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H_∞ Reliable Dynamic Output-Feedback Controller Design for Discrete-Time Singular Systems with Sensor Saturation

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Abstract: In this study, we investigate the H_∞ fault-tolerant control problem for a discrete-time singular system which is subject to external disturbances, actuator faults, and sensor saturation. By assuming that the state variable of the system is unavailable for measurement, and the actuator fault can be described by a Markovian jump process, attention is mainly focused on designing a reliable dynamic output-feedback (DOF) controller able to compensate for the effects of the aforementioned factors on the system stability and performance. Based on the sector non-linear approach to handle the sensor saturation, a new criterion is established to ensure that the closed-loop system is stochastically admissible with a γ level of the H_∞ disturbance rejection performance. The main aim of this work is to develop a procedure for synthesizing the controller gains without any model transformation or decomposition of the output matrix. Therefore, by introducing a slack variable, the H_∞ admissibility criterion is successfully transformed in terms of strict linear matrix inequalities (LMIs). Three practical examples are exploited to test the feasibility and effectiveness of the proposed approach.

Keywords: discrete singular system; actuator failure; sensor saturation; reliable dynamic output feedback; H_∞ control

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1. Introduction

It is well known that when dealing with control design problems, many fundamental issues arising from engineering systems should be considered in the analysis step. The first major issue consists of the synthesis of a feedback control scheme to deal with the practical limitations in the structure of the feedback loops. The sensor saturation introduces a non-linear behavior in the control loop which may disastrously affect the system performances. This explains why the problems of control and filtering with actuator/sensor saturation have been the object of many research studies and significant results have been recently appearing in the literature. To mention a few, the H_∞ output feedback control problem for linear discrete-time systems with sensor nonlinearities was studied in [1–6]. In [7], the networked fuzzy static output feedback control law was designed for discrete-time Takagi–Sugeno fuzzy systems subject to sensor saturation and measurement noise. Based on the non-PDC approach, finite-time H_∞ filtering for a Takagi–Sugeno fuzzy system with uncertain probability sensor saturation was proposed in [8].

The second issue regards the reliability requirement when the system suffers from component failures. This problem introduces the concept of reliable control. The idea consists of producing an adequate controller to sustain the critical functionality of the system despite the occurrence of failures [9,10]. Due to its theoretical and practical significance, the reliable control problem has been extensively studied. In [11], the problem of robust and reliable H_∞ static output feedback control was investigated for discrete-time piecewise-affine systems with delay. The reliable control problem for electronic circuits subject to random actuator faults was studied in [12]. In [13], a reliable fuzzy tracking controller was developed for a near-space hypersonic vehicle using aperiodic measurement information.

The third issue concerns the robustness in the H_∞ sense, i.e., design of a robust controller guaranteeing, in the worst case of external disturbances, the asymptotic stability of the controlled system with an $\mathcal{L}_2$ gain smaller than a prescribed attenuation level $\gamma > 0$ [14–16].

On the other hand, from the viewpoint of developing analytical models, many physical plants exhibit static constraints in their mathematical description. The class of interest in this paper, which can describe this kind of mathematical model, is called singular systems or descriptor systems. Singular systems cover many engineering fields and the control problems regarding this class of systems have attracted a great deal of research attention in the last few decades and many achievements have been made [17–22]. Particularly, discrete-time singular systems have recently received more research value in asymptotic stability, regularity and causality, reliability, and nonfragility [23–29]. Note that, if the state variables are not available for measurement, the static/dynamic output feedback (SOF/DOF) controllers are often investigated as an alternative to control engineering processes. Therefore, DOF is extended for singular systems and has been considered by researchers. For continuous-time singular Markovian jump systems, a dynamic output-feedback controller was synthesized in [30]. In [31], a H_∞ (DOF) controller was designed for a class of discrete-time singular systems and the results were presented in terms of LMI for a particular case of measured states $C_2 = I$. We can emphasize that due to the singular matrix E, the synthesis of the controller parameters becomes difficult. Even though there have been some attempts to consider this problem for continuous singular systems by transforming dynamic output feedback into static output feedback, in [32], or by introducing a particular structure of the LMI variables, in [33], unfortunately, this problem has not been considered yet for discrete singular systems, which motivates this study. Furthermore, it is assumed that the system suffers from sensor saturations and actuator failures with a stochastic behavior described by a Markov process, which is considered as a typical stochastic system to model physical systems with random abrupt variations [9,34,35]. Though the Markov process provides a better description to cope with stochastic actuator failure, the design analysis becomes complex, leading to many computation difficulties. How to reduce the complexity and make the analysis and synthesis easy is the supplementary motivation of this study.

The main objective of this paper is to synthesize a new reliable H_∞ DOF controller for discrete-time singular systems subject to exogenous disturbances, Markovian jump actuator failures, and sensor saturations. The salient features of this work are:

1. it is attractive because the analysis of the controller is conducted for systems operating in real circumstances with exogenous disturbances, stochastic actuator failures, and sensor saturations,
2. the proposed control scheme should be reliable and can accommodate the actuator failures and the sensor nonlinearities,
3. without any model transformation or matrix decomposition, the controller design is carried out by introducing a slack variable to obtain a strict LMI condition,
4. the resulting closed-loop system is able to attenuate the perturbations effects in the H_∞ sense.

The remainder of this paper is organized as follows: Section 2 introduces the problem formulation and some essential preliminaries. The H_∞ stochastic admissibility criterion is developed in Section 3. Section 4 is dedicated to the design of the reliable output-feedback controller for the system under study. To illustrate the effectiveness of the theoretical results, three examples are provided in Section 5. Section 6 concludes the paper and provides the future research direction.

Notation 1. *Throughout this paper, $\mathbb{R}^n$ denotes the n-dimensional Euclidean space, while $\mathbb{R}^{n\times m}$ refers to the set of all $n \times m$ real matrices; matrix $X > 0$ (respectively, $X \geq 0$) is a real symmetric positive definite (respectively, positive semi-definite); $l_2[0, \infty)$ denotes the space of square-summable vectors; $\|.\|$ stands for the Euclidean norm of a vector and its induced norm of a matrix; $\mathbb{E}[.]$ represents the mathematical expectation;* $\mathrm{sym}(X)$ *stands for $X + X^T$; $\lambda()$ denotes the eigenvalue of a matrix; symbol $'*'$ indicates symmetric terms in a symmetric matrix.*

2. Problem Formulation and Preliminaries

Consider a compact discrete-time singular system

$$\begin{cases} Ex(k+1) = (A+\Delta A)x(k) + B_2u^F(k) + B_1w(k) \\ z(k) = C_1x(k) + D_1w(k) \\ y(k) = C_2x(k) + D_2\nu(k) \\ y_s(k) = \sigma(y(k)) \end{cases} \tag{1}$$

where $x(k) \in \mathbb{R}^n$ is the state vector, $u^F(k) \in \mathbb{R}^m$ is the fault control input, $w(k) \in \mathbb{R}^w$ and $\nu(k) \in \mathbb{R}^\nu$ are the disturbance inputs belonging to $L_2[0,\infty)$, $z(k) \in \mathbb{R}^p$ is the controlled output vector, $y_s(k) \in \mathbb{R}^q$ is the saturated signal of the output $y(k)$. Matrices E, A, B_2, B_1, C_1, C_2, D_1, and D_2 are known real constant matrices with suitable dimensions. ΔA is an unknown matrix representing the parametric uncertainties and satisfying $\Delta A = MF(k)N$, where M and N are known real constant matrices of appropriate dimensions, and $F(k)$ is an unknown matrix that satisfies $F^T(k)F(k) \leq I$.

Throughout this paper, it is assumed that:

- matrix $E \in \mathbb{R}^{n\times n}$ may be singular, with $\mathrm{rank}(E) = r < n$;
- system state $x(k)$ is not available for measurement, (A, B) is stabilizable, and (A, C) is detectable;
- saturation function $\sigma(v)$ is defined as

$$\sigma(v) = \begin{bmatrix} \sigma_1(v_1) & \sigma_2(v_2) & \cdots & \sigma_q(v_q) \end{bmatrix}^T \tag{2}$$

with $\sigma_i(v_i) = \mathrm{sign}(v_i)\min\{v_{i,max}, |v_i|\}$, where $v_{i,\max}$ is the i-th element of the saturation level vector v_{max}.

