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A Proposal of a Regular Black Hole Satisfying the Weak Energy Condition

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Abstract: We discuss a black hole generated by some matter fluid, whose stress-energy tensor is known. We show that it is regular and that it satisfies the Weak Energy Condition (WEC) and the null energy condition (NEC). Finally, we look for its asymptotic behavior.

Keywords: regular black holes; fluid approach; energy conditions; dark energy; dark matter

1. Introduction

A typical subject in the black hole literature is the search for regular black holes (RBH). As is well known, the first and most paradigmatic solution of the Einstein equations (EE), the Schwarzschild solution, is singular at the origin. However, singularities are thought to be absent in nature: so their presence indicates a breakout or an incompleteness of the theory.

Among the various approaches to the problem, the most popular ones involve either a modification of the stress-energy tensor or a modification of the Einstein tensor. Any solution should have an asymptotic Schwarzschild behavior, while at the center, they should present a de Sitter core: this is a necessary condition to regularize the metric, known as Sakharov criterion [1]. For a single solution, see [2]. For a review, see [3].

It is worth noticing that such modifications can also involve less trivial approaches, such as $F(R)$ and $F(R,G)$ theories or a non-minimal coupling among geometry and matter. Examples are provided by [4–8]. Of course, the main problem of such theories is their difficulty in providing exact results.

It is also worth noticing that the issue of singularity is not only present in the black hole spacetime, but also in the FLRW cosmological framework (although it takes there a different character). Examples of regularization in this sense are provided by [9,10].

In our work, we take the geometric sector of the theory as in the standard GR. However, contrary to what is typically done, we do not assume the source to be point-like, but we consider an extended distribution. We will show that this arrangement is crucial, in order to avoid singularities.

These proceedings are mainly based on [11].

2. Basic Assumptions

We consider a spherically-symmetric and static symmetry (SSSS), described by the metric:

$$ds^2 = -f(r)dt^2 + \frac{1}{g(r)}dr^2 + r^2 d\Omega^2 \tag{1}$$

where f and g are suitable functions and $d\Omega^2$ is the volume element of a two-sphere of unit radius. We also assume that this metric is generated by some fluid, whose stress-energy tensor is written in the form:

$$T^\mu_\nu = \rho u^\mu u_\nu + (P - P_\perp)\delta^1_\nu \delta^\mu_1 + P_\perp \left(\delta^\mu_\nu + u^\mu\, u_\nu\right) \tag{2}$$

where u_μ is the four-velocity of the fluid, ρ its density, P the radial pressure and $P_\perp$ the transversal pressure. Concerning the density, we ask for its positivity, $\rho \geq 0$; its monotonicity, $\rho' \leq 0$; and its finiteness, $\rho < \infty$.

In the system where the fluid is at rest, the stress-energy tensor reads $T^\mu_\nu = \mathrm{diag}(\rho, P, P_\perp, P_\perp)$. This simplifies the EE, whose independent components become:

$$\frac{d}{dr}\left(r(1-g)\right) = 8\pi r^2 \rho \tag{3}$$

$$\frac{g}{fr}f' + \frac{g-1}{r^2} = 8\pi P \tag{4}$$

Moreover, we write also the conservation equation for the fluid $\nabla_\mu T^\mu_r = 0$, i.e.,

$$P' + \frac{P+\rho}{2}\frac{f'}{f} + \frac{2\left(P - P_\perp\right)}{r} = 0 \tag{5}$$

3. Study of the Solution: No *c*/*r* Term

First, we see that the general solution of Equation (3) contains a Schwarzschild-like singular term c/r. Here, we show that c must be set to zero. For the sake of convenience, we refer to the case of $P = P_\perp$, but things are widely more general.

In the classical limit, outside the star, the metric is just Schwarzschild, with $M = 4\pi \int_0^R r^2 \rho$ the whole physical mass of the star. On the radius R, the external metric has the form:

$$g_{\mathrm{ext}}(R) = 1 - \frac{2M}{R} = 1 - \frac{8\pi}{3}\rho_0 R^2 \tag{6}$$

On the other hand, since we cannot avoid the integration constant, the inner solution reads:

$$g_{\mathrm{int}}(R) = 1 + \frac{c}{r} - \frac{8\pi}{3}\rho_0 R^2 \tag{7}$$

A minimal requirement of the metric is its continuity for any r, specifically for $r = R$: comparing Equations (6) and (7), the consequence $c = 0$ is immediate, so the inner solution is regular.

