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Article

The Truncated Lognormal Distribution as a Luminosity Function for SWIFT-BAT Gamma-Ray Bursts

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Abstract: The determination of the luminosity function (LF) in Gamma ray bursts (GRBs) depends on the adopted cosmology, each one characterized by its corresponding luminosity distance. Here, we analyze three cosmologies: the standard cosmology, the plasma cosmology and the pseudo-Euclidean universe. The LF of the GRBs is firstly modeled by the lognormal distribution and the four broken power law and, secondly, by a truncated lognormal distribution. The truncated lognormal distribution fits acceptably the range in luminosity of GRBs as a function of the redshift.

Keywords: cosmology; observational cosmology; distances, redshifts, radial velocities and spatial distribution of galaxies

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1. Introduction

The number of Gamma-ray bursts (GRBs) for which we know the redshift and the flux is 760, according to the SWIFT-BAT catalog of [1], available at the Centre de Données Astronomiques de Strasbourg (CDS), with the name J/ApJS/207/19. The above catalog gives the hard X-ray flux, the spectral index, the redshift and the X-ray luminosity. The luminosity data of this catalog, which is a theoretical evaluation, are given in the framework of the ΛCDM cosmology with $H_0 = 70\ \mathrm{km\,s^{-1}\,Mpc^{-1}}$, $\Omega_{\mathrm{M}} = 0.3$ and $\Omega_\Lambda = 0.7$. A calibration and a comparison can be done with the models for luminosity here implemented. This large number of observed objects allows applying different cosmologies in order to find the luminosity and the luminosity function (LF) for GRBs. At the moment of writing, the standard cosmology is the ΛCDM cosmology, but other cosmologies, such as the plasma or the pseudo-Euclidean cosmology, can also be analyzed. Once the luminosity is obtained, we can model the LF by adopting the lognormal distribution (see [2,3]) and by a four broken power law.

In the hypothesis that the luminosity of a GRB is due to the early phase of a supernova (SN), the minimum and maximum are due to the various parameters that drive the SN's light curve; see [4].

2. Preliminaries

This section analyses the luminosity in the ΛCDM cosmology, in the plasma cosmology and in the pseudo-Euclidean cosmology. Careful attention should be paid to the multiplicative effects of the main models (3) for the empirical catalogs of SNs (2), which means six different cases to be analyzed; see Figure 1.

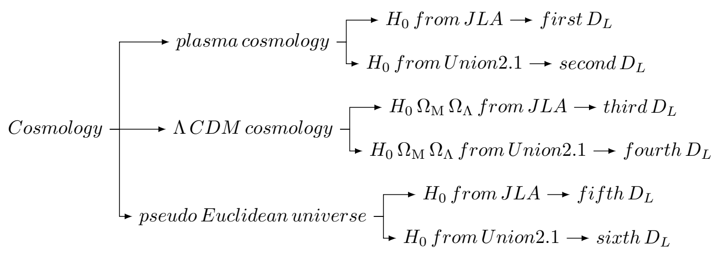

Figure 1. Flowchart for the luminosity distances analyzed here.

2.1. Observed Luminosity

In the framework of the standard cosmology, the received flux, f, is:

$$f = \frac{L}{4\,\pi D_L(z)^2}, \tag{1}$$

where $D_L(z)$ is the luminosity distance, which depends on the parameters of the adopted cosmological model and z is the redshift. As a consequence, the luminosity is:

$$L = 4\pi D_L(z)^2 f. \tag{2}$$

The above formula is then corrected by a k-correction, $k(z,\gamma)$, where:

$$k(z,\gamma) = \frac{\int_{1\mathrm{keV}}^{10^4\mathrm{keV}} C\,E'^{-\gamma}E'dE'}{\int_{15(1+z)\mathrm{keV}}^{150(1+z)\mathrm{keV}} C\,E'^{-\gamma}E'dE'}, \tag{3}$$

where C is a constant and γ is the observed spectral index in energy; see [5] for more details. The corrected luminosity is therefore:

$$L = 4\pi D_L(z)^2 f\,k(z,\gamma). \tag{4}$$

In the case of the survey from the 70 month SWIFT-BAT, the flux f is given in $\frac{fW}{m^2}$ and γ and z are positive numbers; see [1]; Table 1 reports a test GRB.

Table 1. Test Gamma-ray burst (GRB).

SWIFT Name	Flux in $\frac{fW}{m^2}$	γ	z	$\log(L(\mathrm{erg\,s^{-1}}))$
J0017.1+8134	10.12	2.53	3.3660	48.01

2.2. Luminosity in the Standard Cosmology

The luminosity distance, D_L, in the ΛCDM cosmology can be expressed in terms of a Padé approximant, once we provide the Hubble constant, H_0, expressed in $\mathrm{km\,s^{-1}\,Mpc^{-1}}$, the velocity of light, c, expressed in $\mathrm{km\,s^{-1}}$, and the three numbers Ω_M, Ω_K and Ω_Λ; see [6] for more details or Table 2.

A further application of the minimax rational approximation, which is characterized by the two parameters p and q, allows finding a simplified expression for the luminosity distance; see Equations (33a) and (33b) in [6]. The above minimax approximation when $p = 3$, $q = 2$ is:

$$D_{\mathrm{L},3,2} = \frac{p_0 + p_1\,z + p_2\,z^2 + p_3\,z^3}{q_0 + q_1\,z + q_2\,z^2}\,\mathrm{Mpc}, \tag{5}$$

and Table 3 reports the coefficients for the two compilations used here.

Table 2. Numerical values of the ΛCDM cosmology.

Compilation	H_0 in km s^{-1} Mpc^{-1}	Ω_{M}	Ω_{Λ}
Union 2.1	69.81	0.239	0.651
JLA	69.398	0.181	0.538

Table 3. Numerical values of the seven coefficients of the minimax approximation for the Union 2.1 compilation and the JLA compilation.

