Galaxies **2013**, *1*, 261-274; doi:10.3390/galaxies1030261

OPEN ACCESS

galaxies

ISSN 2075-4434

www.mdpi.com/journal/galaxies

Article

A No-Go Theorem for Rotating Stars of a Perfect Fluid without Radial Motion in Projectable Hořava–Lifshitz Gravity

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Received: 16 October 2013; in revised form: 27 November 2013 / Accepted: 10 December 2013 / Published: 16 December 2013

Abstract: Hořava–Lifshitz gravity has covariance only under the foliation-preserving diffeomorphism. This implies that the quantities on the constant-time hypersurfaces should be regular. In the original theory, the projectability condition, which strongly restricts the lapse function, is proposed. We assume that a star is filled with a perfect fluid with no-radial motion and that it has reflection symmetry about the equatorial plane. As a result, we find a no-go theorem for stationary and axisymmetric star solutions in projectable Hořava–Lifshitz gravity under the physically reasonable assumptions in the matter sector. Since we do not use the gravitational action to prove it, our result also works out in other projectable theories and applies to not only strong gravitational fields, but also weak gravitational ones.

Keywords: Hořava–Lifshitz gravity; projectable gravity; rotating star

1. Introduction

Recently, Hořava proposed a power-counting renormalizable gravitational theory [1,2]. The theory is called Hořava–Lifshitz gravity, because it exhibits the Lifshitz-type anisotropic scaling in the ultraviolet:

$$t \to b^z t, \quad x^i \to b x^i \tag{1}$$

where t, x^i, b and z are the temporal coordinate, the spatial coordinates, the scaling factor and the dynamical critical exponent, respectively, and i runs over one, two and three. Since this theory is expected to be renormalizable and unitary, its phenomenological aspects [3,4] and variants [5,6] strenuously have been investigated, including black holes [7–16], dark matter [17,18], dark energy [19], the solar system test [20] and so on.

The field variables in this theory are the lapse function, $N(t)$, the shift vector, $N^i(t,x)$, and the spatial metric, $g_{ij}(t,x)$. Note that the shift vector, N^i, and the spatial metric, g_{ij}, can depend on both t and x^i, but that the lapse function, N, can only do so on t. Since the lapse function, N, can be interpreted as a gauge field associated with the time reparametrization, it is natural to restrict it to be space independent. This assumption, called the projectability condition, is proposed in Hořava's original paper [1] from the view point of quantization. However, the pathological behaviors of the projectability condition, such as the infrared instability and the strong coupling, are found [1,2,21–26], and the theory has been extended to avoid the adverse situation [27,28].

Since higher derivative terms do not contribute at large distances, the action of this theory can recover the apparent form of general relativity if we tune a coupling parameter. In this context, it seems that projectable Hořava–Lifshitz gravity passes astrophysical tests. However, we will show that, actually, this is not true in this paper.

In this theory, black holes have been investigated eagerly, while stars have not been studied so much [29,30]. The comparison of the features of star solutions in Hořava–Lifshitz gravity with the corresponding ones in Einstein gravity would be one of the astrophysical tests for Hořava–Lifshitz gravity. It is important to investigate star solutions, gravitational collapse [31] and the formation of black holes.

The first study of stars in Hořava–Lifshitz gravity was done by Izumi and Mukohyama [29]. They found a no-go theorem that no spherically symmetric and static solution filled a perfect fluid without radial motion exists in this projectable theory under the assumptions that the energy density is a piecewise-continuous and non-negative function of the pressure and that the pressure at the center is positive. Their result is powerful, because it does not depend on the gravitational action.

To construct star solutions, we have to change at least one of their assumptions for the matter sector, the symmetry of spacetime, the projectability and the invariance under the foliation-preserving diffeomorphism. Greenwald, Papazoglou and Wang found spherically symmetric static solutions, which are filled with a perfect fluid with radial motion and a class of an anisotropic fluid in the projectable Hořava–Lifshitz gravity without the detailed balance condition [30].

It seems that static solutions are too simple to describe realistic stars, which are generally rotational. In this paper, we investigate a stationary and axisymmetric star in projectable Hořava–Lifshitz gravity. We find a no-go theorem that the stationary and axisymmetric star filled with a perfect fluid without radial motion in the reflection symmetry about the equatorial plane does not exist under the physically reasonable conditions on the matter sector. Since we do not use the gravitational action to prove it, our result also works out in other projectable theories [5,32] and applies to not only strong gravitational fields, like neutron stars, but also weak gravitational ones, like planets or moons. Our proof implies another ill behavior of the projectability condition if we follow a principle that stars should be described by stationary solutions of a low-energy effective theory. On the other hand, even if we do not follow this principle, our result would be useful to investigate rotating-star solutions in this theory and then to compare the solutions with the corresponding ones in Einstein gravity for astrophysical tests of this theory.