As in [36,37], saturation function (2) can be described by

$$\sigma(v) = H_1v + \phi(v) \tag{3}$$

where $\phi(v) \in [H_1,\ H_2]$ is a nonlinear vector-valued function satisfying the subsequent sector condition [38]

$$\phi(y(k))(\phi(v) - Hv) \leq 0,\ \forall v \in \mathbb{R}^q \tag{4}$$

H_1 and H_2 are known diagonal matrices verifying $0 \leq H_1 < I \leq H_2$ and $H = H_2 - H_1$.

When system (1) operates under actuator failures, the Markov chain is adopted here to model the control signal sent from actuators as:

$$u^F(k) = \mathbf{R}_{r(k)}u(k) \tag{5}$$

where $\mathbf{R}_{r(k)} = \mathrm{diag}\left(R_{1r(k)}, R_{2r(k)}, \cdots, R_{mr(k)}\right)$ is the actuator fault matrix and $R_{sr(k)}$, $s = 1, 2, \ldots m$ is the degradation level of the $'s'$th actuator. $r(k)$ defines a discrete-time Markov process which takes values in a finite set $\mathbb{N} = \left\{1, 2, \ldots, N\right\}$ with a probability matrix $\Pi = \left[\pi_{ij}\right]_{N\times N}$, $(i, j \in \mathbb{N})$. The transition probability π_{ij} is defined as $\pi_{ij} = Pr(r(k+1) = j|r(k) = i)$ and satisfies $\pi_{ij} \geq 0$ and $\sum_{j=1}^N \pi_{ij} = 1$ for each i.

For simplicity of notation, for each $r(k) = i \in \mathbb{N}$, corresponding matrices or vectors relating to $r(k)$ are denoted with the index i. It should be emphasized that this class of systems can describe many physical plants, as is considered in the numerical examples section.

Remark 1. *It should be underscored that the discrete-time homogeneous Markov chain is accepted to cover the cases where the actuator failures have a stochastic feature which can affect many engineering fields, including robotics, aerospace, and missiles [39]. Moreover, the aforementioned failure model provides different cases for particular values of R_{si}, $s = 1, 2, \cdots, m$. The fully operating case occurs for $R_{si} = 1$. The case $R_{si} = 0$ corresponds to the outage case. The actuator faults case corresponds to the case by taking $0 < R_{si} \leq 1$.*

For reliable control purposes, we suggest for system (1) the following full-order dynamic output-feedback controller:

$$\begin{cases} E\hat{x}(k+1) = \hat{A}_i\hat{x}(k) + \hat{B}_i y_s(k) \\ \quad\quad u(k) = \hat{C}_i\hat{x}(k) + \hat{D}_i y_s(k) \end{cases} \tag{6}$$

where $\hat{x}(k) \in \mathbb{R}^n$ is the controller state, $\hat{A}_i$, $\hat{B}_i$, $\hat{C}_i$, and $\hat{D}_i$ are the controller gains with appropriate dimensions to be determined later.

Combining (1) and (6), the augmented closed-loop system under failure is represented by the following dynamic model:

$$\begin{cases} \bar{E}\bar{x}(k+1) = (\bar{A}_i + \Delta\bar{A})\bar{x}(k) + \bar{B}_{1i}\varpi(k) + \bar{B}_{\phi i}\phi(y(k)) \\ \quad\quad z(k) = \bar{C}_1\bar{x}(k) + \bar{D}_1\varpi(k) \end{cases} \tag{7}$$

where $\bar{x}^T(k) = \begin{bmatrix} \hat{x}^T(k) & x^T(k) \end{bmatrix}^T$, $\varpi^T(k) = \begin{bmatrix} w^T(k) & v^T(k) \end{bmatrix}^T$, $\Delta\bar{A} = \bar{M}\Delta\bar{N}$ and

$$\begin{aligned} &\bar{A}_i = \begin{bmatrix} \hat{A}_i & \hat{B}_i H_1 C_2 \\ B_2 \mathbf{R}_i \hat{C}_i & A + B_2 \mathbf{R}_i \hat{D}_i H_1 C_2 \end{bmatrix}, \ \bar{B}_{1i} = \begin{bmatrix} 0 & \hat{B}_i H_1 D_2 \\ B_1 & B_2 \mathbf{R}_i \hat{D}_i H_1 D_2 \end{bmatrix}, \\ &\bar{B}_{\phi i} = \begin{bmatrix} \hat{B}_i \\ B_2 \mathbf{R}_i \hat{D}_i \end{bmatrix}, \ \bar{C}_1 = \begin{bmatrix} 0 & C_1 \end{bmatrix}, \ \bar{D}_1 = \begin{bmatrix} D_1 & 0 \end{bmatrix}, \ \bar{C}_2 = \begin{bmatrix} 0 & C_2 \end{bmatrix}, \\ &\bar{M} = \begin{bmatrix} 0 \\ M \end{bmatrix}, \ \bar{N} = \begin{bmatrix} 0 & N \end{bmatrix} \end{aligned} \tag{8}$$

Remark 2. *The DOF control design problem for discrete-time singular systems has been investigated in [31] using the following controller:*

$$\begin{cases} E\hat{x}(k+1) = \hat{A}\hat{x}(k) + \hat{B}x(k) \\ \quad\quad u(k) = \hat{C}\hat{x}(k) \end{cases} \tag{9}$$

It consists of a particular controller model applied to a system with measured states $C_2 = I$. In addition, compared to the proposed controller, the matrix $\hat{D}$ is null.

Problem 1. *For a given singular system (1), the main problem addressed in this paper is to design a DOF controller (6) such that the closed-loop system defined in (7) is stochastically admissible with H_∞ performance, i.e., under a zero initial condition, $\mathbb{E}\{\sum_{k=0}^{\infty} z^T(k)z(k)\} < \gamma^2 \sum_{k=0}^{\infty} \varpi^T(k)\varpi(k)$, for all $0 \neq \varpi(k) \in L_2[0, \infty)$.*

Before proceeding, we recall the concept of stochastic admissibility for a nominal singular Markovian jump system, defined as

$$Ex(k+1) = A_i x(k) \tag{10}$$

Definition 1 ([40,41]).

1. *Pair (E, A_i) is said to be regular, if* $\det(zE - A_i)$ *is not identically zero for each $i \in \mathbb{N}$;*
2. *Pair (E, A_i) is said to be causal if $deg\Big(det(zE - A_i)\Big) = rank(E)$ for each $i \in \mathbb{N}$;*
3. *System (1) is said to be stochastically stable, if for any initial state (r_0, x_0), the condition $\mathbb{E}\Big\{ \sum_{k=0}^{\infty} \|x(k))\|^2 | r_0, x_0 \Big\} < \infty$ is satisfied;*
4. *System (1) is said to be stochastically admissible, if it is regular, causal, and stochastically stable.*

The following Lemmas are introduced to be used in the controller design procedure.

Lemma 1 ([42])**.** *Let $Q = Q^T$, M and N be real matrices of appropriate dimensions. The condition $Q + MF(k)N + N^T F^T(k) M^T < 0$ holds, for any $F(k)$ satisfying $F^T(k)F(k) \leq I$, if and only if, for any scalar, $\varepsilon > 0$, $Q + \varepsilon MM^T + \varepsilon^{-1} N^T N < 0$.*

Lemma 2 ([43])**.** *For given real matrices Q, N, and M with appropriate dimensions, the following inequality*

$$Q + \mathrm{sym}(NM^T) < 0 \tag{11}$$

is fulfilled if the following condition holds:

$$\begin{bmatrix} Q & N \\ N^T & 0 \end{bmatrix} + \mathrm{sym}\left\{ \begin{bmatrix} H \\ F \end{bmatrix} \begin{bmatrix} M^T & -I \end{bmatrix} \right\} < 0 \tag{12}$$

3. H_∞ Performance and Admissibility Analysis

In this section, the focus is on to the admissibility and H_∞ performance analysis of a closed-loop system (7).