Coefficient	Union 2.1	JLA
p_0	0.3597252600	0.4429883062
p_1	5.612031882	6.355991909
p_2	5.627811123	5.405310650
p_3	0.05479466285	0.04413321265
q_0	0.010587821	0.0129850304
q_1	0.1375418627	0.1546989174
q_2	0.1159043801	0.1097492834

The monochromatic luminosity, X-band (14–195 keV), without $k - z$ correction, $\log(L_{3,2})_b$ according to Equation (2) is:

$$\log(L_{3,2}(\mathrm{erg\,s^{-1}}))_b = 0.43429\,\ln\left(1.1964\,\frac{\mathit{fluxfwm2}\,(16.6843+(194.6669+(1878.8341+180.34010\,z)z)z)^2}{(0.08644+(0.2578-0.00849\,z)z)^2}\right) +38.0\ \ \mathit{Union}\ 2.1. \tag{6}$$

In the case of a test GRB with the parameters as in Table 1, the above formula gives $\log(L) = 48.13$ against $\log(L_{SWIFT}) = 48.01$ of the SWIFT-BAT catalog. The goodness of the approximation is evaluated through the percentage error, η, which is:

$$\eta = \frac{\left|\log(L_{3,2}(\mathrm{erg\,s^{-1}}))_b - \log(L_{SWIFT})\right|}{\log(L_{SWIFT})} \times 100, \tag{7}$$

and over all of the elements of the SWIFT-BAT catalog $2.28\,10^{-5}\,\% \leq \eta \leq 0.295\%$. We now report an expression for the luminosity of a GRB, Equation (4), based on the minimax approximation when the Union 2.1 compilation is considered:

$$\log(L_{3,2}(\mathrm{erg\,s^{-1}})) = 41.5647 + 0.4342\,\ln\left(32522\,\frac{\mathit{fluxfwm2}\,(z+10.3144)^2\left(z^2+0.10378\,z+0.0089695\right)^2}{(-0.08644-0.2578\,z+0.008491\,z^2)^2}\right) + \left(\frac{-1+0.5\,\gamma}{(1+z)^2\left((15+15\,z)^{-\gamma}-100\,(150+150\,z)^{-\gamma}\right)}\right)\ \mathit{Union}\ 2.1, \tag{8}$$

where fluxfwm2 is the flux expressed in $\frac{fW}{\mathrm{m}^2}$.

In the case of a test GRB with the parameters as in Table 1, the above formula gives $\log(L) = 54.512$, which means a bigger luminosity of ≈6 decades with respect to the band luminosity. Figure 2 reports the luminosity-redshift distribution for the SWIFT-BAT survey, as well as a theoretical lower curve, which can be found by inserting the minimum flux in Equation (8).

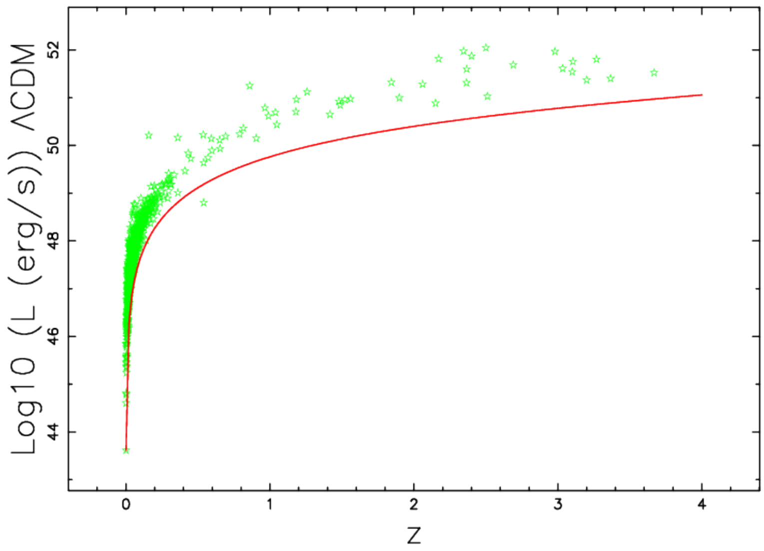

Figure 2. Luminosity in the ΛCDM cosmology versus redshift for 784 GRB as given by the 70-month SWIFT-BAT survey (green points) and the theoretical curve for the lowest luminosity at a given redshift (red curve); see Equation (8).

Another useful quantity is the angular diameter distance, D_A, which is:

$$D_A = \frac{D_L}{(1+z)^2}, \tag{9}$$

(see [7]), and therefore:

$$D_{A,3,2} = \frac{D_{L,3,2}}{(1+z)^2}. \tag{10}$$

2.3. *Luminosity in the Plasma Cosmology*

The distance d in the plasma cosmology has the following dependence:

$$d(z) = \frac{\ln(z+1)\, c}{H_0}, \tag{11}$$

see [8–11] and Table 4.

Table 4. Numerical values of H_0 in km s^{-1} Mpc^{-1} (plasma cosmology) for the Union 2.1 compilation and the JLA compilation.

Union 2.1	JLA
$H_0 = 74.2 \pm 0.24$	$H_0 = 74.45 \pm 0.2$

The monochromatic luminosity, X-band (14–195 keV), is:

$$\log(L(z)) = \frac{\ln\left(19531902.82\, fluxfwm2\ (\ln(1+z))^2\right)}{\ln(10)} + 38. \tag{12}$$

In the case of a test GRB with the parameters as in Table 1, the above formula gives $\log(L) = 46.63$, which is a lower value than the $\log(L_{SWIFT}) = 48.01$ of the SWIFT-BAT catalog.

The luminosity in the case of the absence of absorption is:

$$L(z) = 4\pi d(z)^2 f\, k(\gamma), \tag{13}$$

where the $k(\gamma)$ correction is:

$$k(\gamma) = \frac{\int_{1\mathrm{keV}}^{10^4\mathrm{keV}} C\, E'^{-\gamma} E' dE'}{\int_{15\mathrm{keV}}^{150\mathrm{keV}} C\, E'^{-\gamma} E' dE'}. \tag{14}$$

There is no relativistic correction in the denominator because the plasma cosmology is both static and Euclidean. Figure 3 reports the luminosity in the plasma cosmology as a function of the redshift, as well as the theoretical luminosity.

