n\in\mathbb{N}}$ be a fixed subsequence of $(g_n)_{n\in\mathbb{N}}$. By (2) and the decomposition $1_{\{\|h_n\|<n\}}h_n = h_n - (h_n - 1_{\{\|h_n\|<n\}}h_n)$ we get

$$\forall\, j \in \mathbb{N}^*, \qquad \frac{1}{n}\sum_{i=1}^{n}\langle 1_{\{\|h_i\|<i\}}h_i(\omega), x_j\rangle \to \varphi_j(\omega) \qquad a.e. \tag{5}$$

As $(\langle 1_{\{\|h_n\|<n\}}h_n, x_j\rangle)_{n\in\mathbb{N}}$ is UI for each $j \in \mathbb{N}^*$, it follows by (5) and the Lebesgue–Vitali's theorem that for each $A \in \mathcal{F}$

$$\forall\, j \in \mathbb{N}^*, \qquad \frac{1}{n}\sum_{i=1}^{n}\int_A \langle 1_{\{\|h_i\|<i\}}h_i, x_j\rangle \, d\mu \to \int_A \varphi_j \, d\mu. \tag{6}$$

On the other hand, by **(i)**

$$\forall l \in L_E^\infty(\mu), \qquad \frac{1}{n}\sum_{i=1}^{n}\int_A \langle 1_{\{\|h_i\|<i\}}h_i, l\rangle \, d\mu \to \int_\Omega \langle f, l\rangle \, d\mu, \tag{7}$$

so in particular for each $A \in \mathcal{F}$ and $x_j \in D$ we have

$$\frac{1}{n}\sum_{i=1}^{n}\int_A \langle 1_{\{\|h_i\|<i\}}h_i, x_j\rangle \, d\mu \to \int_A \langle f, x_j\rangle \, d\mu, \tag{8}$$

then by (6) and (8) we get

$$\forall\, j \in \mathbb{N}^*, \qquad \varphi_j(\omega) = \langle f(\omega), x_j\rangle \qquad a.e. \tag{9}$$

and therefore by (4)

$$\forall\, j \in \mathbb{N}^*, \qquad \langle \frac{1}{n}\sum_{i=1}^{n} h_i(\omega), x_j\rangle \to \langle f(\omega), x_j\rangle \qquad a.e. \tag{10}$$

Finally, by (3), $(\frac{1}{n}\sum\limits_{i=1}^{n} h_i(.))_{n\in\mathbb{N}}$ is pointwise bounded *a.e.*; this, along with the density of D, yields

$$\forall\, x \in E, \qquad \langle \frac{1}{n}\sum_{i=1}^{n} h_i(\omega), x\rangle \to \langle f(\omega), x\rangle \qquad a.e.$$

So the proof is complete. □

An immediate application of Theorem 1, we have the following criteria for $\sigma(L^1_{E'}[E], L^\infty_E(\mu))$ compactness in $L^1_{E'}[E]$, which generalizes Lemma 3.

Theorem 3. *Every uniformly integrable set in $L^1_{E'}[E]$ is sequentially relatively compact for the topology $\sigma(L^1_{E'}[E], L^\infty_E(\mu))$.*

Proof. Let H be an UI set in $L^1_{E'}[E]$ and $(f_n)_{n\in\mathbb{N}}$ a sequence in H. As $(f_n)_{n\in\mathbb{N}}$ is bounded, by Theorem 1 (i) there is a function f in $L^1_{E'}[E]$ and a subsequence $(h_n)_{n\in\mathbb{N}}$ of $(f_n)_{n\in\mathbb{N}}$, such that $(1_{\{\|h_n\|<n\}}h_n)_{n\in\mathbb{N}}$ converges $\sigma(L^1_{E'}[E], L^\infty_E(\mu))$ to f in $L^1_{E'}[E]$ and $(1_{\{\|h_n\|\geq n\}}h_n)_{n\in\mathbb{N}}$ converges *a.e.* to 0 in E'. As $(h_n)_{n\in\mathbb{N}}$ is UI, $(1_{\{\|h_n\|\geq n\}}h_n)_{n\in\mathbb{N}}$ converges strongly to 0 in $L^1_{E'}[E]$ and hence $(h_n)_{n\in\mathbb{N}}$ converges $\sigma(L^1_{E'}[E], L^\infty_E(\mu))$ to f in $L^1_{E'}[E]$. Then H is $\sigma(L^1_{E'}[E], L^\infty_E(\mu))$-sequentially relatively compact. □

It is well known that the Komlós type results can be used to develop weak compactness criteria in $L^1_E(\mu)$. Using this argument, we now provide some weak compactness results in $L^1_{E'}[E]$.