Theorem 1. *For a given scalar $\gamma > 0$, the closed-loop system (7) is stochastically admissible with an H_∞ performance γ, if inequality* (13) *is satisfied for some positive scalars ε_{1i}, ε_{2i}, and matrices $P_i > 0$, S_i, G_1, and G_2.*

$$\Phi_i = \begin{bmatrix} \Phi_{11i} & \Phi_{12i} & G_1 \bar{B}_{\phi i} + \varepsilon_{1i} H \bar{C}_2 & G_1 \bar{B}_{1i} & \bar{C}_1^T & G_1 \bar{M} \\ * & \Phi_{22i} & G_2 \bar{B}_{\phi} & G_2 \bar{B}_{1i} & 0 & G_2 \bar{M} \\ * & * & -2\varepsilon_{1i} I & 0 & 0 & 0 \\ * & * & * & -\gamma^2 I & \bar{D}_1^T & 0 \\ * & * & * & * & -I & 0 \\ * & * & * & * & * & -\varepsilon_{2i} I \end{bmatrix} < 0 \tag{13}$$

where

$$\begin{aligned} \Phi_{11i} &= -\bar{E}^T P_i \bar{E} + sym(G_1 \bar{A}_i) + \varepsilon_{2i} \bar{N}^T \bar{N}, \\ \Phi_{12i} &= -G_1 + \bar{A}_i^T G_2^T \\ \Phi_{22i} &= -\bar{R}^T S_i \bar{R} - sym(G_2) + X_i \\ X_i &= \sum_{j=1}^{N} \pi_{ij} P_j \end{aligned} \tag{14}$$

$\bar{R}$ is any matrix satisfying $\bar{R}\bar{E} = 0$ and $\mathrm{rank}(\bar{R}) = 2n - 2r$.

Proof. Under the condition of Theorem 1, we shall prove that system (7) with $\Delta \bar{A} = 0$ is stochastically admissible with H_∞ performance. From (13), it can be easily verified that

$$\Psi_i = \begin{bmatrix} -\bar{E}^T P_i \bar{E} + sym(G_1 \bar{A}_i) & -G_1 + \bar{A}_i^T G_2^T \\ * & -\bar{R}^T S_i \bar{R} - sym(G_2) \end{bmatrix} < 0 \tag{15}$$

Performing the congruence transformation to (15) by $\left[I, \bar{A}_i^T\right]^T$ yields

$$-\bar{E}^T P_i \bar{E} - \text{sym}\, \bar{A}_i^T (\bar{R}^T S_i \bar{R}) \bar{A}_i < 0 \tag{16}$$

In addition, because rank$(\bar{E}) = 2r < 2n$, there always exist two nonsingular matrices $\hat{M}$ and $\hat{N}$ so that $\hat{E} = \hat{M}\bar{E}\hat{N} = \begin{bmatrix} I_{2r} & 0 \\ 0 & 0 \end{bmatrix}$.

Define

$$\begin{aligned} &\hat{A}_i = \hat{M}\bar{A}_i\hat{N} = \begin{bmatrix} \hat{A}_{11i} & \hat{A}_{12i} \\ \hat{A}_{21i} & \hat{A}_{22i} \end{bmatrix}, \qquad \hat{R} = \bar{R}\hat{M}^{-1} = \begin{bmatrix} \hat{R}_1 & \hat{R}_2 \end{bmatrix} \\ &\hat{P}_i = \hat{M}^{-T} P_i \hat{M}^{-1} = \begin{bmatrix} \hat{P}_{11i} & \hat{P}_{12i} \\ * & \hat{P}_{22i} \end{bmatrix}. \end{aligned} \tag{17}$$

From $\bar{R}\bar{E} = 0$, it can be verified that $\hat{R}\hat{E} = 0$ and $\hat{R}_1 = 0$.

Pre- and post-multiplying (16) by $\hat{N}^T$ and $\hat{N}$, respectively, in light of (17) results in

$$\begin{bmatrix} \star & \star \\ \star & \hat{A}_{22i}^T \hat{R}_2^T S_i \hat{R}_2 \hat{A}_{22i} \end{bmatrix} < 0 \tag{18}$$

where $\star$ will not be used in the following development. It is readily concluded that $\hat{A}_{22i}$ is nonsingular and pair $(\bar{E}, \bar{A}_i)$ is regular and casual, according to Definition 1.

To prove the stochastic stability of system (7), the following Lyapunov function is selected:

$$V(k) = \bar{x}^T(k)\bar{E}^T P_{r_k} \bar{E}\bar{x}(k) \tag{19}$$

letting $\Delta V(k)$ be the forward difference of $V(k)$. Then, along the trajectories of the system (7), we have

$$\begin{aligned} \mathbb{E}\{\Delta V(k)\} &= \mathbb{E}\Big\{V(k+1) - V(k) | x(k), r_k = i\Big\} \\ &= \mathbb{E}\Big\{\bar{x}^T(k+1)\bar{E}^T X_i \bar{E}\bar{x}(k+1)\Big\} \\ &- \bar{x}^T(k)\bar{E}^T P_i \bar{E}\bar{x}(k) \end{aligned} \tag{20}$$

Furthermore, given the constraint $\bar{R}\bar{E} = 0$, the following null equations are true for appropriate matrices G_1, G_1, and S_i:

$$2\xi^T(k)\begin{bmatrix} G_1^T & G_2^T & 0 \end{bmatrix}^T \begin{bmatrix} \bar{A}_i & -I & \bar{B}_{\phi i} \end{bmatrix} \xi(k) = 0 \tag{21}$$

$$-\bar{x}^T(k+1)\bar{E}^T S_i^T \bar{R}\bar{E}\bar{x}(k+1) = 0 \tag{22}$$

where $\xi(k) = \text{col}\Big\{\bar{x}(k),\ \bar{E}\bar{x}(k+1),\ \phi(y(k))\Big\}$.

Moreover, in view of (4), it can be established that

$$-2\varepsilon_{1i}\mathbb{E}\Big\{\phi^T(y(k))(\phi(y(k)) - H\bar{C}_2\bar{x}(k))\Big\} \geq 0 \tag{23}$$

where ε_{1i} is a positive scalar.

Substituting (21)–(23) into (20), one can obtain

$$\mathbb{E}\{\Delta V(k)\} \leq \mathbb{E}\Big\{\xi^T(k)\bar{\Psi}_i\xi(k)\Big\} \tag{24}$$

where

$$\bar{\Psi}_i = \begin{bmatrix} -\bar{E}^T P_i \bar{E} + sym(G_1 \bar{A}_i) & \Phi_{12i} & G_1 \bar{B}_{\phi i} + \varepsilon_{1i} H \bar{C}_2 \\ * & \Phi_{22i} & G_2 \bar{B}_\phi \\ * & * & -2\varepsilon_{1i} I \end{bmatrix} \tag{25}$$

By virtue of inequality (13), it can be deduced that $\bar{\Psi}_i < 0$.

From (24), one can obtain

$$\mathbb{E}\{\Delta V(k)\} \leq \varphi \mathbb{E}\Big\{\|\xi(k)\|^2\Big\} \tag{26}$$

where $\varphi < 0$ denotes the largest eigenvalue of $\bar{\Psi}_i$, for all $i \in \mathbb{N}$. Then, from (26) results

$$\mathbb{E}\Big\{\sum_0^\infty \xi(k)\|^2\Big\} \leq \frac{1}{\varphi}\mathbb{E}\Big\{\sum_0^\infty \Delta V(k)\Big\} \leq -\frac{1}{\varphi} V(0) < \infty \tag{27}$$

So, according to Definition 1, system (7) is stochastically admissible.

To investigate the H_∞ performance for system (7), the following index is introduced:

$$J = \mathbb{E}\Big\{\sum_{k=0}^{\infty}\Big(z^T(k)z(k) - \gamma^2 \varpi^T(k)\varpi(k)\Big)\Big\} \tag{28}$$

Define $\zeta(k) = \text{col}\Big\{\xi(k),\ \varpi(k)\Big\}$ and $J_{zw}(k) = z^T(k)z(k) - \gamma^2 \varpi^T(k)\varpi(k)$.