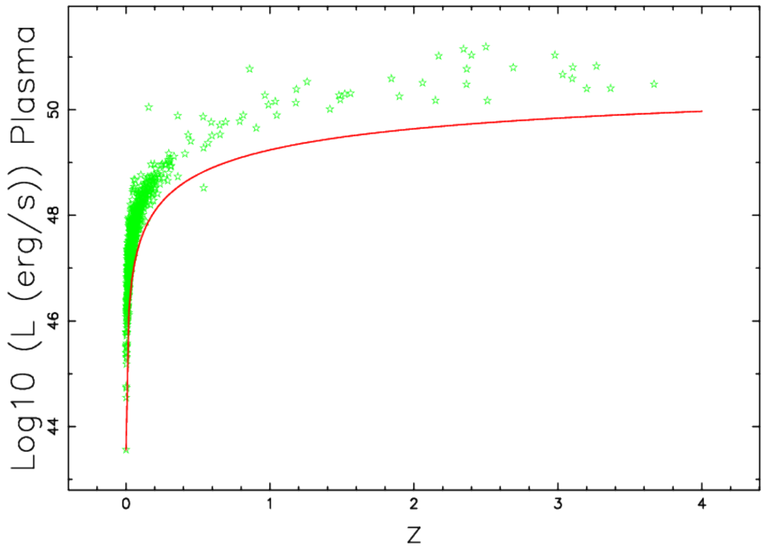

Figure 3. Luminosity in the plasma cosmology versus redshift for 784 GRBs as given by the 70-month SWIFT-BATsurvey (green points) and the theoretical curve for the lowest luminosity at a given redshift (red curve); see Equation (14).

2.4. Luminosity in the Pseudo-Euclidean Cosmology

The distance d in the pseudo-Euclidean cosmology has the following dependence:

$$d(z) = \frac{zc}{H_0}, \tag{15}$$

and we used $H_0 = 67.93\ \mathrm{km\,s^{-1}\,Mpc^{-1}}$; see Table 5.

Table 5. Numerical values of H_0 in $\mathrm{km\,s^{-1}\,Mpc^{-1}}$ (pseudo-Euclidean cosmology) for the Union 2.1 compilation and the JLA compilation when the redshift covers the range $[0, 0.1]$.

Union 2.1	**JLA**
$H_0 = 67.93 \pm 0.38$	$H_0 = 67.51 \pm 0.42$

The above formula gives approximate results up to $z \ll 1.0$. The monochromatic luminosity, X-band (14–195 keV), is:

$$L(z) = 4\pi d(z)^2 f, \tag{16}$$

where the $k(z)$ correction is absent or:

$$\log(L(z)) = \frac{\ln\left(19531902.82\, \mathit{fluxfwm2}\, z^2\right)}{\ln(10)} + 38. \tag{17}$$

2.5. High versus Low z

The differences between the four distances used here, which are the luminosity distance and the angular-diameter distance in the ΛCDM, the plasma cosmology distance and the pseudo-Euclidean cosmology distance, can be outlined in terms of a percentage difference, Δ. As an example for D_{A},

$$\Delta = \frac{|D_{\mathrm{L}}(z) - D_{\mathrm{A}}(z)|}{D_{\mathrm{L}}(z)} \times 100. \tag{18}$$

Figure 4 reports the four distances, and for $z \leq 0.05$, the three percentage differences are lower than 10%. In the framework of the two Euclidean distances, the plasma and the pseudo-Euclidean one, for $z \leq 0.15$, the percentage difference is lower than 10%.

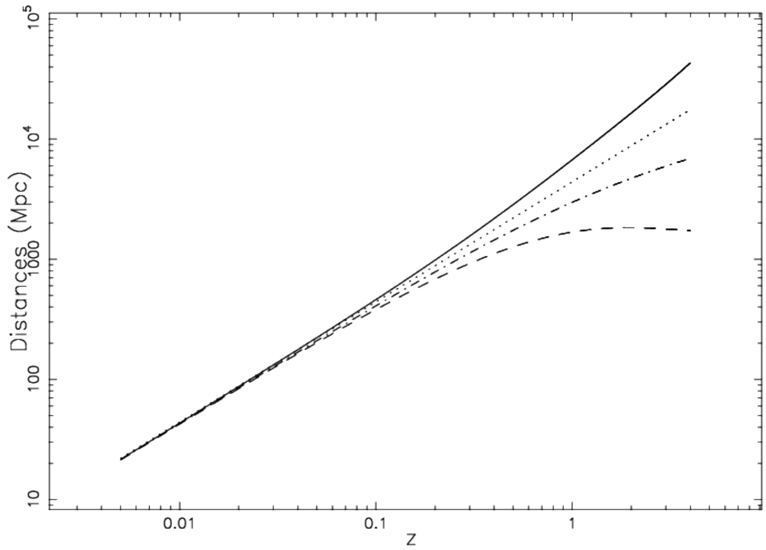

Figure 4. The distances adopted here: luminosity distance, D_L, in ΛCDM (full line), angular-diameter distance, D_A, in ΛCDM (dash line), plasma cosmology distance, d (dot-dash-dot-dash line), and pseudo-Euclidean cosmology distance (dotted line).

Therefore, the boundary between low and high z can be fixed at $z = 0.05$.

3. Two Existing Distributions

This section reviews the four broken power law distribution and the lognormal distribution and derives an analytical expression for the number of GRBs for a given flux in the linear and non-linear cases.

3.1. The Four Broken Power Law Distribution

The four broken power law has the following piecewise dependence:

$$p(L) \propto L^{\alpha_i}, \tag{19}$$

each of the four zones being characterized by a different exponent α_i. In order to have a PDF normalized to unity, one must have:

$$\sum_{i=1,4} \int_{L_i}^{L_{i+1}} c_i L^{\alpha_i} dL = 1. \tag{20}$$

For example, we start with c_1=1: c_2 will be determined by the following equation:

$$c_1(L_2 - \epsilon)^{\alpha_1} = c_2(L_2 + \epsilon)^{\alpha_2}, \tag{21}$$

where ϵ is a small number, e.g., $\epsilon = \frac{L_2}{10^{+8}}$. This PDF is characterized by nine parameters and takes values L in the interval $[L_1, L_5]$.