This paper is organized as follows. In Section 2, we shall describe the definitions, the basic equations and the properties of Hořava–Lifshitz gravity. In Section 3, we give the main result that there are no stationary and axisymmetric star solutions with a perfect fluid, which does not have the radial component of the four-velocity under a set of reasonable assumptions in the matter sector. In Section 4, we summarize and discuss our result. In Appendix A, we show the explicit expression for the equation of motion. In Appendix B, we show the triad components of the extrinsic curvature tensor. In this paper, we use the units in which $c = 1$.

2. Properties of Hořava–Lifshitz Gravity

In this section, we shall describe the definitions, the basic equations and the properties of Hořava–Lifshitz gravity. Hořava–Lifshitz gravity does not have general covariance, since the Lifshitz-type anisotropic scaling treats time and space differently. Instead, this theory is invariant under the foliation-preserving diffeomorphism:

$$t \to \tilde{t}(t),\ x^i \to \tilde{x}^i(t, x^j) \tag{2}$$

This means that the foliation of the spacetime manifold by constant-time hypersurfaces has a physical meaning. Thus, the quantities on the constant-time hypersurfaces, such as the extrinsic curvature tensor and the shift vector, must be regular.

It is useful to describe the line element in the Arnowitt–Deser–Misner (ADM) form [33]:

$$ds^2 = -N^2 dt^2 + g_{ij}(dx^i + N^i dt)(dx^j + N^j dt) \tag{3}$$

The action proposed by Hořava [1] is given by:

$$I = I_g + I_m \tag{4}$$

$$\begin{aligned} I_g &= \int dt d^3x \sqrt{g} N \left\{ \frac{2}{\kappa^2}(K^{ij}K_{ij} - \lambda K^2) - \frac{\kappa^2}{2\omega^4} C_{ij}C^{ij} + \frac{\kappa^2 \mu}{2\omega^2} \varepsilon^{ijk} R_{il} D_j R^l_k - \frac{\kappa^2 \mu^2}{8} R_{ij} R^{ij} \right. \\ &\quad \left. + \frac{\kappa^2 \mu^2}{8(1-3\lambda)} \left(\frac{1-4\lambda}{4} R^2 + \Lambda_W R - 3\Lambda_W^2 \right) \right\} \end{aligned} \tag{5}$$

where I_m is the matter action, R is the Ricci scalar of g_{ij}, R_{ij} is the Ricci tensor of g_{ij}, D_i is the covariant derivative compatible with g_{ij}, K_{ij} is the extrinsic curvature of a constant-time hypersurface, defined by:

$$K_{ij} = \frac{1}{2N}(\partial_t g_{ij} - D_i N_j - D_j N_i) \tag{6}$$

$K = g^{ij} K_{ij}$, C_{ij} is the Cotton tensor, defined by:

$$C^{ij} = \varepsilon^{ikl} D_k \left(R^j_l - \frac{1}{4} R \delta^j_l \right) \tag{7}$$

$\varepsilon^{ikl} = \epsilon^{ikl}/\sqrt{g}$ is the antisymmetric tensor, which is covariant with respect to g_{ij}, and $\kappa, \omega, \mu, \lambda$ and Λ_W are constant parameters. We can rewrite the gravitational action (5):

$$I_g = \int dt d^3x \sqrt{g} N [\alpha(K^{ij}K_{ij} - \lambda K^2) + \beta C_{ij}C^{ij} + \gamma \varepsilon^{ijk} R_{il} D_j R^l_k + \zeta R_{ij} R^{ij} + \eta R^2 + \xi R + \sigma] \tag{8}$$

where parameters $\alpha, \beta, \gamma, \zeta, \eta, \xi$ and σ are given by:

$$\alpha = \frac{2}{\kappa^2}, \quad \beta = -\frac{\kappa^2}{2\omega^4}, \quad \gamma = \frac{\kappa^2\mu}{2\omega^2}, \quad \zeta = -\frac{\kappa^2\mu^2}{8},$$
$$\eta = \frac{\kappa^2\mu^2}{8(1-3\lambda)}\frac{1-4\lambda}{4}, \quad \xi = \frac{\kappa^2\mu^2}{8(1-3\lambda)}\Lambda_W, \quad \sigma = \frac{\kappa^2\mu^2}{8(1-3\lambda)}(-3\Lambda_W^2) \tag{9}$$

If we take $\lambda = 1$ to recover the apparent form of general relativity and the apparent Lorentz invariance, we can compare this action to that of general relativity. Then, we obtain:

$$\alpha = \frac{1}{16\pi G}, \quad \xi = \alpha, \quad \sigma = -2\Lambda\alpha \tag{10}$$

where Λ is the cosmological constant and G is Newton's constant.