Working within a more general density profile, the argument does not change. In this case, indeed, Equation (3) generates an automatically continuous solution, and apparently, there is no argument to avoid the c/r term:

$$g(r) = 1 - \frac{c}{r} - \frac{8\pi}{r}\int r^2 \rho \tag{8}$$

However, Equation (8) must also fulfill the classical limit, when $\rho \to \rho_0 \theta(R-r)$. In this limit, $\frac{8\pi}{r}\int r^2\rho$ reduce to $2M/r$ for $r > R$ and to $\frac{8\pi}{3}\rho_0 r^2$ for $r < R$. The key point is that c does not reduce to anything, since c is an arbitrary integration constant: in other words, if $c \neq 0$ even in Equation (8), it would be preserved in its classical limit, when the density reduces to $\rho_0\theta(R-r)$. However, this, as we saw, would make the classical metric not continuous, and this is not admissible.

4. Buchdahl Limit

Now, we show that the maximally-symmetric $P = P_\perp$ case is not able to solve the so-called Buchdahl limit [12,13], i.e., the divergence of the central pressure; moreover, when we set R lower than the Schwarzschild radius, the central pressure acquires an imaginary part.

If $P = P_\perp$, the field equations cannot easily be solved exactly, so we take an approximation: around the origin, we write the density as:

$$\rho(r \to 0) = \rho_0 - \kappa r^n + \ldots \tag{9}$$

where ρ_0 is the central density, $\kappa > 0$ is some parameter and $n > 0$. Equation (9) directly descends from our requests about the density. Thus said, the pressure field equation reads:

$$\frac{2P'}{r} + 8\pi \frac{P^2 + \left(\frac{4}{3}\rho_0 - \frac{n+4}{n+3}\kappa r^n\right) P + (\rho_0 - \kappa r^n)\left(\frac{1}{3}\rho_0 - \frac{\kappa}{n+3} r^n\right)}{1 - \frac{8\pi}{3}\rho_0 r^2 + \frac{8\pi}{n+3}\kappa r^{n+2}} = 0 \tag{10}$$

The only way to hope for an analytical solution is to further approximate this object taking the lowest orders in r. The resulting equation is:

$$\frac{2P'}{P^2 + \frac{4}{3}\rho_0 P + \frac{1}{3}\rho_0^2} + \frac{8\pi r}{1 - \frac{8\pi}{3}\rho_0 r^2}\left[1 - \frac{1}{n+3}\left[\frac{(n+4)P + \frac{n+6}{3}\rho_0}{P^2 + \frac{4}{3}\rho_0 P + \frac{1}{3}\rho_0^2} + \frac{8\pi r^2}{1 - \frac{8\pi}{3}\rho_0 r^2}\right]\kappa r^n\right] = 0 \tag{11}$$

However, since we are working in the limit $r \to 0$, the pressure appearing in the square parentheses is just the central pressure P_0: any other correction is suppressed at the lowest order. This allows us to integrate both sides of Equation (11) in order to get the solution:

$$P(r \to 0) = \rho_0 \frac{e^{8\pi\kappa\rho_0 A r^{n+2}}\sqrt{1 - \frac{8\pi}{3}\rho_0 r^2} - \sqrt{1 - \frac{8\pi}{3}\rho_0 R^2}}{3\sqrt{1 - \frac{8\pi}{3}\rho_0 R^2} - e^{8\pi\kappa\rho_0 A r^{n+2}}\sqrt{1 - \frac{8\pi}{3}\rho_0 r^2}} \tag{12}$$

where A is a constant. Equation (12) allows one to calculate the central pressure for a generic density profile; and surprisingly, we have:

$$P(r = 0) = \rho_0 \frac{1 - \sqrt{1 - \frac{8\pi}{3}\rho_0 R^2}}{3\sqrt{1 - \frac{8\pi}{3}\rho_0 R^2} - 1} \tag{13}$$

which is exactly the same as the classical case. It may sound strange, but it is just a natural consequence of the request $\rho(r \to 0) = \rho_0 - \kappa r^n + ...$, where the dominant term is the constant density of the classical limit. This means that, under the symmetry assumption $P = P_\perp$, there is no way to avoid the Buchdahl limit, and pressure becomes infinite at $R = \frac{9}{8}R_S$, independently of the density profile of the star (here, $R_S \equiv 2M$ is the Schwarzschild radius).

Finally, if $R < \frac{9}{8}R_S$, one sees that the central pressure becomes negative, and if $R < R_S$, it acquires an imaginary part, which cannot be set to zero by any means:

$$P_0(R < R_S) = -\frac{\rho_0}{2\left(\frac{9R_S}{8R} - 1\right)}\left(1 - \frac{3R_S}{4R} \pm \frac{i}{2}\left|\sqrt{1 - \frac{R_S}{R}}\right|\right) \tag{14}$$

Things look quite and crucially different working with $f = g$. In this case, we have $P = -\rho$ and $P_\perp = P + \frac{1}{2}P'r$: so, a suitable choice of the density profile prevents any risk of divergence and fully avoids the Buchdahl limit. This is a crucial result.

5. A Model of RBH Satisfying the WEC

We excluded the maximal symmetry assumption ($P = P_\perp$) as a source of generating black holes, so focusing on the matter assumption ($P = -\rho$). We now choose the density profile so that it reduces to the classical density $\rho(r) = \rho_0\theta(R - r)$ and prevents any divergence. A smart choice is:

$$\rho(r) = \rho_0 \frac{1 + e^{-\lambda R^3}}{1 + e^{\lambda(r^3 - R^3)}} \tag{15}$$

where λ plays the role of quantum-like deformation; in the limit $\lambda \to \infty$, density (15) reduces to the classical one, as required; the use of cubic powers in the exponentials is chosen just for calculative convenience.

With this choice, the metric becomes:

$$\begin{aligned} g(r) &= 1 - \frac{8\pi}{r}\rho_0 \left(1 + e^{-\lambda R^3}\right) \int \frac{r^2}{1 + e^{\lambda(r^3 - R^3)}} dr \\ &= 1 - \frac{8\pi\rho_0}{3r} \left(1 + e^{-\lambda R^3}\right) \left(r^3 - \frac{1}{\lambda} \ln\left(\frac{1 + e^{\lambda\left(r^3 - R^3\right)}}{1 + e^{-\lambda R^3}}\right)\right) \end{aligned} \tag{16}$$

which reduces to the classical solution (Schwarzschild outside + de Sitter inside) when $\lambda \to \infty$:

$$g(r > R; \lambda \to \infty) = 1 - \frac{2M}{r} + o(\lambda^{-1}) \tag{17}$$

$$g(r < R; \lambda \to \infty) = 1 - \frac{8\pi\rho_0}{3} r^2 + o(\lambda^{-1}) \tag{18}$$

Depending on the values of the parameters, this object can build a black hole, an extremal black hole or a star. This has been verified by an easy numerical proof.

6. Energy Conditions

We anticipated that the black hole metric (16) fulfills the WEC; here is the time to prove it. In order to have a complete discussion, we also see what happens in the case of the null energy condition (NEC), the dominant energy condition (DEC) and the strong energy condition (SEC).

The NEC only consists of the positivity of the scalar $T_\mu^\nu k_\nu k^\mu$, where k_μ is a general null vector. Using our stress-energy tensor within the $P = -\rho$ and SSSS framework, we have:

$$\begin{aligned} T_\mu^\nu k_\nu k^\mu &= -(\rho + P_\perp)(k_0 k^0 + k_1 k^1) \\ &= \frac{1}{r^2}(\rho + P_\perp)\left(k_2^2 + \frac{k_3^2}{\sin^2\theta}\right) \\ &\geq 0 \end{aligned} \tag{19}$$

meaning that $\rho + P_\perp \geq 0$, i.e.,

$$\rho - \left(\rho + \frac{1}{2}\rho' r\right) = -\frac{1}{2}\rho' r \geq 0 \tag{20}$$

and this is easily verified, since $\rho' \leq 0$ was one of our original requests on the density.