3.2. Lognormal Distribution

Let L be a random variable taking values L in the interval $[0, \infty]$; the lognormal probability density function (PDF), following [12] or formula (14.2)′ in [13], is:

$$PDF(L; L^*, \sigma) = \frac{\sqrt{2} e^{-\frac{1}{2} \frac{1}{\sigma^2} \left(\ln\left(\frac{L}{L^*}\right)\right)^2}}{2\, L\sigma\, \sqrt{\pi}}, \tag{22}$$

where $L^* = \exp \mu_{LN}$ and $\mu_{LN} = \ln L^*$. The mean luminosity is:

$$E(L; L^*, \sigma) = L^*\, e^{\frac{1}{2}\sigma^2}, \tag{23}$$

and the variance, Var, is:

$$Var(L^*, \sigma) = e^{\sigma^2} \left(-1 + e^{\sigma^2}\right) L^{*2}. \tag{24}$$

The distribution function (DF) is:

$$DF(L; L^*, \sigma) = \frac{1}{2} + \frac{1}{2} \operatorname{erf}\left(\frac{1}{2} \frac{\sqrt{2}\, (\ln(L) - \ln(L^*))}{\sigma}\right), \tag{25}$$

where $\operatorname{erf}(z)$ is the error function; see [14]. A luminosity function for GRB, PDF_{GRB}, can be obtained by multiplying the lognormal PDF by Φ^*, which is the number of GRB per unit volume, Mpc^3 units for unit time, y units,

$$\Phi(L; L^*, \sigma) = \Phi^* \frac{\sqrt{2} e^{-\frac{1}{2} \frac{1}{\sigma^2} \left(\ln\left(\frac{L}{L^*}\right)\right)^2}}{2\, L\sigma\, \sqrt{\pi}} \frac{number}{\mathrm{Mpc}^3\, \mathrm{y}}. \tag{26}$$

A numerical value for the constant Φ^* can be obtained by dividing the number of GRBs, N_{GRB}, observed in a time, T, in a given volume V by the volume itself and by T, which is the time over which the phenomena are observed, in the case of SWIFT-BAT, 70 months; see [1],

$$\Phi^* = \frac{N_{GRB}}{V\, T} \mathrm{Mpc}^{-3} \mathrm{yr}^{-1}, \tag{27}$$

where the volume is different in the three cosmologies,

$$V = \frac{4}{3}\pi\left(\frac{cz}{H_0}\right)^3 \mathrm{Mpc}^3 \qquad pseudo-Euclidean\, cosmology \tag{28a}$$

$$V = \frac{4}{3}\pi\left(\frac{\ln(z+1)\, c}{H_0}\right)^3 \mathrm{Mpc}^3 \qquad plasma\, cosmology \tag{28b}$$

$$V = \frac{4}{3}\pi(D_{\mathrm{A},3,2})^3\, \mathrm{Mpc}^3 \qquad \Lambda\, CDM\, cosmology, \tag{28c}$$

where $D_{\mathrm{A},3,2}$ has been defined in Equation (10). The parameters of the fit for the four broken power law's PDF are reported in Table 6 when the luminosity is taken with the $k(z)$ correction; Figure 5.

The parameters of the fit for the lognormal PDF are reported in Table 7 when the luminosity is taken with the $k(z)$ correction.

Table 6. The 9 parameters of the four broken power laws for the ΛCDM cosmology where Equation (8) was used and the two parameters of the Kolmogorov–Smirnov (K–S) test D and P_{KS}.

Name	
L_1 in $\frac{L^*}{10^{51}\mathrm{erg\,s^{-1}}}$	4×10^{-8}
L_2 in $\frac{L^*}{10^{51}\mathrm{erg\,s^{-1}}}$	5×10^{-7}
L_3 in $\frac{L^*}{10^{51}\mathrm{erg\,s^{-1}}}$	6.3×10^{-6}
L_4 in $\frac{L^*}{10^{51}\mathrm{erg\,s^{-1}}}$	7.9×10^{-5}
L_5 in $\frac{L^*}{10^{51}\mathrm{erg\,s^{-1}}}$	9.8×10^{-4}
α_1	1.2
α_2	0.54
α_3	−0.23
α_4	−2.74
D	0.063
P_{KS}	0.507

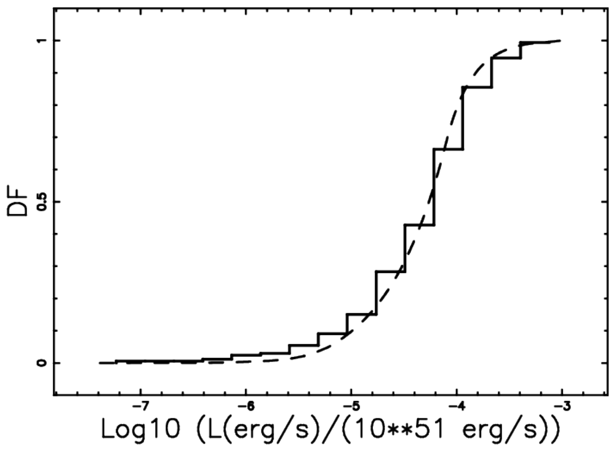

Figure 5. Observed DF (step-diagram) for GRB luminosity and superposition of the four broken power laws' DFs (line), the case of ΛCDM cosmology with the parameters as in Table 6.

Table 7. The 3 parameters of the luminosity function (LF) as modeled by the lognormal distribution for z in $[0, 0.02]$ with the Union 2.1 data and the two parameters of the K–S test D and P_{KS}. In the case of the plasma cosmology and the ΛCDM cosmology, we used the luminosity as given by Equations (13) and (8), respectively.

Parameter	Plasma Cosmology	ΛCDM Cosmology
$\frac{L^*}{10^{51}\mathrm{erg\,s^{-1}}}$	3.516×10^{-5}	4.055×10^{-5}
σ	1.42	1.42
$\frac{\Phi^*}{\mathrm{Mpc^{-3}yr^{-1}}}$	7.2524×10^{-8}	1.025×10^{-5}
D	0.089	0.090
P_{KS}	0.131	0.127

The case of LF modeled by a lognormal PDF with L as represented by a monochromatic luminosity in the X-band (14–195 keV) is reported in Table 8.

The goodness of the fit with the lognormal PDF has been assessed by applying the Kolmogorov–Smirnov (K–S) test [15–17]. The K–S test, as implemented by the FORTRAN subroutine KSONE in [18], finds the maximum distance, D, between the theoretical and the observed DF, as well as the significance level, P_{KS}; see Formulas 14.3.5 and 14.3.9 in [18]; the values of $P_{KS} \geq 0.1$ indicate that the fit is acceptable; see Table 7 for the results.

In the case of the ΛCDM cosmology, Figure 6 reports the lognormal DF, with the parameters as in Table 7.

Table 8. The 3 parameters of the LF, the case of the X-band (14–195 keV), as modeled by the lognormal distribution for z in $[0, 0.02]$ with the Union 2.1 data and the two parameters of the K–S test, D and P_{KS}. In the case of the plasma cosmology and the pseudo-Euclidean cosmology, we used the luminosity as given by Equations (12) and (16), respectively.