Under the infinitesimal coordinate transformation:

$$\delta t = f(t), \qquad \delta x^i = \zeta^i(t, x) \tag{11}$$

g_{ij}, N^i and N transform as:

$$\delta g_{ij} = f\partial_t g_{ij} + \mathcal{L}_\zeta g_{ij} \tag{12}$$
$$\delta N^i = \partial_t(N^i f) + \partial_t\zeta^i + \mathcal{L}_\zeta N^i \tag{13}$$
$$\delta N_i = \partial_t(N_i f) + g_{ij}\partial_t\zeta^j + \mathcal{L}_\zeta N_i \tag{14}$$
$$\delta N = \partial_t(Nf) \tag{15}$$

where $\mathcal{L}_\zeta$ is the Lie derivative along $\zeta^i(t, x)$. $\mathcal{L}_\zeta g_{ij}$ and $\mathcal{L}_\zeta N^i$ are given by:

$$\mathcal{L}_\zeta g_{ij} = g_{jk}D_i\zeta^k + g_{ik}D_j\zeta^k \tag{16}$$
$$\mathcal{L}_\zeta N^i = \zeta^k D_k N^i - N^k D_k\zeta^i \tag{17}$$

By the variation of the action with respect to N, we get the Hamiltonian constraint:

$$H_{g\perp} + H_{m\perp} = 0 \tag{18}$$

where:

$$\begin{aligned} H_{g\perp} &\equiv -\frac{\delta I_g}{\delta N} \\ &= \int dx^3\sqrt{g}\left[(\alpha K^{ij}K_{ij} - \lambda K^2) - \beta C_{ij}C^{ij} - \gamma\varepsilon^{ijk}R_{il}D_jR^l_k - \zeta R_{ij}R^{ij} - \eta R^2 - \xi R - \sigma\right] \end{aligned} \tag{19}$$

and:

$$H_{m\perp} \equiv -\frac{\delta I_m}{\delta N} = \int dx^3\sqrt{g}T_{\mu\nu}n^\mu n^\nu \tag{20}$$

Here, n^μ is defined as:

$$n_\mu dx^\mu = -Ndt, \quad n^\mu\partial_\mu = \frac{1}{N}(\partial_t - N^i\partial_i) \tag{21}$$

Notice that due to the projectability condition $N = N(t)$, the Hamiltonian constraint is global in Hořava–Lifshitz gravity, while it is local in general relativity.

From the variation of the action with respect to N^i, we obtain the momentum constraint:

$$\mathcal{H}_{gi} + \mathcal{H}_{mi} = 0 \tag{22}$$

where:

$$\mathcal{H}_{gi} \equiv -\frac{1}{\sqrt{g}}\frac{\delta I_g}{\delta N^i} = -2\alpha D^j(K_{ij} - \lambda K g_{ij}) \tag{23}$$

$$\mathcal{H}_{mi} \equiv -\frac{1}{\sqrt{g}}\frac{\delta I_m}{\delta N^i} = T_{i\mu}n^\mu \tag{24}$$

By the variation of the action with respect to g_{ij}, we get the equation of motion:

$$\mathcal{E}_{gij} + \mathcal{E}_{mij} = 0 \tag{25}$$

where:

$$\mathcal{E}_{gij} \equiv g_{ik}g_{jl}\frac{2}{N\sqrt{g}}\frac{\delta I_g}{\delta g_{kl}} \tag{26}$$

$$\mathcal{E}_{mij} \equiv g_{ik}g_{jl}\frac{2}{N\sqrt{g}}\frac{\delta I_m}{\delta g_{kl}} = T_{ij} \tag{27}$$

The explicit expression for the equation of motion is given in Appendix A.