Following the same reasoning as developed previously, and using the following null equation:

$$2\zeta^T(k)\begin{bmatrix} G_1^T & G_2^T & 0 & 0 \end{bmatrix}^T \begin{bmatrix} \bar{A}_i & -I & \bar{B}_{\phi i} & \bar{B}_{1i} \end{bmatrix}\zeta(k) = 0 \tag{29}$$

it can be established from (13) that

$$\Delta V(k) + J_{zw}(k) = \zeta^T(k)\bar{\Phi}_i\zeta(k) < 0 \tag{30}$$

where

$$\bar{\Phi}_i = \begin{bmatrix} \Phi_{11i} & \Phi_{12i} & G_1\bar{B}_{\phi i} + \varepsilon_{1i} H \bar{C}_2 & G_1 \bar{B}_{1i} & \bar{C}_1^T \\ * & \Phi_{22i} & G_2 \bar{B}_\phi & G_2 \bar{B}_{1i} & 0 \\ * & * & -2\varepsilon_{1i} I & 0 & 0 \\ * & * & * & -\gamma^2 I & \bar{D}_1^T \\ * & * & * & * & -I \end{bmatrix} \tag{31}$$

Under the zero initial condition, it is uncomplicated to see that

$$J \leq \mathbb{E}\sum_{k=0}^{\infty}\Big\{\Delta V(k) + J_{zw}\Big\} < 0 \tag{32}$$

Hence, system (7) is stochastically admissible with H_∞ performance γ.

Now, suppose that $\Delta\bar{A} \neq 0$. In the same way, we have

$$\bar{\Phi}_i + \text{sym}\Big(\Gamma_1^T F(k) \Gamma_2\Big) < 0 \tag{33}$$

where

$$\Gamma_1 = \begin{bmatrix} (G_1\bar{M})^T & (G_2\bar{M})^T & 0 & 0 & 0 \end{bmatrix}, \qquad \Gamma_2 = \begin{bmatrix} \bar{N} & 0 & 0 & 0 & 0 \end{bmatrix}$$

Then, in agreement with Lemma 1, inequality (13) holds. This concludes the proof. □

Remark 3. *The sufficient criterion derived in Theorem 1 shows the existence of a dynamic output-feedback controller such that the closed-loop system is stochastically admissible with H_∞ performance. Nevertheless, condition (13) shows bilinear matrix inequality (BMI) terms with respect to the matrices $\hat{A}_i$, $\hat{B}_i$, $\hat{C}_i$, and $\hat{D}_i$. Unlike the method proposed in [44,45] where the model transformation is used to linearize the BMI conditions, our design approach is based on the introduction of an auxiliary variable U_i to separate the LMI variables by setting $\mathbf{K}_i = U_i^{-1}\mathbf{F}_i$, where $\mathbf{K}_i$ is* *the augmented form of the controller matrices. The next section shows in detail the controller design procedure, where Lemma 2 plays an important role to linearize condition (13) and overcome nonlinear terms.*

4. H_∞ Controller Design

In the sequel, we focus on developing a method to synthesize controller gains $\hat{A}_i$, $\hat{B}_i$, $\hat{C}_i$, and $\hat{D}_i$ so that the closed-loop system (7) is robustly admissible with H_∞ disturbance attenuation γ.

Theorem 2. *Given prescribed scalars $\gamma > 0$ and β, if there exist matrices $P_i > 0$, $J > 0$, G_1, G_2, U_i, $\mathbf{F}_i$ and scalars $\varepsilon_{1i} > 0$ and $\varepsilon_{2i} > 0$ such that the following LMI holds*

$$\bar{\Phi}_i = \begin{bmatrix} \hat{\Phi}_i + \mathbb{I}_1^T J \mathbb{I}_1 & \Gamma_{12i} & 0 \\ * & -\beta\,\mathrm{sym}(\Gamma_{22i}) & \beta\Gamma_{23i} \\ * & * & -J \end{bmatrix} < 0 \tag{34}$$

where

$$\hat{\Phi}_i = \begin{bmatrix} \hat{\Phi}_{11i} & \hat{\Phi}_{12i} & \hat{\Phi}_{13i} & \hat{\Phi}_{14i} & \bar{C}_1^T & G_1\bar{M} & \bar{N} \\ * & \hat{\Phi}_{22i} & \hat{\Phi}_{23i} & \hat{\Phi}_{24i} & 0 & G_2\bar{M} & 0 \\ * & * & -2\varepsilon_{1i}I & 0 & 0 & 0 & 0 \\ * & * & * & -\gamma^2 I & \bar{D}_1^T & 0 & 0 \\ * & * & * & * & -I & 0 & 0 \\ * & * & * & * & * & -\varepsilon_{2i}I & 0 \\ * & * & * & * & * & * & -\varepsilon_{2i}I \end{bmatrix}$$

$$\begin{aligned} &\hat{\Phi}_{11i} = -\bar{E}^T P_i \bar{E} + sym(G_1\mathbf{A} + \mathbf{B}\mathbb{R}_i\mathbf{F}_i\mathbf{C}), \\ &\hat{\Phi}_{12i} = -G_1 + (G_2\mathbf{A} + \mathbf{B}\mathbb{R}_i\mathbf{F}_i\mathbf{C})^T \\ &\hat{\Phi}_{22i} = -\bar{R}^T S_i \bar{R} - sym(G_2) + X_i \\ &\hat{\Phi}_{13i} = \mathbf{B}\mathbb{R}_i\mathbf{F}_i\mathbb{I} + \varepsilon_{1i}H\bar{C}_2 \qquad \hat{\Phi}_{23i} = \mathbf{B}\mathbb{R}_i\mathbf{F}_i\mathbb{I} \\ &\hat{\Phi}_{14i} = G_1\mathbf{B}_1 + \mathbf{B}\mathbb{R}_i\mathbf{F}_i\mathbf{D} \qquad \hat{\Phi}_{24i} = G_2\mathbf{B}_1 + \mathbf{B}\mathbb{R}_i\mathbf{F}_i\mathbf{D} \\ &\Gamma_{21i} = \begin{bmatrix} \mathbf{B}^T\mathbf{B}\mathbf{F}_i\mathbf{C} & 0 & \mathbf{B}^T\mathbf{B}\mathbf{F}_i\mathbb{I} & \mathbf{B}^T\mathbf{B}\mathbf{F}_i\mathbf{D} & 0 & 0 & 0 \end{bmatrix} \\ &\Gamma_{22i} = \mathbf{B}^T\mathbf{B}U_i \\ &\Gamma_{23i} = \begin{bmatrix} (G_1\mathbf{B}\mathbb{R}_i - \mathbf{B}\mathbb{R}_iU_i)^T & (G_2\mathbf{B}\mathbb{R}_i - \mathbf{B}\mathbb{R}_iU_i)^T \end{bmatrix} \\ &J = \begin{bmatrix} J_{11} & J_{12} \\ * & J_{22} \end{bmatrix} \mathbb{I}_1 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \mathbb{I} = \begin{bmatrix} 0 & I \end{bmatrix}^T \end{aligned} \tag{35}$$

$$\begin{aligned} &\mathbf{A} = \begin{bmatrix} 0 & 0 \\ 0 & A \end{bmatrix}, \mathbf{B} = \begin{bmatrix} I & 0 \\ 0 & B_2 \end{bmatrix}, \mathbf{C} = \begin{bmatrix} I & 0 \\ 0 & H_1C_2 \end{bmatrix}, \mathbf{K}_i = \begin{bmatrix} \hat{A}_i & \hat{B}_i \\ \hat{C}_i & \hat{D}_i \end{bmatrix}, \\ &\mathbb{R}_i = \begin{bmatrix} I & 0 \\ 0 & \mathbf{R}_i \end{bmatrix} \mathbf{B}_1 = \begin{bmatrix} 0 & 0 \\ B_1 & 0 \end{bmatrix}, \mathbf{D} = \begin{bmatrix} 0 & 0 \\ 0 & H_1D_2 \end{bmatrix}, \end{aligned} \tag{36}$$

then, the closed-loop system (7) is stochastically admissible with H_∞ norm bounded γ. Furthermore, the controller gain is computed as $\mathbf{K}_i = U_i^{-1}\mathbf{F}_i$.