Parameter	Plasma Cosmology	Pseudo-Euclidean Cosmology
$\frac{L^*}{10^{51} \mathrm{erg\,s^{-1}}}$	5.9×10^{-9}	7.12×10^{-9}
σ	1.42	1.42
$\frac{\Phi^*}{\mathrm{Mpc^{-3} yr^{-1}}}$	1.01×10^{-5}	9.88×10^{-6}
D	0.089	0.089
P_{KS}	0.13	0.129

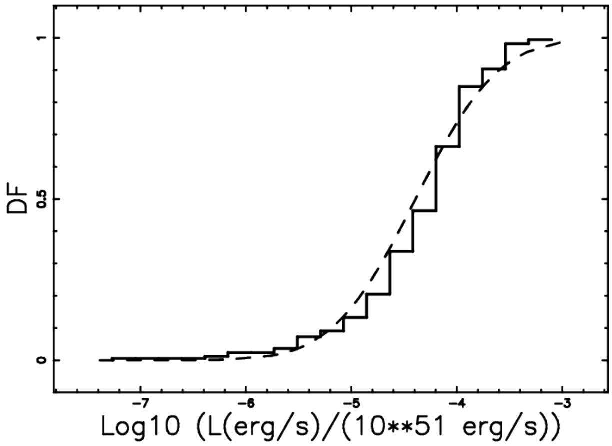

Figure 6. Observed distribution function (DF) (step-diagram) for GRB luminosity and superposition of the lognormal DF (line), the case of the ΛCDM cosmology with the parameters as in Table 7.

In the case of the ΛCDM cosmology, Figure 7 reports a comparison between the empirical distribution and the lognormal PDF, and Figure 6 reports the lognormal DF, with the parameters as in Table 7.

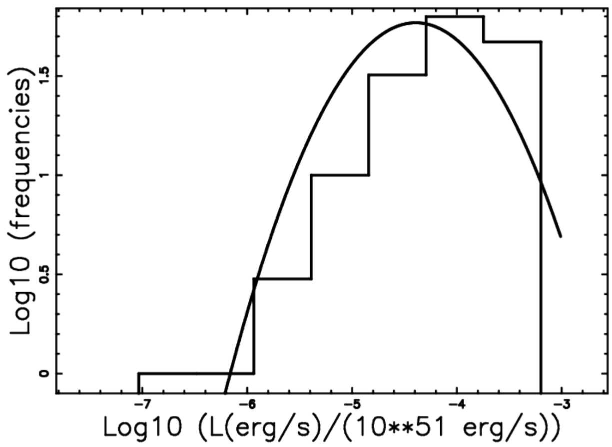

Figure 7. Log-log histogram (step-diagram) of GRB luminosity and superposition of the lognormal PDF (line), the case of the pseudo-Euclidean cosmology with the parameters as in Table 7.

The case of the plasma and pseudo-Euclidean cosmologies is covered in Figures 8 and 9, respectively.

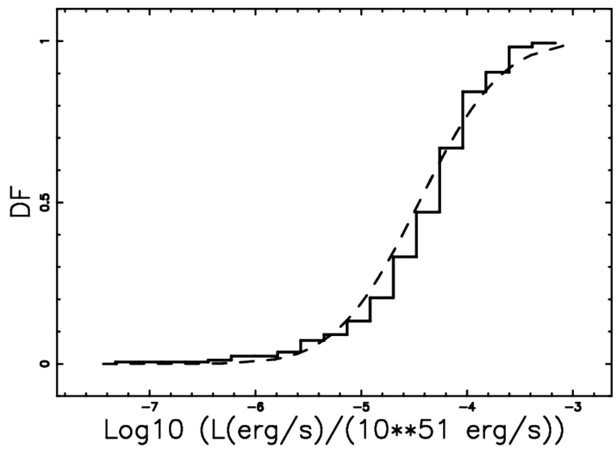

Figure 8. Observed DF (step-diagram) for GRB luminosity and superposition of the lognormal DF (line), the case of the plasma cosmology with the parameters as in Table 7.

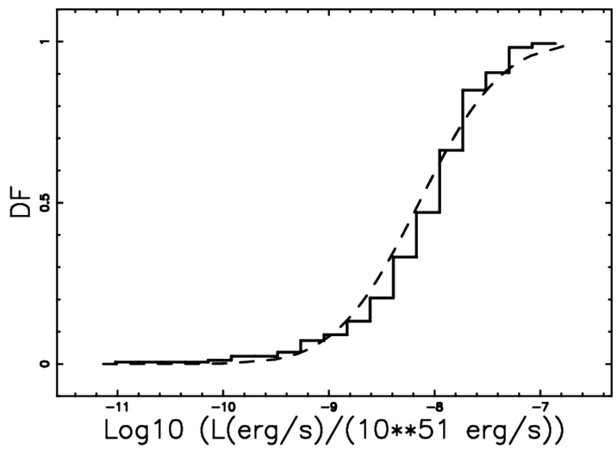

Figure 9. Observed DF (step-diagram) for GRB monochromatic luminosity, X-band (14–195 keV), and superposition of the lognormal DF (line), the case of the pseudo-Euclidean cosmology with the parameters as in Table 7.

3.3. The Linear Case

We assume that the flux, f, scales as $f = \frac{L}{4\pi r^2}$, according to Equation (15):

$$r = \frac{zc}{H_0}, \tag{29}$$

and:

$$z = \frac{r\,H_0}{c}. \tag{30}$$

The relation between the two differentials dr and dz is:

$$dr = \frac{c\,dz}{H_0}. \tag{31}$$

The joint distribution in z and f for the number of galaxies is:

$$\frac{dN}{d\Omega dz df} = \frac{1}{4\pi}\int_0^\infty 4\pi r^2 dr \Phi(\frac{L}{L^*})\delta(z-(\frac{r\,H_0}{c}))\delta(f-\frac{L}{4\pi r^2}), \tag{32}$$

where δ is the Dirac delta function. We now introduce the critical value of z, z_{crit}, which is:

$$z_{crit}^2 = \frac{H_0^2 L^*}{4\pi f c^2}. \tag{33}$$

The evaluation of the integral over luminosities and distances gives:

$$\frac{dN}{d\Omega dz df} = F(z; f, \Phi^*, L^*, \sigma) = \frac{z^2 c^3 \sqrt{2} e^{-\frac{1}{2}\frac{1}{\sigma^2}\left(\ln\left(\frac{z^2}{z_{crit}{}^2}\right)\right)^2} \Phi^*}{2\sqrt{\pi} H_0{}^3 f \sigma}, \tag{34}$$

where $d\Omega$, dz and df represent the differential of the solid angle, the redshift and the flux, respectively, and Φ^* is the normalization of the lognormal LF for GRB. The number of GRBs in z and f as given by the above formula has a maximum at $z = z_{pos-max}$, where:

$$z_{pos-max} = e^{\frac{1}{2}\sigma^2} z_{crit}, \tag{35}$$

which can be re-expressed as:

$$z_{pos-max} = \frac{e^{\frac{1}{2}\sigma^2}\sqrt{L^*} H_0}{2\sqrt{\pi}\sqrt{f} c}. \tag{36}$$

Figure 10 reports the observed and theoretical number of GRBs with a given flux as a function of the redshift.