We discuss now the Weak Energy Condition (WEC). Always referring to our stress-energy tensor (2) and considering the timelike vector X_μ, it reads:

$$\begin{aligned} T_\mu^\nu X_\nu X^\mu &= -(\rho + P_\perp)(X_0 X^0 + X_1 X^1) - P_\perp \left|X_\mu X^\mu\right| \\ &= \frac{1}{r^2}(\rho + P_\perp)\left(X_2^2 + \frac{X_3^2}{\sin^2\theta}\right) + \rho \left|X_\mu X^\mu\right| \\ &\geq 0 \end{aligned} \tag{21}$$

consisting of two sufficient conditions:

$$\rho \geq 0 \tag{22}$$

$$\rho + \left(-\rho - \frac{1}{2}\rho' r\right) \geq 0 \tag{23}$$

The first one is easily satisfied, since is just one of our requests on the density. The second one holds from the discussion of the NEC.

For the DEC, we should prove, together with the WEC, that the vector $-T^{\mu}_{\beta}Y^{\beta}$ is causal and future-directed, where Y^{β} is any causal and future-directed vector. We already showed that the WEC holds, so we only need to check if:

$$-T^{0}_{\beta}Y^{\beta} \geq 0 \tag{24}$$

$$\begin{aligned} g_{\mu\nu}(-T^{\mu}_{\beta}Y^{\beta})(-T^{\nu}_{\alpha}Y^{\alpha}) &= \rho^2 Y_0 Y^0 + P^2 Y_1 Y^1 + P_{\perp}(Y_2 Y^2 + Y_3 Y^3) \\ &= -\left(\frac{1}{r^2}(\rho^2 - P_{\perp}^2)\left((Y_2)^2 + \frac{(Y_3)^2}{\sin^2\theta}\right) + \rho^2 \left|Y_{\beta}Y^{\beta}\right|\right) \\ &\leq 0 \end{aligned} \tag{25}$$

The first equation easily reads $\rho Y^0 \geq 0$, and since ρ and Y^0 are both positive, it is immediately satisfied. Regarding the second, it is not difficult to see that this corresponds to the condition $\rho^2 - P_{\perp}^2 \geq 0$, i.e., since we know the relation between ρ and $P_{\perp}$,

$$\rho \geq -\frac{1}{4}\rho' r \tag{26}$$

In general, it is not possible to establish if this holds or not, since it depends on how $\rho' r$ behaves w.r.t. ρ. We can however discuss it in the case of our density (15): we get:

$$1 + \left(1 - \frac{3}{4}\lambda r^3\right) e^{\lambda(r^3 - R^3)} \geq 0 \tag{27}$$

and it is not difficult to see that, for $r \to \infty$, this is badly violated. Therefore, contrary to the NEC and the WEC, the DEC is violated.

The only thing we are left with is seeing whether the SEC is satisfied or not. In other words, we need to see if $\left(T^{\nu}_{\mu} - \frac{1}{2}\delta^{\nu}_{\mu}\right) X_{\nu} X^{\mu} \geq 0$, where T is the trace of the stress-energy tensor and X_{μ} any causal vector. We have:

$$\begin{aligned} \left(T^{\nu}_{\mu} - \frac{1}{2}\delta^{\nu}_{\mu}\right) X_{\nu} X^{\mu} &= -(\rho + P_{\perp})(X_0 X^0 + X_1 X^1) - \frac{1}{2}(\rho - P)\left|X_{\mu}X^{\mu}\right| \\ &= (\rho + P_{\perp})\left((X_2)^2 + \frac{(X_3)^2}{\sin^2\theta}\right) + P_{\perp}\left|X_{\mu}X^{\mu}\right| \\ &\geq 0 \end{aligned} \tag{28}$$