Lemma 5. *Let $(f_n)_{n\in\mathbb{N}}$ be a uniformly integrable sequence in $L^1_{E'}[E]$. Assume that $(f_n)_{n\in\mathbb{N}}$ is w-K-converge a.e. to a function f, then $(f_n)_{n\in\mathbb{N}}$ converges weakly to f in $L^1_{E'}[E]$.*

Proof. By a general criterion for weak convergence sequence in Banach space ([17], Corollary 2) it is enough to prove that for every subsequence $(f'_n)_{n\in\mathbb{N}}$ of $(f_n)_{n\in\mathbb{N}}$ there exist $g_n \in co\left\{f'_i : i \geq n\right\}$ which weakly converges to f in $L^1_{E'}[E]$. Let $(f'_n)_{n\in\mathbb{N}}$ be a subsequence of $(f_n)_{n\in\mathbb{N}}$, by the hypothesis

$$\frac{1}{n}\sum_{i=1}^{n} f'_i(\omega) \to f(\omega) \text{ weakly in } E' \ a.e.$$

so the sequence $(g_n)_{n\in\mathbb{N}}$ defined by $g_n = \frac{1}{n+1}\sum\limits_{i=n}^{2n} f'_i \in co\left\{f'_i : i \geq n\right\}$ and

$$g_n = \frac{2n}{n+1}\frac{1}{2n}\sum_{i=1}^{2n} f'_i - \frac{n-1}{n+1}\frac{1}{n-1}\sum_{i=1}^{n-1} f'_i$$

w-converges $a.e.$ to f. On the other hand $(g_n)_{n\in\mathbb{N}}$ is UI in $L^1_{E'}[E]$, hence by ([1], Theorem 4.5) it converges weakly to f in $L^1_{E'}[E]$. □

The next result is a different version of Theorem 1, which deals with the weak convergence. Recall that $\mathcal{R}wc(E')$ denoted the set of nonempty closed convex subsets of E', such that their intersection with any closed ball is weakly compact.

Theorem 4. *Let $(f_n)_{n\in\mathbb{N}}$ be a bounded sequence in $L^1_{E'}[E]$. Suppose that there exist a $\mathcal{R}wc(E')$-valued multifunction Γ, such that $f_n(\omega) \in \Gamma(\omega)$ for a.e. $\omega \in \Omega$ and for all $n \in \mathbb{N}$. Then, there exists a function f in $L^1_{E'}[E]$ and a subsequence $(g_n)_{n\in\mathbb{N}}$ of $(f_n)_{n\in\mathbb{N}}$, such that for every subsequence $(h_n)_{n\in\mathbb{N}}$ of $(g_n)_{n\in\mathbb{N}}$ the following holds:*

- ***(j)*** *$(\frac{1}{n}\sum\limits_{i=1}^{n} h_i)_{n\in\mathbb{N}}$ w-converges a.e. to f;*
- ***(jj)*** *$(1_{\{\|h_n\|<n\}}h_n)_{n\in\mathbb{N}}$ converges $\sigma(L^1_{E'}[E], (L^1_{E'}[E])')$ (weakly) to f in $L^1_{E'}[E]$ and $(h_n - 1_{\{\|h_n\|<n\}}h_n)_{n\in\mathbb{N}}$ converges a.e. to 0 in E'.*

Proof. (j) As $(f_n)_{n\in\mathbb{N}}$ is bounded in $L^1_{E'}[E]$, by Theorem 1 (ii) there is f in $L^1_{E'}[E]$ and a subsequence $(g_n)_{n\in\mathbb{N}}$ of $(f_n)_{n\in\mathbb{N}}$, such that

$$(g_n)_{n\in\mathbb{N}} \text{ w*-K-converges } a.e. \text{ to } f. \tag{11}$$

Applying Komlós theorem to $(\|g_n(.)\|)_{n\in\mathbb{N}}$ and by extracting a subsequence if necessary, we may suppose that there exists a real integrable function φ, such that

$$(\|g_n(.)\|)_{n\in\mathbb{N}} \text{ K-converges } a.e. \text{ to } \varphi. \tag{12}$$