By the invariance of the gravitational action and the matter action under the infinitesimal transformation (11), we get the energy conservation:

$$N\partial_t H_{\alpha\perp} + \int dx^3 \left(N^i\partial_t(\sqrt{g}\mathcal{H}_{\alpha i}) + \frac{N\sqrt{g}}{2}\mathcal{E}_\alpha^{\ ij}\partial_t g_{ij}\right) = 0 \tag{28}$$

and the momentum conservation:

$$0 = \frac{1}{N}(\partial_t - N^j D_j)\mathcal{H}_{\alpha i} + K\mathcal{H}_{\alpha i} - \frac{1}{N}\mathcal{H}_{\alpha j}D_i N^j - D^j\mathcal{E}_{\alpha ij} \tag{29}$$

where α represents g or m.

In the next section, we will only use the momentum conservation of the matter to show that no stationary and axisymmetric star solution exists. Therefore, our result does not depend on the gravitational action.

3. No Stationary and Axisymmetric Star Solutions

In this section, we show a no-go theorem for stationary and axisymmetric star solutions in projectable Hořava–Lifshitz gravity. To prove it, we assume that a star is filled with a perfect fluid, which does not have the radial component of the four-velocity, that it has the reflection symmetry about the equatorial plane, that the energy density is a piecewise-continuous and non-negative function of the pressure, that the pressure is a continuous function of r and that the pressure at the center of the star is positive.

3.1. Stationary and Axisymmetric Configuration

We consider stationary and axisymmetric configurations with the timelike and spacelike Killing vectors, respectively, given by:

$$t^\mu \partial_\mu = \partial_t \tag{30}$$

$$\phi^\mu \partial_\mu = \partial_\phi \tag{31}$$

Under the stationary configurations, the lapse function, N, does not depend on t. In the original theory, the projectability condition $N = N(t)$ is proposed [1]. This condition means that the lapse function, N, does not depend on the spatial coordinates, x^i, but only can do so on the temporal coordinate, t. Thus, the lapse function, N, is a constant.

The timelike Killing vector, t^μ, implies everywhere:

$$N^2 - N_i N^i > 0 \tag{32}$$

The spacelike Killing vector, ϕ^μ, implies that:

$$\phi^\mu \phi_\mu = g_{\phi\phi} \tag{33}$$

is a geometrical invariant.

As a part of the gauge condition, we take:

$$g_{r\theta} = g_{r\phi} = 0 \tag{34}$$

Under this gauge condition, the general form for the spatial line element is described by [34]:

$$dl^2 = \psi^4 [A^2 dr^2 + \frac{r^2}{B^2} d\theta^2 + r^2 B^2 (\sin\theta d\phi + \xi d\theta)^2] \tag{35}$$

where ψ, A, B and ξ are functions of r and θ, but neither t nor ϕ for stationarity and axisymmetry.

Now we assume that the spacetime has a rotation axis, where $\sin\theta = 0$. This means:

$$\phi^\mu \phi_\mu = g_{\phi\phi} = 0 \tag{36}$$

there [35].

3.2. Triad Components of Shift Vector

We define triad basis vectors $\{\boldsymbol{e}_{(i)}\}$. $\boldsymbol{e}_{(1)}$ is along the radial direction; $\boldsymbol{e}_{(3)}$ is along the axial Killing vector and $\boldsymbol{e}_{(2)}$ is fixed by the orthonormality and the right-hand rule. The coordinate components for the orthonormal triad are:

$$e^i_{(1)} = \frac{1}{\psi^2} \left[\frac{1}{A}, 0, 0 \right] \tag{37}$$

$$e^i_{(2)} = \frac{1}{\psi^2} \left[0, \frac{B}{r}, -\frac{\xi B}{r \sin\theta} \right] \tag{38}$$

$$e^i_{(3)} = \frac{1}{\psi^2} \left[0, 0, \frac{1}{rB\sin\theta} \right] \tag{39}$$

where we have used the spatial line element (35). The projection of the shift vector on the triad is related to its coordinate components by:

$$N_{(1)} = \frac{N_r}{\psi^2 A} \tag{40}$$

$$N_{(2)} = \frac{N_\theta B}{\psi^2 r} - \frac{N_\phi \xi B}{\psi^2 r \sin\theta} \tag{41}$$

$$N_{(3)} = \frac{N_\phi}{\psi^2 r B \sin\theta} \tag{42}$$

3.3. Regularity Conditions at the Origin

Here, we give the regularity conditions of the shift vector, N^i, near the origin. A tensorial quantity is regular at $r = 0$ if and only if all its components can be expanded in non-negative integer powers of x, y and z in locally Cartesian coordinates, defined by:

$$x \equiv r \sin\theta \cos\phi \tag{43}$$

$$y \equiv r \sin\theta \sin\phi \tag{44}$$

$$z \equiv r \cos\theta \tag{45}$$

The Lie derivative of the shift vector, N^i, along the spacelike Killing vector vanishes, or:

$$N^i{}_{,j}\phi^j - \phi^i{}_{,j}N^j = 0 \tag{46}$$

In locally Cartesian coordinates, the spacelike Killing vector is written as:

$$\phi^i\partial_i = -y\partial_x + x\partial_y \tag{47}$$

Then, its components of Equation (46) are:

$$-N^x_{,x}y + N^x_{,y}x + N^y = 0 \tag{48}$$

$$-N^y_{,x}y + N^y_{,y}x - N^x = 0 \tag{49}$$

$$-N^z_{,x}y + N^z_{,y}x = 0 \tag{50}$$

The general regular solution of these equations is:

$$N^x = F_1(z, \rho^2)x - F_2(z, \rho^2)y \tag{51}$$

$$N^y = F_1(z, \rho^2)y + F_2(z, \rho^2)x \tag{52}$$

$$N^z = F_3(z, \rho^2) \tag{53}$$

where F_1, F_2 and F_3 are independent and regular functions, which depend on z and $\rho^2 \equiv x^2 + y^2$.

Now, transforming N^i back to the spherical coordinates, r, θ and ϕ, we get the spherical components:

$$\frac{N^r}{r} = \sin^2\theta F_1 + \frac{1}{r}\cos\theta F_3 \tag{54}$$

$$\frac{N^\theta}{\sin\theta} = \cos\theta F_1 - \frac{F_3}{r} \tag{55}$$

$$N^\phi = F_2 \tag{56}$$

On the rotation axis ($\sin\theta = 0$), thus, we obtain:

$$N^\theta = 0 \tag{57}$$

Using Equations (35), (37)–(39) and (54)–(56), the triad components are given by:

$$N_{(1)} = \psi^2 A(r\sin^2\theta F_1 + \cos\theta F_3) \tag{58}$$

$$N_{(2)} = \frac{\psi^2}{B}\sin\theta(r\cos\theta F_1 - F_3) \tag{59}$$

$$N_{(3)} = \psi^2 B\sin\theta(r\xi\cos\theta F_1 - \xi F_3 + rF_2) \tag{60}$$

Here, we additionally assume the reflection symmetry about the equatorial plane $z = 0$ or $\theta = \pi/2$. Then, N^x and N^y must be even functions of z, and N^z must be an odd function of z. This implies that F_1, F_2 must be even functions of z, and F_3 must be an odd function of z. Since N^r is an odd function of z on the rotation axis ($\sin\theta = 0$), we get:

$$N^r = 0 \tag{61}$$

at the origin.

3.4. Matter Sector and Momentum Conservation

For simplicity, we assume that the matter consists of a perfect fluid. The stress-energy tensor is given by:

$$T_{\mu\nu} = (\rho + P)u_\mu u_\nu + P g_{\mu\nu} \tag{62}$$

where P and ρ represent the pressure and the energy density, respectively. We assume the four-velocity given by:

$$\begin{aligned} u^\mu \partial_\mu &= \frac{1}{D}(t^\mu + \omega\phi^\mu)\partial_\mu \\ &= \frac{1}{D}\partial_t + \frac{\omega}{D}\partial_\phi \end{aligned} \tag{63}$$

where:

$$D \equiv (N^2 - N_i N^i - 2\omega N_\phi - \omega^2 g_{\phi\phi})^{\frac{1}{2}} \tag{64}$$

is the normalization factor and ω is a function of r and θ. For the four-velocity, u^μ, to be timelike, we shall have $N^2 - N_i N^i - 2\omega N_\phi - \omega^2 g_{\phi\phi} > 0$.