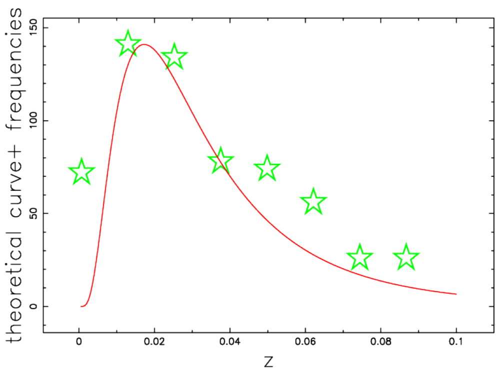

Figure 10. The GRBs of the SWIFT-BAT catalog with $3.1 \frac{fW}{m^2} \leq f \leq 150.54 \frac{fW}{m^2}$, which means $\langle f \rangle = 76.82 \frac{fW}{m^2}$, are organized in frequencies versus spectroscopic redshift (green stars). The redshift covers the range $[0, 0.1]$; the maximum frequency in the observed GRBs is at $z = 0.019$, $\chi^2 = 5925$; and the number of bins is eight. The full red line is the theoretical curve generated by $\frac{dN}{d\Omega dz df}(z)$ as given by the application of the lognormal LF, which is Equation (34), in the pseudo-Euclidean cosmology with the parameters as in Table 7.

The theoretical maximum as given by Equation (35) is at $z = 0.017$, with the parameters as in Table 7, against the observed $z = 0.019$. The theoretical mean redshift of GRBs with flux f can be deduced from Equation (34):

$$\langle z \rangle = \frac{\int_0^\infty z\, F(z; f, L^*, \Phi^*, \sigma) dz}{\int_0^\infty F(z; f, L^*, \Phi^*, \sigma) dz}. \tag{37}$$

The above integral does not have an analytical expression and should be numerically evaluated. The above formula with parameters as in Figure 10 gives a theoretical/numerical $\langle z \rangle = 0.0368$ against the observed $\langle z \rangle = 0.0385$. The quality of the fit in the number of GRBs with a given flux depends on the chosen flux, the interval of the flux in which the frequencies are evaluated and the number of histograms. A larger number of available GRBs will presumably increase the goodness of the fit.

3.4. The Non-Linear Case

We assume that $f = \frac{L}{4\pi r^2}$ and:

$$z = e^{(H_0\, r/c)} - 1, \tag{38}$$

where r is the distance; in our case, d is as represented by the non-linear Equation (11). The relation between dr and dz is:

$$dr = \frac{c dz}{(z+1)\, H_0}. \tag{39}$$

The joint distribution in z and f for the number of galaxies is:

$$\frac{dN}{d\Omega dz df} = \frac{1}{4\pi} \int_0^\infty 4\pi r^2 dr \Phi(\frac{L}{L^*}) \delta(z - (e^{(H_0\, r/c)} - 1)) \delta(f - \frac{L}{4\pi r^2}), \tag{40}$$

where δ is the Dirac delta function.

The evaluations of the integral over luminosities and distances give:

$$\frac{dN}{d\Omega dz df} = \frac{(\ln(z+1))^2\, c^3 \sqrt{2} e^{-\frac{1}{2}\frac{1}{\sigma^2}\left(\ln\left(\frac{(\ln(z+1))^2}{z_{crit}{}^2}\right)\right)^2} \Phi^*}{2\sqrt{\pi} H_0{}^3 f \sigma\ (z+1)}. \tag{41}$$

The above formula has a maximum at $z = z_{pos-max}$, where:

$$z_{pos-max} = e^{4\,\frac{W\left(\frac{1}{4}\sigma^2 z_{crit} e^{\frac{1}{2}\sigma^2}\right)}{\sigma^2}} - 1, \tag{42}$$

where $W(x)$ is the Lambert W function; see [14]. The above maximum can be re-expressed as:

$$z_{pos-max} = e^{4\,\frac{1}{\sigma^2} W\left(\frac{1}{8}\frac{\sigma^2 \sqrt{L^*} H_0 e^{\frac{1}{2}\sigma^2}}{\sqrt{\pi}\sqrt{fc}}\right)} - 1. \tag{43}$$

Figure 11 reports the observed and theoretical number of GRBs with a given flux as a function of the redshift.

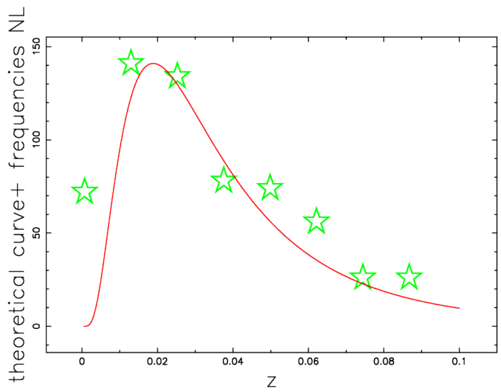

Figure 11. Frequencies of GRBs at a given flux as a function of the redshift; parameters as in Figure 10. The full red line is the theoretical curve generated by $\frac{dN}{d\Omega dz df}(z)$ as given by the application of the lognormal LF, which is Equation (41), in the plasma cosmology with the parameters as in Table 8, $\chi^2 = 6193$.

In the case of the plasma cosmology, the theoretical maximum as given by Equation (42) is at $z = 0.0188$, with the parameters as in Table 7, against the observed $z = 0.019$. The theoretical averaged redshift of GRBs is $\langle z \rangle = 0.041$ against the observed $\langle z \rangle = 0.0385$.

4. The Truncated Lognormal Distribution

This section derives the normalization and the mean for a truncated lognormal PDF. This truncated PDF fits the high redshift behavior of the LF for GRBs.