Proof. Using the matrices in (36), the closed-loop matrices can be written as

$$\bar{A}_i = \mathbf{A} + \mathbf{B}\mathbb{R}_i\mathbf{K}_i\mathbf{C},\ \bar{B}_{1i} = \mathbf{B}_1 + \mathbf{B}\mathbb{R}_i\mathbf{K}_i\mathbf{D},\ \bar{B}_{\phi i} = \mathbf{B}\mathbb{R}_i\mathbf{K}_i\mathbb{I}, \tag{37}$$

Moreover, using the fact $\mathbf{K}_i = U_i^{-1}\mathbf{F}_i$, we can easily verify for any $G_l,\ l = 1, 2$ that

$$\begin{aligned} G_l\mathbf{B}\mathbb{R}_i\mathbf{K}_i\mathbf{C} &= G_l\mathbf{B}\mathbb{R}_i\mathbf{K}_i\mathbf{C} + \mathbf{B}\mathbb{R}_i\mathbf{F}_i\mathbf{C} - \mathbf{B}\mathbb{R}_i\mathbf{F}_i\mathbf{C} \\ &= \mathbf{B}\mathbb{R}_i\mathbf{F}_i\mathbf{C} + (G_1\mathbf{B}\mathbb{R}_i - \mathbf{B}\mathbb{R}_i U_i)U_i^{-1}\mathbf{F}_i\mathbf{C} \\ G_l\mathbf{B}\mathbb{R}_i\mathbf{K}_i\mathbf{D} &= \mathbf{B}\mathbb{R}_i\mathbf{F}_i\mathbf{D} + (G_1\mathbf{B}\mathbb{R}_i - \mathbf{B}\mathbb{R}_i U_i)U_i^{-1}\mathbf{F}_i\mathbf{D} \\ G_l\mathbf{B}\mathbb{R}_i\mathbf{K}_i\mathbb{I} &= \mathbf{B}\mathbb{R}_i\mathbf{F}_i\mathbb{I} + (G_1\mathbf{B}\mathbb{R}_i - \mathbf{B}\mathbb{R}_i U_i)U_i^{-1}\mathbf{F}_i\mathbb{I} \end{aligned} \tag{38}$$

Assume that inequality (34) holds. Thus, a feasible solution verifies that G_1, G_2, and U_i are nonsingular.

Define $\mathbb{W} = \begin{bmatrix} I & 0 & \mathbb{I}_1^T \\ 0 & \frac{1}{\beta}I & 0 \end{bmatrix}^{\mathbb{T}}$. By performing the congruence transformation to (34) by $\mathbb{W}$, we obtain

$$\begin{bmatrix} \hat{\Phi}_i & (\Gamma_{23i}\mathbb{I}_1)^T + (\frac{1}{\beta}\mathbf{B}^T\mathbf{B}\mathrm{Y}_{21i})^T & \\ * & -\frac{1}{\beta}\,\mathrm{sym}(\mathbf{B}^T\mathbf{B}U_i) = \hat{\Phi}_i & (\Gamma_{23i}\mathbb{I}_1)^T \\ * & 0 & \end{bmatrix} + \mathrm{sym}\left\{ \begin{bmatrix} 0 \\ \frac{1}{\beta}\mathbf{B}^T\mathbf{B}U_i \end{bmatrix} U_i^{-1} \begin{bmatrix} \mathrm{Y}_{21i} & -I \end{bmatrix} \right\} < 0 \tag{39}$$

where

$$\mathrm{Y}_{21i} = \begin{bmatrix} \mathbf{F}_i\mathbf{C} & 0 & \mathbf{F}_i\mathbb{I} & \mathbf{F}_i\mathbf{D} & 0 & 0 & 0 \end{bmatrix} \tag{40}$$

According to Lemma 2, inequality (39) is equivalent to

$$\Phi_i = \hat{\Phi}_i + \mathrm{sym}\left((\Gamma_{23i}\mathbb{I}_1)^T U_i^{-1}\mathrm{Y}_{21i} \right) < 0 \tag{41}$$

Considering (38), it can be concluded from Theorem 1 that the designed controller makes the closed-loop system in (7) stochastically admissible with H_∞ disturbance attenuation level γ. □

Remark 4. *Compared with existing results in [31,45,46], the key merit of the proposed control design scheme lies in its simplicity and lower conservativeness. In fact, contrary to our method, the suggested one in [45] needs a particular structure of matrices G_i to synthesize the controller gains. In [46], the SVD decomposition technique with a particular structure of G_i is also adopted. Additionally, the strategy used in [31] requires many scalars to tune. However, the LMI in (34) can be solved easily by selecting only one parameter and using any LMI software.*

Remark 5. *Note that condition (34) is a strict LMI if the tuning parameter β is well chosen.*

Remark 6. *Since the LMI in Theorem 2 is linear in the scalar γ^2, it can be considered as an optimization variable for the following convex optimization problem to reduce the attenuation level bound:*

$$minimise\, v = \gamma^2, subject\ to\ LMI\ (34) \tag{42}$$

The solution of this problem determines the optimal H_∞ performance as $\gamma^* = \sqrt{v}$.

Remark 7. *In order to design the reliable controller, it is assumed that the transition matrix of the Markov process, that characterizes the actuator faults, is completely known. Nevertheless, this assumption is very restrictive and a Markov chain with partly unknown transition probabilities should considered as future work [21,35].*

5. Numerical Examples

In this section, three simulation examples are provided to test the effectiveness of the developed control scheme.

Example 1. *Referring to [31], consider a singular system (1) with the following parameters:*

$$E = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, A = \begin{bmatrix} 2.5 & 1.0 & 0.3 \\ 2.6 & 1.2 & 0.65 \\ 1.7 & -0.2 & 0.7 \end{bmatrix}, B_2 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, B_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix},$$

$$C_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, D_2 = \begin{bmatrix} 0.1 \\ 0.1 \\ 0.1 \end{bmatrix}, C_1 = \begin{bmatrix} 0.1 & 0.1 & 0.1 \end{bmatrix}, D_1 = 0.1$$

In this example, let $V_{yj,max} = 0.5,\ j = 1,2,3,\ H_1 = \text{diag}(0.25, 0.25, 0.25)$, and $H_2 = I$.

To study the effect of actuator failures, we inspect the scenario where the failure may occur with a 40% reduction in signal amplitude with the transition probability matrix chosen as

$$\Pi = \begin{bmatrix} 0.8 & 0.2 \\ 0.3 & 0.7 \end{bmatrix} \tag{43}$$

Applying Theorem 2 with $\beta = 0.3$ and $R = diag(R_0, R_0),\ R_0 = \begin{bmatrix} 0 & 0.1 & 0 \\ 0 & 0.1 & 0 \\ 0 & 0.1 & 0 \end{bmatrix}$, the corresponding controller can be designed with the following calculated parameters:

$$\begin{aligned}
A_{c1} &= \begin{bmatrix} -0.15341 & -0.21305 & -0.37074 \\ -0.14988 & -0.20081 & -0.21483 \\ -0.16871 & -0.09083 & 0.11124 \end{bmatrix} \\
B_{c1} &= \begin{bmatrix} -0.17902 & -0.18786 & -0.32215 \\ -0.89931 & -0.84344 & -0.90925 \\ 0.052493 & 0.021098 & -0.053026 \end{bmatrix} \\
C_{c1} &= \begin{bmatrix} -0.37758 & -0.33222 & -0.42294 \end{bmatrix} \\
D_{c1} &= \begin{bmatrix} -0.13684 & -0.16389 & 0.27953 \end{bmatrix} \\
A_{c2} &= \begin{bmatrix} -0.0854 & -0.15738 & -0.31616 \\ -0.13127 & -0.17726 & -0.17805 \\ -0.17357 & -0.091193 & 0.11952 \end{bmatrix} \\
B_{c2} &= \begin{bmatrix} -0.16169 & -0.1628 & -0.32657 \\ -0.86849 & -0.83448 & -0.95778 \\ 0.037549 & 0.021162 & -0.087656 \end{bmatrix} \\
C_{c2} &= \begin{bmatrix} -0.41221 & -0.38024 & -0.48807 \end{bmatrix} \\
D_{c2} &= \begin{bmatrix} -0.17282 & -0.19162 & 0.38845 \end{bmatrix}
\end{aligned} \tag{44}$$

The minimum H_∞ level γ is obtained as 0.1.