Let $(h_n)_{n\in\mathbb{N}}$ be a fixed subsequence of $(g_n)_{n\in\mathbb{N}}$ and set $S_n := \frac{1}{n}\sum\limits_{i=1}^{n} h_i$. There exists by (12) $\mathcal{N} \in \mathcal{F}$ with $\mu(\mathcal{N}) = 0$, such that for all $\omega \in \Omega\backslash\mathcal{N}$

$$\|S_n(\omega)\| \leq \frac{1}{n}\sum_{i=1}^{n}\|h_i(\omega)\| \to \varphi(\omega),$$

hence $(S_n(\omega))_{n\in\mathbb{N}}$ is bounded and $S_n(\omega) \in K(\omega)$ where

$$K(\omega) = \Gamma(\omega) \cap (\sup_n \|S_n(\omega)\|)\overline{B}_{E'}$$

is convex weakly compact in E' since Γ is $\mathcal{R}wc(E')$-valued. By (11) there exists $\mathcal{N}' \in \mathcal{F}$ with $\mu(\mathcal{N}') = 0$, such that for all $\omega \in \Omega \backslash \mathcal{N}'$, $(S_n(\omega))_{n\in\mathbb{N}}$ w*-converges to $f(\omega)$. Hence, for all $\omega \in \Omega \backslash (\mathcal{N} \cup \mathcal{N}')$, every w-convergent subsequence of $(S_n(\omega))_{n\in\mathbb{N}}$ converges to $f(\omega)$. As $(S_n(\omega))_{n\in\mathbb{N}}$ is w-relatively compact in E', we conclude that $(S_n(\omega))_{n\in\mathbb{N}}$ w-converges to $f(\omega)$.

(jj) Applying Lemma 2 to the bounded sequence $(g_n)_{n\in\mathbb{N}}$, yields the existence of a subsequence $(g'_n)_{n\in\mathbb{N}}$ of $(g_n)_{n\in\mathbb{N}}$, such that

$$(1_{\{\|h_n\|<n\}} h_n)_{n\in\mathbb{N}} \text{ is UI,} \tag{13}$$

and

$$(h_n - 1_{\{\|h_n\|<n\}} h_n)_{n\in\mathbb{N}} \text{ converge } a.e. \text{ to } 0 \text{ in } E' \tag{14}$$

for every further subsequence $(h_n)_{n\in\mathbb{N}}$ of $(g'_n)_{n\in\mathbb{N}}$. Let $(h_n)_{n\in\mathbb{N}}$ be a fixed subsequence of $(g'_n)_{n\in\mathbb{N}}$, we will show that $(1_{\{\|h_n\|<n\}} h_n)_{n\in\mathbb{N}}$ converges weakly to f in $L^1_{E'}[E]$. By **(j)**, the sequence $(h_n)_{n\in\mathbb{N}}$ w-K-converges *a.e.* to f, and by (14), and the decomposition $1_{\{\|h_n\|<n\}} h_n = h_n - (h_n - 1_{\{\|h_n\|<n\}} h_n)$ we can see that $(1_{\{\|h_n\|<n\}} h_n)_{n\in\mathbb{N}}$ also w-K-converges *a.e.* to f. Now, as $(1_{\{\|h_n\|<n\}} h_n)_{n\in\mathbb{N}}$ is UI, by Lemma 4, $(1_{\{\|h_n\|<n\}} h_n)_{n\in\mathbb{N}}$ converges weakly to f in $L^1_{E'}[E]$. Finally, take $(g'_n)_{n\in\mathbb{N}}$ instead of $(g_n)_{n\in\mathbb{N}}$ in **(j)**, then $(g'_n)_{n\in\mathbb{N}}$ and f satisfy **(j)** and **(jj)**. $\square$

We finish this work with the following result (compare with Proposition 5.1 in [1]).