We set $\alpha = m$, and then, the momentum conservation equation (29) of the matter becomes:

$$0 = -\frac{1}{N}N^j D_j(T_{i\mu}n^\mu) + K T_{i\mu}n^\mu - \frac{1}{N}T_{j\mu}n^\mu D_i N^j - D^j T_{ij} \tag{65}$$

After some calculation, we obtain the r component:

$$0 = -P_{,r} + \frac{\rho + P}{D^2}\left\{\frac{1}{2}(N_i N^i)_{,r} + \omega N_{\phi,r} + \frac{1}{2}\omega^2 g_{\phi\phi,r} + \frac{N_{,r}}{N}N^r N_r + \frac{N_{,\theta}}{N}N^\theta N_r\right\} \tag{66}$$

Now, we use the projectability condition $N = N(t)$. As we mentioned above, the projectability condition means that the lapse function, N, does not depend on the spatial coordinates, x^i, but only can do on the temporal coordinate, t. Thus, the r component of the momentum conservation equation (66) becomes:

$$0 = -P_{,r} + \frac{\rho + P}{D^2}\left\{\frac{1}{2}(-N^2 + N_i N^i)_{,r} + \omega N_{\phi,r} + \frac{1}{2}\omega^2 g_{\phi\phi,r}\right\} \tag{67}$$

We do not use the θ and ϕ components to prove that no stationary and axisymmetric star exists.

Here, we concentrate on the r component of the momentum conservation of the matter on the rotation axis $\sin\theta = 0$. On the rotation axis, $g_{\phi\phi}$ and $g_{\phi\phi,r}$ vanish from Equation (36). From Equation (42), the regularity of the triad component of the shift vector, $N_{(3)}$, implies:

$$N_\phi = 0 \tag{68}$$

on the rotation axis. Thus, $N_{\phi,r} = 0$. Thus, the r component of the momentum conservation equation (67) on the rotation axis becomes:

$$0 = -P_{,r} - \frac{1}{2}\frac{(\rho + P)(N^2 - N_i N^i)_{,r}}{N^2 - N_i N^i} \tag{69}$$

3.5. Contradiction of Momentum Conservation

We assume that the star has the reflection symmetry about the equatorial plane $\theta = \frac{\pi}{2}$, that the energy density, ρ, is a piecewise-continuous and non-negative function of the pressure, P, that the pressure, P is a continuous function of r and that the pressure at the center of the star $P_c \equiv P(r = 0)$ is positive. Thus, $\rho + P$ is a piecewise-continuous function of r. We have assumed that the energy density, ρ, is non-negative everywhere and that the pressure at the center, P_c, is positive; hence, $\rho + P$ is positive at the center. We define r_s as the minimal value of r for which at least one of $(\rho + P)|_{r=r_s}$, $\lim_{r\to r_s-0}(\rho + P)$ and $\lim_{r\to r_s+0}(\rho + P)$ is nonpositive.

Dividing the momentum conservation equation (69) by $\frac{1}{2}(\rho + P)$, we have:

$$\left\{\log\left(N^2 - N_i N^i\right)\right\}_{,r} = -2\frac{P_{,r}}{\rho + P} \tag{70}$$

Under the assumption that the energy density is a function of the pressure, $\rho = \rho(P)$, integrating the momentum conservation equation (70) over the interval $0 \leq r < r_s$, we obtain:

$$\log\left(N^2 - N_i N^i\right)\big|_{r=r_s} - \log\left(N^2 - N_i N^i\right)\big|_{r=0} = -2\int_{P_c}^{P_s} \frac{dP}{\rho + P} \tag{71}$$

where $P_s \equiv P(r = r_s)$.

The definition of r_s implies that at least one of $(\rho + P)|_{r=r_s}$, $\lim_{r\to r_s-0}(\rho + P)$ and $\lim_{r\to r_s+0}(\rho + P)$ is nonpositive. Since we have assumed that $P(r)$ is a continuous function and that ρ is non-negative everywhere, $P_s = \lim_{r\to r_s-0} P = \lim_{r\to r_s+0} P$ is non-positive. Thus, we get:

$$P_s \leq 0 < P_c \tag{72}$$

This implies that the right-hand side of Equation (71) is positive. However, the left-hand side of Equation (71) is nonpositive, since we have the projectability condition $N = N(t)$ and we obtain from Equations (57), (61) and (68):

$$N_i N^i\big|_{r=0} = 0 \tag{73}$$

at the center of the star. This contradicts that the right-hand side of Equation (71) is positive.

4. Discussion and Conclusions

Hořava–Lifshitz gravity is only covariant under the foliation-preserving diffeomorphism. This means that the foliation of the spacetime manifold by the constant-time hypersurfaces has a physical meaning. As a result, the regularity condition at the center of a star is more restrictive than the one in a theory that has general covariance.