Herein, a further comparison of feasibility results is performed between the works of [31,44,47] and the present study for $\gamma = 0.1$ (see Table 1).

Table 1. Comparison of the feasibility results by different methods for $\gamma = 0.1$.

	Methods
Theorem 2	(44)
Theorem 2 in [31]	Infeasible
Theorem 3 in [47]	Infeasible
Theorem 7 in [44]	Infeasible

To test the effectiveness of the proposed control scheme, simulation studies are performed with initial condition $x(0) = \begin{bmatrix} 0.1745 & 0.3491 & 3 \end{bmatrix}$. Figures 1–3 show the convergence behaviors of actual and ideal measurements of the system, while Figures 4–6 demonstrate, respectively, the failure mode signal, the control input $u(k)$ response, and the curve of the ratio $\gamma(k) = \frac{\sqrt{\sum_0^\infty z^T(k)z(k)}}{\sqrt{\sum_0^\infty w^T(k)w(k)}}$ under a zero initial condition. From this figure, it can be easily verified that the ratio is less than the prescribed disturbance attenuation level of 0.1.

To further demonstrate the merit of the proposed control scheme, a comparison is performed with the method proposed in [31]. Figure 7 displays the output response. From this figure, it is clear that, under the saturation and failure constraints, the controller suggested in [31] is not able to stabilize the system. However, the synthesized control law is effective in stabilizing the discrete-time singular system with satisfactory performances in spite of the external disturbances, sensor saturation, and stochastic actuator failure.

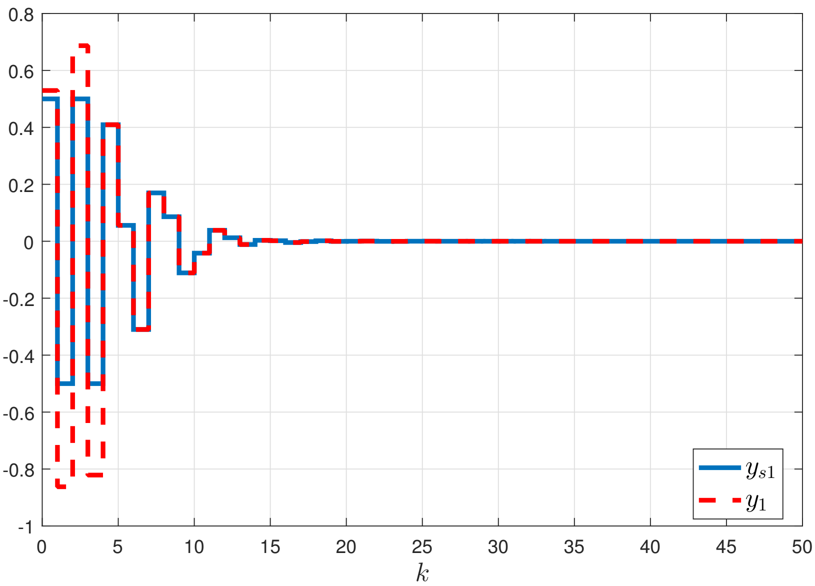

Figure 1. Actual and ideal measurements of y_1.

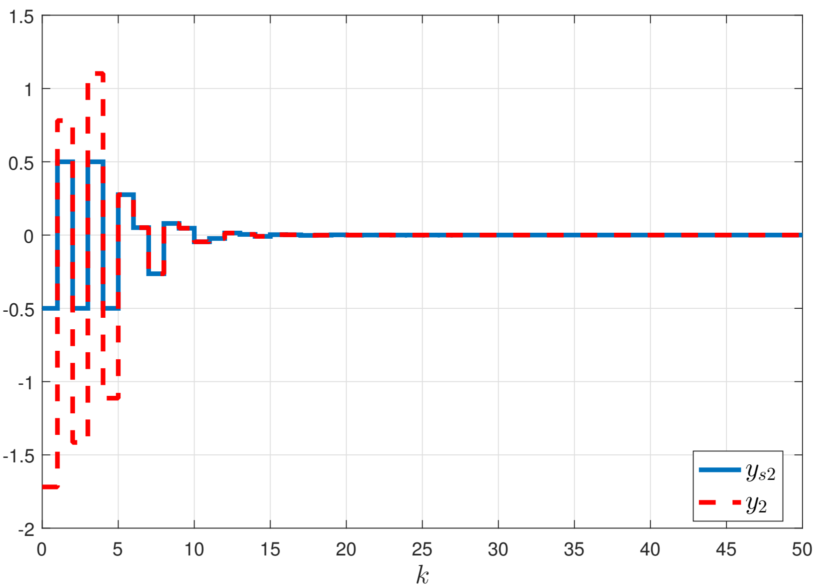

Figure 2. Actual and ideal measurements of y_2.

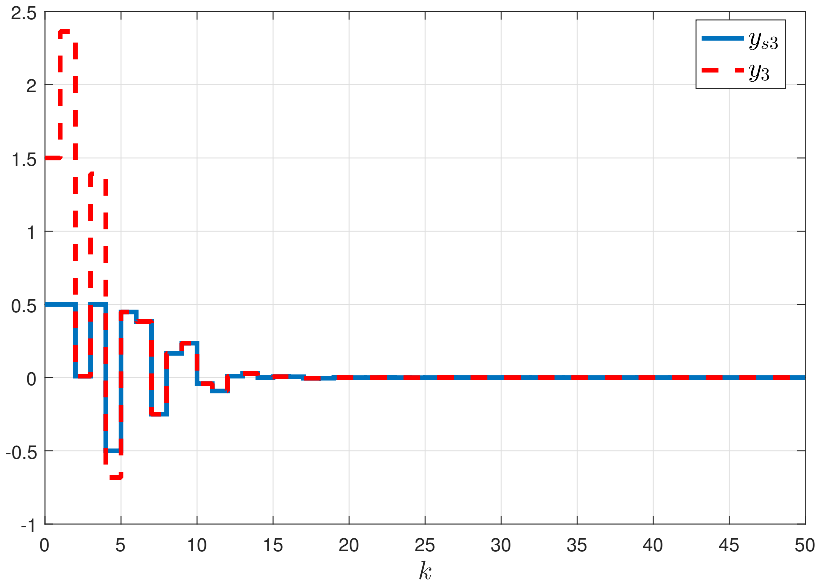

Figure 3. Actual and ideal measurements of y_3.

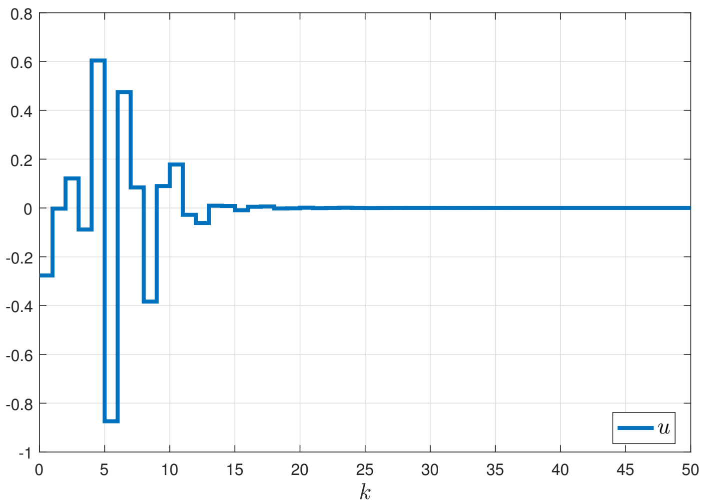

Figure 4. Control response $u(k)$.