Corollary 1. *Suppose that Γ is a $\mathcal{R}wc(E')$-valued multifunction on Ω and H is a UI set in $L^1_{E'}[E]$, such that $f(\omega) \in \Gamma(\omega)$ for a.e. $\omega \in \Omega$ and for all $f \in H$, then H is relatively weakly compact in $L^1_{E'}[E]$.*

Proof. By Eberlein–Smulian's theorem, the conclusion to be derived is equivalent with H being sequentially relatively weakly compact. Let $(f_n)_{n\in\mathbb{N}}$ be a bounded sequence in H. Since $f_n(\omega) \in \Gamma(\omega)$ for *a.e.* $\omega \in \Omega$ and for all $n \in \mathbb{N}$, by Theorem 3 (jj) there is a function f in $L^1_{E'}[E]$ and a subsequence $(h_n)_{n\in\mathbb{N}}$ of $(f_n)_{n\in\mathbb{N}}$, such that $(1_{\{\|h_n\|<n\}} h_n)_{n\in\mathbb{N}}$ converges weakly to f in $L^1_{E'}[E]$, and $(1_{\{\|h_n\|\geq n\}} h_n)_{n\in\mathbb{N}}$ converges *a.e.* to 0 in E'. On the other hand, since $(h_n)_{n\in\mathbb{N}}$ is UI, $(1_{\{\|h_n\|\geq n\}} h_n)_{n\in\mathbb{N}}$ converges strongly to 0 in $L^1_{E'}[E]$, and then $(h_n)_{n\in\mathbb{N}}$ converges weakly to f in $L^1_{E'}[E]$. Hence, H is sequentially relatively weakly compact in $L^1_{E'}[E]$. $\square$

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References

1. Benabdellah, H.; Castaing, C. Weak compactness and convergences in $L^1_{E'}[E]$. *Adv. Math. Econ* **2001**, *3*, 1–44.
2. Bourras, A.; Castaing, C.; Guessous, M. Olech-types lemma and Visintin-types theorem in Pettis integration and $L^1_{E'}[E]$. *Josai Math. Monogr.* **1999**, *1*, 1–26.
3. Castaing, C.; Raynaud de Fitte, P.C.; Valadier, M. *Young Measures on Topological Spaces. With Applications in Control Theory and Probability*; Kluwer Academic: Dordrecht, The Netherlands, 2004.
4. Castaing, C.; Saadoune, M. Komlós type convergence for random variables and random sets with applications to minimization problems. *Adv. Math. Econ.* **2007**, *10*, 1–29.
5. Balder, E.J. New sequential compactness results for spaces of scalarly integrable functions. *J. Math. Anal. Appl.* **1990**, *151*, 1–16. [CrossRef]

6. Chakraborty, N.D.; Choudhury, T. Convergence theorems for Pettis integrable functions and regular methods of summability. *J. Math. Anal. Appl.* **2009**, *359*, 95–105. [CrossRef]
7. Saadoune, M. A new extension of Komlós' Theorem in infinite dimensions. Application: Weak compactness in $L_1(X)$. *Port. Math.* **1998**, *55*, 113.
8. Diestel, J.; Uhl, J., Jr. *Vector Measures*; AMS Mathematical Surveys; American Mathematical Society: Providence, RI, USA, 1977.
9. Ionescu-Tulcea, A.; Ionescu-Tulcea, C. *Topics in the Theory of Lifting*; Springer: New York, NY, USA, 1969.
10. Komlós, J. A generalization of a problem of Steinhauss. *Acta Math. Acad. Sci. Hung.* **1967**, *18*, 217–223. [CrossRef]
11. Balder, E.J. Infinite-dimensional extension of a theorem of Komlós. *Proba. Theory Related Fields* **1989**, *81*, 185–188. [CrossRef]
12. Balder, E.J.; Hess, C. Two Generalizations of Komlós' Theorem with Lower Closure-Type Applications. *J. Convex Anal.* **1996**, *3*, 25–44.
13. Dehaj, A.; Guessous, M. Permutation-Invariance in Komlós' Theorem for Hilbert-Space Valued Random Variables. *J. Convex Anal.* **2021**, *28*, in press.
14. Garling, D.J.H. Subsequence principles for vector-valued random variables. *Math. Proc. Camb. Philos. Soc.* **1979**, *86*, 301–311. [CrossRef]
15. Guessous, M. An elementary proof of Komlós-Revesz theorem in Hilbert spaces. *J. Convex Anal.* **1997**, *4*, 321–332.
16. Castaing, C.; Guessous, M. Convergence in $L_1(\mu, X)$. *Adv. Math. Econ.* **1999**, *1*, 17–37.
17. Ülger, A. Weak compactness in $L_1(X)$. *Proc. Am. Math. Soc.* **1991**, *113*, 143–149.