4.1. Basic Equations

Let X be a random variable taking values x in the interval $[x_l, x_u]$; the truncated lognormal (TL) PDF is:

$$TL(x; m, \sigma, x_l, x_u) = \frac{\sqrt{2} e^{-\frac{1}{2}\frac{1}{\sigma^2}\left(\ln\left(\frac{x}{m}\right)\right)^2}}{\sqrt{\pi}\sigma\left(-\mathrm{erf}\left(\frac{1}{2}\frac{\sqrt{2}}{\sigma}\ln\left(\frac{x_l}{m}\right)\right) + \mathrm{erf}\left(\frac{1}{2}\frac{\sqrt{2}}{\sigma}\ln\left(\frac{x_u}{m}\right)\right)\right)x}. \tag{44}$$

Its expected value is:

$$E(m, \sigma, x_l, x_u) = \frac{e^{\frac{1}{2}\sigma^2} m\left(\mathrm{erf}\left(\frac{1}{2}\frac{\sqrt{2}(\sigma^2 + \ln(m) - \ln(x_l))}{\sigma}\right) - \mathrm{erf}\left(\frac{1}{2}\frac{\sqrt{2}(\sigma^2 + \ln(m) - \ln(x_u))}{\sigma}\right)\right)}{\mathrm{erf}\left(\frac{1}{2}\frac{\sqrt{2}(-\ln(x_l) + \ln(m))}{\sigma}\right) - \mathrm{erf}\left(\frac{1}{2}\frac{\sqrt{2}(-\ln(x_u) + \ln(m))}{\sigma}\right)}. \tag{45}$$

The distribution function is:

$$DF(x; m, \sigma, x_l, x_u) = \frac{-\mathrm{erf}\left(\frac{1}{2}\frac{\sqrt{2}}{\sigma}\ln\left(\frac{x}{m}\right)\right) + \mathrm{erf}\left(\frac{1}{2}\frac{\sqrt{2}}{\sigma}\ln\left(\frac{x_l}{m}\right)\right)}{\mathrm{erf}\left(\frac{1}{2}\frac{\sqrt{2}}{\sigma}\ln\left(\frac{x_l}{m}\right)\right) - \mathrm{erf}\left(\frac{1}{2}\frac{\sqrt{2}}{\sigma}\ln\left(\frac{x_u}{m}\right)\right)}. \tag{46}$$

The four parameters that characterize the truncated lognormal distribution can be found with the maximum likelihood estimators (MLE) and by the evaluation of the minimum and maximum elements of the sample. The LF for GRB as given by the truncated lognormal, $\Phi_T(L; L^*, \sigma, L_l, L_u)$, is therefore:

$$\Phi_T(L; L^*, \sigma, L_l, L_u) = \Phi^* TL(L; L^*, \sigma, L_l, L_u)\frac{number}{Mpc^3\, yr}, \tag{47}$$

where L^* is the scale, L_l the lower bound in luminosity, L_u the upper bound in luminosity and Φ^* is given by Equation (27).

4.2. Applications at High z

The LF for GRBs as modeled by a truncated lognormal DF is reported in Figure 12 in the case of the ΛCDM cosmology and in Figure 13 in the case of the plasma cosmology without a $k(z)$ correction; the data are as in Table 9.

Table 9. The 5 parameters of the LF as modeled by the truncated lognormal distribution for z in $[0, 0.02]$ and the two parameters of the K–S test D and P_{KS}. We analyzed the case of the ΛCDM cosmology where the luminosity is given by Equation (8) in the second column, the case of the plasma cosmology and the case of the X-band (14–195 keV) without $k(z)$ correction, where the luminosity is given by Equation (12), the third column.

Parameter	ΛCDM Cosmology	Plasma Cosmology
$\frac{L_l}{10^{51}\mathrm{erg\,s^{-1}}}$	4.11×10^{-8}	6.11×10^{-12}
$\frac{L_u}{10^{51}\mathrm{erg\,s^{-1}}}$	9.8×10^{-4}	1.42×10^{-7}
$\frac{L^*}{10^{51}\mathrm{erg\,s^{-1}}}$	4.05×10^{-5}	5.9×10^{-9}
σ	1.42	1.42
$\frac{\Phi^*}{\mathrm{Mpc^{-3}yr^{-1}}}$	$1.02\, 10^{-5}$	$1.01\, 10^{-5}$
D	0.084	0.084
P_{KS}	0.177	0.18

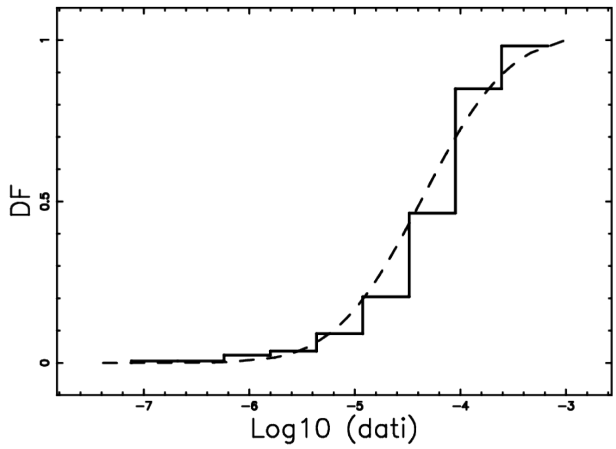

Figure 12. Observed DF (step-diagram) for GRB luminosity and superposition of the truncated lognormal DF (line), the case of the ΛCDM cosmology with the parameters as in Table 9.

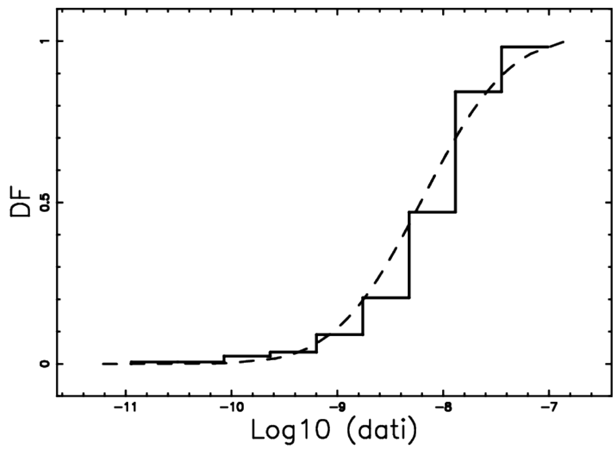

Figure 13. Observed DF (step-diagram) for GRB luminosity and superposition of the truncated lognormal DF (line), the case of the plasma cosmology without $k(z)$ correction with the parameters as in Table 9.