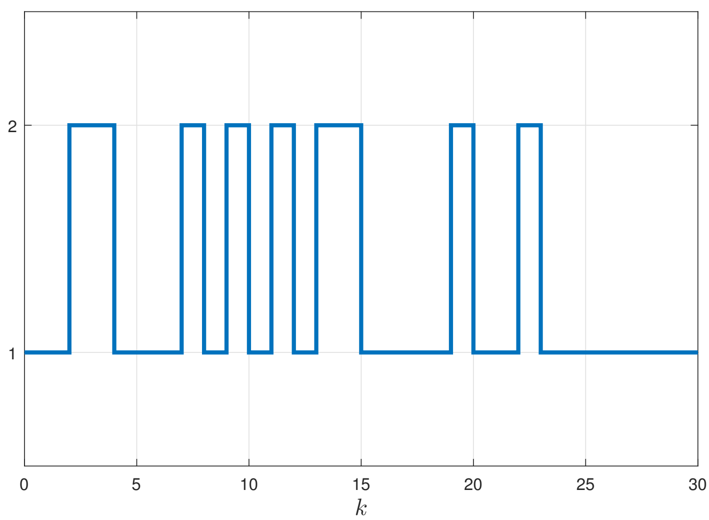

Figure 5. Failure modes.

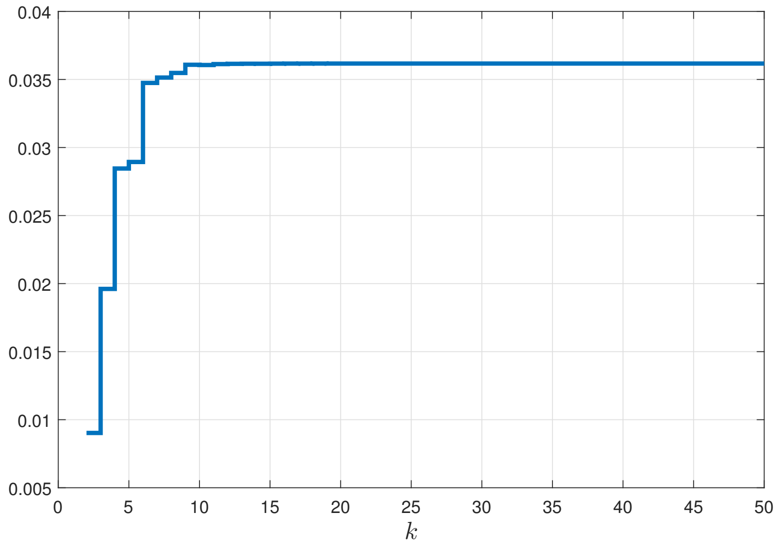

Figure 6. Response of the ratio $\gamma(k)$.

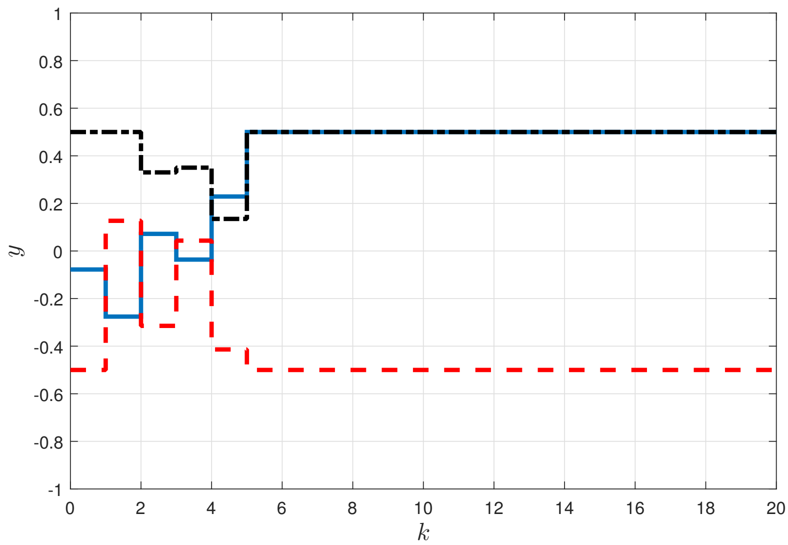

Figure 7. Output response with controller in [31].

Example 2. *Figure 8 shows a hydraulic system with three tanks. The linearized discrete-time singular model of this system is borrowed from [48] and given as:*

$$\left\{\begin{aligned} \begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 0\end{bmatrix}\begin{bmatrix}q_1(k+1)\\ q_2(k+1)\\ q_3(k+1)\end{bmatrix} &= \begin{bmatrix}0.9692 & 0 & 0\\ 0.0095 & 0.9867 & 0\\ 1 & 2.3328 & 1\end{bmatrix}\begin{bmatrix}q_1(k)\\ q_2(k)\\ q_3(k)\end{bmatrix} \\ &\quad + \begin{bmatrix}0.056\\ 0.003\\ 0\end{bmatrix}u(k) + \begin{bmatrix}0.02\\ 0.01\\ 0\end{bmatrix}w(k) \\ y(k) &= q_2(k) + 0.3\nu(k) \end{aligned}\right. \tag{45}$$

where vector $q(k)$ represents volumes in the tanks, $u(k)$ is pump flow, $w(k)$ is plant noise, and $\nu(k)$ is measurement noise.

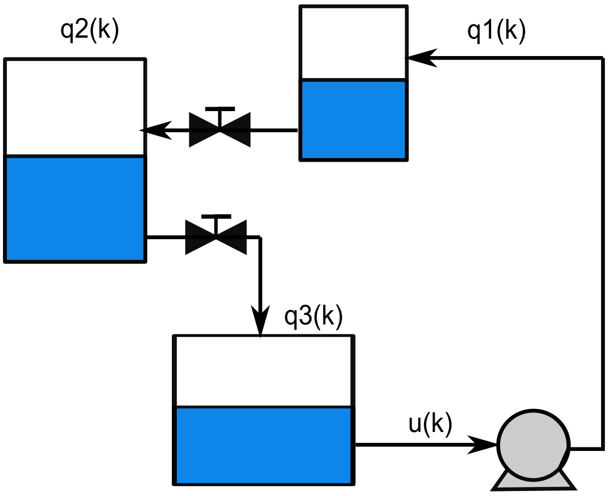

Figure 8. Hydraulic system.

Assume that the sensor is subject to saturation with the following parameters

$$V_{y,max} = 2.5\ H_1 = 0.3 \text{ and } \ H_2 = 1. \tag{46}$$

and suppose that actuator failure may occur with a 70% reduction in signal amplitude.

The external disturbance $w(k)$ is assumed to be

$$w(k) = \begin{cases} 1 & 360 \leq k \leq 375 \\ -1 & 450 \leq k \leq 475 \\ 0 & \text{otherwise} \end{cases} \tag{47}$$

while the measurement noise is selected as $v(k) = 0.1sin(k+2)(0.5 - unifrnd(0,1,1,1))$

In addition, we choose

$$\Pi = \begin{bmatrix} 0.9 & 0.1 \\ 0.4 & 0.6 \end{bmatrix} \tag{48}$$

Then, the essential objective to be achieved is to design a reliable DOF controller (6) such that, under given sensor saturation and actuator failure parameters, the admissibility of the closed-loop system as well as the H_∞ performance are satisfied. To this end, setting $\beta = 0.2$, $R = \text{diag}(R_0, R_0)$, $R_0 = \text{diag}(0,0,1)$, LMIs in Theorem 2 can be solved with the following controller gains:

$$\begin{aligned} A_{c1} &= \begin{bmatrix} 0.059347 & -0.10817 & -0.17411 \\ -0.10817 & 0.059347 & -0.17411 \\ 2.1971 & 2.1971 & 2.6273 \end{bmatrix} \quad B_{c1} = \begin{bmatrix} -0.50677 \\ -0.50676 \\ 6.7524 \end{bmatrix} \\ C_{c1} &= \begin{bmatrix} -2.5088 & -2.5088 & -2.6299 \end{bmatrix} \quad D_{c1} = -6.6406 \\ A_{c2} &= \begin{bmatrix} 0.049448 & -0.16593 & -0.16196 \\ -0.16593 & 0.049448 & -0.16196 \\ 0.97856 & 0.97856 & 1.0638 \end{bmatrix} \quad B_{c2} = \begin{bmatrix} -0.57122 \\ -0.57122 \\ 3.1622 \end{bmatrix} \\ C_{c2} &= \begin{bmatrix} -1.4123 & -1.4123 & -1.2975 \end{bmatrix} \quad D_{c2} = -3.4328 \end{aligned} \tag{49}$$

It is also worth pointing out that the associated minimum H_∞ performance index is computed as $\gamma_{min} = 0.1$.