Under the assumption that a star is filled with a perfect fluid that has no radial motion, that it has reflection symmetry about the equatorial plane and that the matter sector obeys the physically reasonable conditions, we have shown that the momentum conservation is incompatible with the projectability condition and the regularity condition at the center for stationary and axisymmetric configurations. Since we have not used the gravitational action to prove it, our result is also true in other projectable theories [5,32]. Note that our result is true under not only strong-gravity circumstances, like neutron stars, but also weak-gravity ones, like planets or moons. However, it is not certain that star solutions can exist in non-projectable theories. Since we have used the covariance under the foliation-preserving diffeomorphism, the projectability condition and the assumptions of the matter sector to prove the no-go theorem for stationary and axisymmetric stars, our proof will not apply if we do not assume all the above.

Izumi and Mukohyama found that no spherically symmetric and static solution filled with a perfect fluid without radial motion exists in this theory under the assumption that the energy density is a piecewise-continuous and non-negative function of the pressure and that the pressure at the center is positive [29]. They concluded that a spherically symmetric star should include a time-dependent region near the center. Although we cannot deny that stars should be described by dynamical configurations, the fact that we cannot find simple stationary and axisymmetric star solutions with the four-velocity generated by the Killing vectors will be an unattractive feature of this theory.

Greenwald, Papazoglou and Wang found static spherically symmetric solutions with a perfect fluid plus a heat flow along the radial direction and with a class of an anisotropic fluid under the assumption that the spatial curvature is constant in a projectable theory without the detailed balance condition [30], although it is doubtful that the constant-spatial-curvature solutions represent realistic stars. This, however, implies that rotating star solutions with a perfect fluid plus a radial heat flow and with an anisotropic fluid can also exist.

We might get star solutions by introducing an exotic matter with a negative pressure, but it seems that the physical justification to introduce it is difficult.

Our result does not imply the non-existence of rotation star solutions in this theory. However, it would be useful to investigate rotating-star solutions in this theory and then to compare the solutions with the corresponding ones in Einstein gravity for astrophysical tests of this theory. Furthermore, although we do not disprove the existence of rotation star solutions with radial motion, it is doubtful whether such star solutions describe realistic astrophysical stars.

Recently, the property of matter in the non-projectable version of the extended Hořava–Lifshitz gravity [28] at both classical and quantum levels has been investigated by Kimpton and Padilla [36]. Although the gravity sector in Hořava–Lifshitz has been investigated eagerly, the matter sector has not, relatively. It is left as future work to answer the question of whether or not the no-go theorem applies at a quantum level.

Acknowledgments

The authors would like to thank M. Saijo, U. Miyamoto, S. Kitamoto, N. Shibazaki, T. Kuroki, S. Mukohyama, K. Izumi, M. Nozawa, R. Nishikawa, M. Shimano, H. Nemoto, S. Kamata, S. Okuda and T. Wakabayashi for valuable comments and discussion. Naoki Tsukamoto thanks the Yukawa Institute for Theoretical Physics at Kyoto University, where this work was initiated during the YITP-W-11-08 on "Summer School on Astronomy and Astrophysics 2011". Tomohiro Harada was supported by the Grant-in-Aid for Scientific Research Fund of the Ministry of Education, Culture, Sports, Science and Technology, Japan [Young Scientists (B) 21740190].

Conflicts of Interest

The authors declare no conflict of interest.

Appendix

A. Explicit Expression for Equation of Motion

After a long straightforward calculation, we obtain the explicit expression for the equation of motion:

$$\begin{aligned}
&\alpha\left[\frac{N}{2}K^{lm}K_{lm}g^{ij}-2NK^{im}K_m^j-\frac{1}{\sqrt{g}}(\sqrt{g}K^{ij})-D_p(K^{ip}N^j)-D_p(K^{pj}N^i)+D_p(K^{ij}N^p)\right]\\
&-\alpha\lambda\left[\frac{N}{2}K^2g^{ij}-2NKK^{ij}-\frac{1}{\sqrt{g}}(\sqrt{g}Kg^{ij})-D_p(Kg^{ip}N^j)-D_p(Kg^{jp}N^i)+D_p(KN^pg^{ij})\right]\\
&+\beta\Big[-\frac{1}{2}NC^{kl}C_{kl}g^{ij}+2NC^{jl}C_l^i+2\varepsilon^{pkl}R_l^jD_k(NC_p^i)-\varepsilon^{pki}D_mD^jD_k(NC_p^m)-\varepsilon^{pkl}D_lD^jD_k(NC_p^i)\\
&+\varepsilon^{pkj}D^lD_lD_k(NC_p^i)+\varepsilon^{pkl}g^{ij}D_mD_lD_k(NC_p^m)-\varepsilon^{kil}D_p(NC_k^jR_l^p)-\varepsilon^{pkl}D_k(NC_p^jR_l^i)+\varepsilon^{pil}D_k(NC_p^kR_l^j)\,\Big]\\
&+\gamma\Big[\varepsilon^{pqk}D_pD^i(ND_qR_k^j+\frac{1}{2}R_k^jD_qN)+\varepsilon^{jqk}D_lD^i(ND_qR_k^l+\frac{1}{2}R_k^lD_qN)-\varepsilon^{iqk}D^lD_l(ND_qR_k^j+\frac{1}{2}R_k^jD_qN)\\
&-\varepsilon^{pqk}g^{ij}D_pD_l(ND_qR_k^l+\frac{1}{2}R_k^lD_qN)+\varepsilon^{pqk}R_k^jD_q(ND_qR_p^i)+\varepsilon^{ikp}D_l(NR_p^lR_k^j)\Big]\\
&+\zeta\left[\frac{1}{2}NR_{kl}R^{kl}g^{ij}-2NR^{il}R_l^j+2D_kD^j(NR^{ki})-D^lD_l(NR^{ij})-g^{ij}D_kD_l(NR^{kl})\right]\\
&+\eta\left[\frac{1}{2}NR^2g^{ij}-2NRR^{ij}+2D^iD^j(NR)-2g^{ij}D^lD_l(NR)\right]\\
&+\xi\left[\frac{1}{2}NRg^{ij}-NR^{ij}+D^jD^iN-g^{ij}D^lD_lN\right]+\sigma N\frac{1}{2}g^{ij}+(i\leftrightarrow j)+\frac{2}{\sqrt{g}}\frac{\delta I_m}{\delta g_{ij}}=0
\end{aligned}\qquad\text{(A1)}$$

where $(i\leftrightarrow j)$ means the terms, i and j, exchanged each other.

B. Triad Components of Extrinsic Curvature Tensor

In this theory, the triad components of the extrinsic curvature tensor also should be regular. The Lie derivative of g_{ij} along N^i is:

$$\begin{aligned}
\mathcal{L}_{\boldsymbol{N}}g_{ij} &= D_jN_i+D_iN_j\\
&= g_{ik}N^k_{\ ,j}+g_{jk}N^k_{\ ,i}+g_{ij,k}N^k
\end{aligned}\qquad\text{(A2)}$$

The extrinsic curvature tensor (6) and (A2) yield:

$$\frac{dg_{ij}}{dt}-N^k_{\ ,i}g_{jk}-N^k_{\ ,j}g_{ki}=2NK_{ij}\qquad\text{(A3)}$$

where:

$$\frac{d}{dt} \equiv \frac{\partial}{\partial t} - N^i \frac{\partial}{\partial x^i} \tag{A4}$$

By projecting Equation (A3) onto the triad (37)–(39), we obtain the following equations [34]:

$$NK_{(1)(1)} = -N^r_{,r} + \frac{1}{A}\frac{dA}{dt} + \frac{2}{\psi}\frac{d\psi}{dt} \tag{A5}$$

$$\frac{2NK_{(1)(2)}}{\sin\theta} = \frac{AB}{r}N^r{}_{,X} - \frac{r}{AB\sin\theta}N^\theta_{,r} \tag{A6}$$

$$\frac{2NK_{(1)(3)}}{\sin\theta} = -\frac{rB}{A}[N^\phi_{,r} + \frac{\xi}{\sin\theta}N^\theta_{,r}] \tag{A7}$$

$$NK_{(2)(2)} = \frac{1}{r}\frac{dr}{dt} + \frac{2}{\psi}\frac{d\psi}{dt} - \frac{1}{B}\frac{dB}{dt} - N^\theta{}_{,\theta} \tag{A8}$$

$$NK_{(3)(3)} = \frac{1}{r}\frac{dr}{dt} + \frac{2}{\psi}\frac{d\psi}{dt} + \frac{1}{B}\frac{dB}{dt} - \frac{\cos\theta}{\sin\theta}N^\theta \tag{A9}$$

$$2NK_{(2)(3)} = B^2\frac{d\xi}{dt} + (1-X^2)B^2(N^\phi{}_{,X} + \frac{\xi}{\sin\theta}N^\theta{}_{,X}) \tag{A10}$$

where:

$$X \equiv \cos\theta \tag{A11}$$

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