In order to model evolutionary effects, a variable upper bound in luminosity, L_u, has been introduced:

$$L_u = 1.25(1+z)^2\,10^{51} \quad \mathrm{erg\,s^{-1}}, \tag{48}$$

see Equation (7) in [5]; conversely, the lower bound, L_l, was already fixed by Equation (8). A second evolutionary correction is:

$$\sigma = \sigma_0(1+z)^2, \tag{49}$$

where σ_0 is the evaluation of σ at $z \approx 0$; see Table 9.

Figure 14 reports a comparison between the theoretical average luminosity and the observed average luminosity for the ΛCDM cosmology.

In the case of the plasma cosmology, the variable upper bound in luminosity, L_u, is:

$$L_u = 1.25(1+z)^2\,10^{47} \quad \mathrm{erg\,s^{-1}}, \tag{50}$$

and Figure 15 reports a comparison between the theoretical average luminosity and the observed average luminosity for the plasma cosmology.

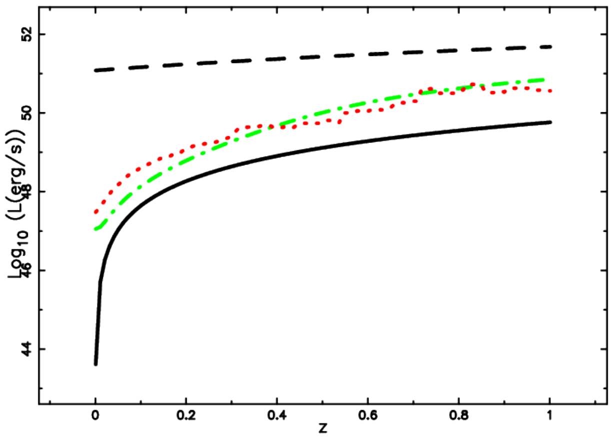

Figure 14. Average observed luminosity in the ΛCDM cosmology versus redshift for 784 GRBs (red points), the theoretical average luminosity for truncated lognormal LF as given by Equation (45) (dot-dash-dot green line), the theoretical curve for the lowest luminosity at a given redshift (see Equation (12)) (full black line) and the empirical curve for the highest luminosity at a given redshift (dashed black line) (see Equation (50)).

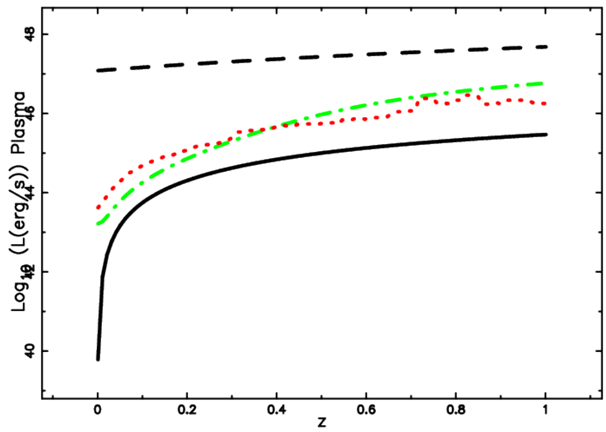

Figure 15. Average observed luminosity in the plasma cosmology without $k(z)$ correction versus redshift for 784 GRBs (red points), the theoretical average luminosity for truncated lognormal LF as given by Equation (45) (dot-dash-dot green line), the theoretical curve for the lowest luminosity at a given redshift (see Equation (8)) (full black line) and the empirical curve for the highest luminosity at a given redshift (dashed black line) (see Equation (50)).

5. Conclusions

5.1. Luminosity

The evaluation of the luminosity is connected with the evaluation of the luminosity distance, which is different for every adopted cosmology: the ΛCDM and plasma cosmologies cover the range in z $[0-4]$, and the pseudo-Euclidean cosmology covers the limited range in z, $[0-0.15]$.

The application of a correction for the luminosity over all of the γ range, which is $[1\,\text{kev} - 10^4\,\text{kev}]$, allows speaking of the extended luminosity of a GRB; in the case of ΛCDM (see Equation (8)), which depends on the three observable parameters, $fluxfwm2$, z and γ. An analytical formula for the luminosity in ΛCDM without corrections is given as a function of the two observable parameters $fluxfwm2$ and z (see Equation (6)), which can be tested on the SWIFT-BAT catalog of [1].

5.2. Lognormal Luminosity Function

We analyzed the widely-used lognormal PDF as an LF for GRBs; see Section 3.2. We derived an expression for the maximum in the number of GRBs for a given flux, which is Equation (35) in the linear case (pseudo-Euclidean universe) (see also Figure 10) and Equation (42) in the non-linear case (plasma cosmology) (see also Figure 11).

5.3. Four Broken Power Law Luminosity Function

The four broken power law PDF gives the best statistical results for the LF of GRBs; see Table 6. The weak point of this LF is in the number of parameters, which is nine, against the four of the truncated lognormal LF or two of the lognormal LF.

5.4. Maximum in Flux

The maximum in the joint distribution in redshift and energy flux density is modeled here in the case of a pseudo-Euclidean universe adopting a standard technique originally developed for galaxies; see Formula (5.132) in [19] and our Formula (34). In the case of the plasma cosmology, the maximum has been found by analogy; see our Formula (34). In the case of the ΛCDM cosmology, the redshift as a function of the luminosity has a complex behavior (see Formula (66) in [10]), and the analysis has been postponed to future research. The above complexity has been considered in an example of a simpler plasma cosmology rather than in the ΛCDM cosmology.

5.5. Evolutionary Effects

The LF for GRBs at high z is well modeled by a truncated lognormal PDF; see Section 4.1. The lower bound for the luminosity is fixed by the decrease in the range of observable luminosities and the higher bound by a standard assumption; see Equation (48). A further refinement of the truncated lognormal model for the GRBs at high z is obtained by introducing a cosmological correction for σ; see Equation (49); see Figure 12 for the case of the ΛCDM cosmology and Figure 15 for the case of the plasma cosmology. In other words, the ΛCDM cosmology and the plasma cosmology are indistinguishable in the range of redshifts analyzed here, $0 \leq z \leq 4$.

Conflicts of Interest: The author declares no conflict of interest.

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