This imposes the two conditions $\rho + P_{\perp} \geq 0$ and $P_{\perp} \geq 0$, which respectively read:

$$-\frac{1}{2}\rho' r \geq 0 \tag{29}$$

$$\rho + \frac{1}{2}\rho' r \leq 0 \tag{30}$$

The first one is satisfied due to the NEC. On the other hand, we cannot discuss the second immediately, since ρ and ρ' have opposite signs and may have very different forms. However, if we want to verify that the SEC is violated, it is enough to show that it is for some region of space. In order to do this, focus around the origin: in this limit, the density reads $\rho(r \to 0) = \rho_0 - \kappa r^n + ...$ with $\kappa > 0$ is the first non-vanishing coefficient in the expansion of the density. So equation (30) reads:

$$\kappa \geq \frac{2\rho_0}{(n+2)r^n}\,. \tag{31}$$

However, this should hold for any r around the origin, and it is clear that it cannot be: the SEC is violated.

We conclude the section noting that the present discussion does not violate the Penrose–Hawking theorem [14–16]. Indeed the theorem, in its most general form, only requires the violation of some condition on energy: it does not specify which one (the WEC or another) is violated, and we saw that DEC and SEC are actually not fulfilled.

7. Limit $R \to 0$: Asymptotical in Time

First, we study the limit $R \to 0$. The limit can be performed exactly, and we get:

$$g(r; R \to 0) = 1 - \frac{16\pi\rho_0}{3r}\left(r^3 - \frac{1}{\lambda}\ln\left(\frac{1+e^{\lambda r^3}}{2}\right)\right) \tag{32}$$

In a similar way to the general case, we prove this is still an RBH, which carries the finite mass $M_0 \equiv \frac{4}{3}\pi\left((\ln 2/\lambda)^{1/3}\right)^3 (2\rho_0)$.

This is the same result if we study the same problem in a dynamical framework, and we focus on the asymptotic $t \to \infty$ regime. In this case, the static density (15) is substituted by:

$$\rho(r,t) = \rho_0(t)\frac{1+e^{-\lambda R^3(t)}}{1+e^{\lambda(r^3-R^3(t))}} \tag{33}$$

only assuming the invariance on λ (which is expected to be a quantum-generated parameter, so independent of the spacetime variables).

We are not able to solve the whole EE in the dynamical framework, so we are just satisfied with studying the asymptotic regime. In that case, $u_0u^0 \simeq -1$ and $u_1u^1 \simeq 0$. With this approximation, one sees that the EE are almost their static version (3)–(5). This proves that we are allowed to write g as:

$$g(r,t) = g_\infty(r) + h(r,t) \tag{34}$$

where g_∞ is the asymptotic solution and $h \ll g_\infty$ is some (small) time-dependent correction. The interesting thing is that, if one assumes the star evaporating, g_∞ turns out to be:

$$g(r,t; t \to \infty) = 1 - \frac{16\pi\rho_0}{3r}\left(r^3 - \frac{1}{\lambda}\ln\left(\frac{1+e^{\lambda r^3}}{2}\right)\right) \tag{35}$$

which is the same result of Equation (33).

8. Conclusions

We discussed how to build an RBH working within the framework of GR, only using some exotic fluid instead of standard matter or a vacuum. We showed that the standard maximal symmetry assumption $P = P_\perp$ is not able to produce only an RBH, since there is no way to avoid or circumnavigate the Buchdahl limit, independently of the specific choice of the density profile. We then take $P = -\rho$ as the state equation.

Regarding the energy conditions, we found that two of them are satisfied, namely the NEC and the WEC. However, since the model has been built within the framework of GR, we expect also the Penrose–Hawking theorem to hold. This actually happens, since we saw that the DEC and the SEC are violated and they are enough to fulfill the theorem.

Finally, considering the dynamical framework, we proved that the asymptotic limit of the time-dependent solution is equivalent to the solution in the static case for $R \to 0$. We speculate that such dark energy black holes, carrying some mass at any time, may comprise a candidate for dark matter.

Conflicts of Interest: The author declares no conflict of interest.

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