At this point, simulation studies are implemented with the initial condition $x(0) = [3\ 3\ -10]$ to test the effectiveness of the design procedure and the results are shown in Figures 9–11. Among them, Figure 9 depicts the responses of the system outputs (ideal and saturated). The evolution of control input is displayed in Figure 10. The outputs (ideal and saturated) and control responses are plotted in Figures 9 and 10, respectively. As expected, $y(k)$ converges to the origin even the saturation affects the sensors. Evidently, the simulation result shows that, under the measurement noise, sensor saturation, and actuator failure shown in Figure 11, the closed-loop system is stable with a satisfactory performance and provides potent verification of the effectiveness of the proposed control scheme.

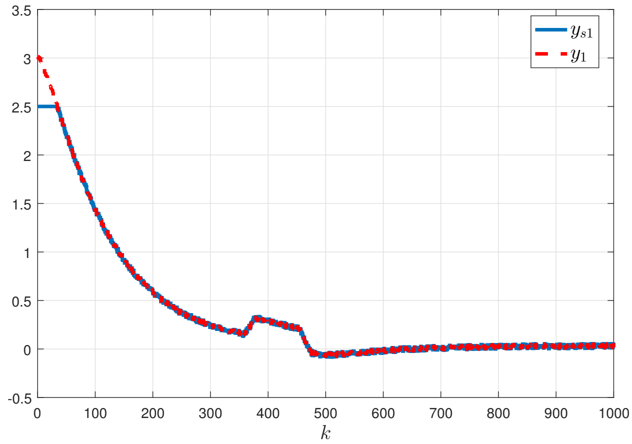

Figure 9. Actual and ideal measurements of y.

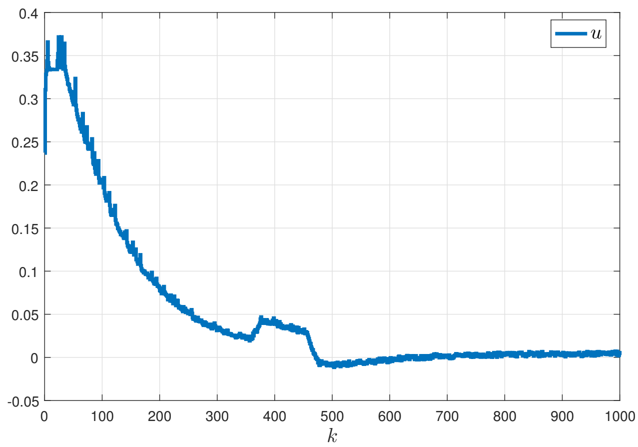

Figure 10. Control response $u(k)$.

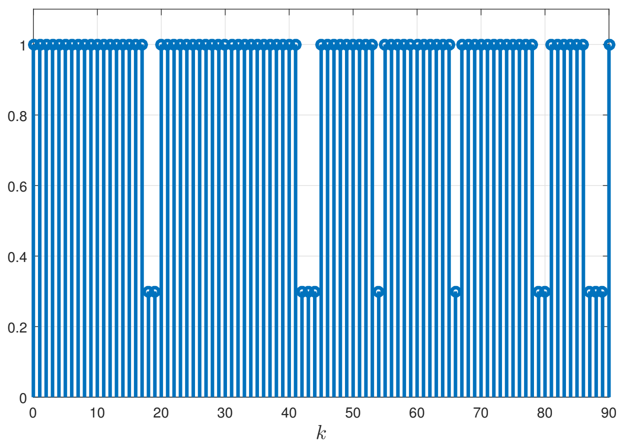

Figure 11. Failure modes.

Example 3. *In this example, a mechanical system that consists of a disc rolling on a surface without slipping, is shown in Figure 12. The disc associated with a spring and a damper are fixed to a wall. The spring has the coefficient K, and the damper has a damping coefficient b. This system is described by the following discrete-time singular model with T_s = 0.05 s:*

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} x(k+1) = \begin{bmatrix} 1 & T_s & 0 & 0 \\ -T_s\frac{K}{m} & 1 - T_s\frac{b}{m} & 0 & \frac{T_s}{m} \\ 0 & 1 & -r & 0 \\ \frac{K}{m} & \frac{b}{m} & 0 & -(\frac{r^2}{J} + \frac{1}{m}) \end{bmatrix} x(k) + \begin{bmatrix} 0 \\ 0 \\ 0 \\ \frac{r}{J} \end{bmatrix} \big(u(k) + f(x(k))\big) + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} w(k)$$
$$y(k) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix} x(k) \quad (50)$$

where m is the mass, r is the radius, and J is the inertia of the disc. For the state variables, x_1 is the position, x_2 is the translational velocity of the center of the disc along the surface, x_3 is the angular velocity of the disc, and x_4 is the contact force between the disc and the surface. The control input u is the torque applied to the disc. The parameters of the system are given in Table 2.

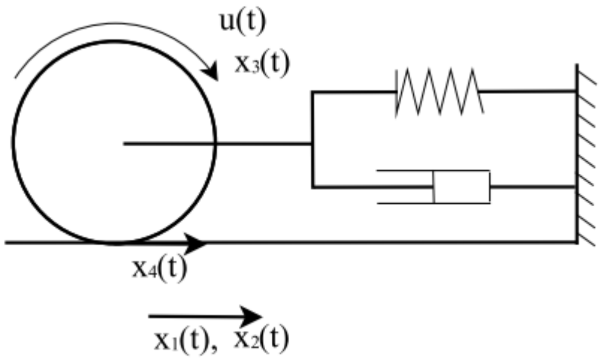

Figure 12. Rolling disc.

Table 2. Parameters of the system.

Parameter	Value	Unit
K	100	[Nm^{-1}]
b	30	[Ns/m]
m	40	[kg]
J	3.2	[kgm^{-2}]
r	10	[cm]

The parametric uncertainties are $F(k) = 0.8sin(5k)$ and

$$M = \begin{bmatrix} 0 & 0 & 0 & 0.01 \end{bmatrix}^T, \quad N = \begin{bmatrix} 0.1 & 0 & 0 & 0 \end{bmatrix}$$

The sensors are subject to saturation with the corresponding parameters $H_1 = \text{diag}(0.5, 0.5)$ and $H_2 = I_2$.

To study the effect of actuator failures, we inspect the scenario where the failure may occur with 30% and 80% reductions in signal amplitude with the transition probability matrix chosen as

$$\Pi = \begin{bmatrix} 0.73 & 0.17 & 0.1 \\ 0.35 & 0.5 & 0.15 \\ 0.15 & 0.2 & 0.65 \end{bmatrix} \tag{51}$$

Let $\beta = 0.8$, $R = \text{diag}(R_0, R_0)$, $R_0 = \text{diag}(0, 0, 1, 1)$. By resorting to the Yalmip toolbox with Sdpt3 solver, Theorem (2) provides a feasible solution with the following parameters:

$$\begin{aligned}
A_{c1} &= \begin{bmatrix} 1.0735 & 0.71228 & 0.63569 & 0.63569 \\ 0.71228 & 1.0735 & 0.63569 & 0.63569 \\ -2.4279 & -2.4279 & -0.96954 & -2.9125 \\ -2.4279 & -2.4279 & -2.9125 & -0.96954 \end{bmatrix} \\
B_{c1} &= \begin{bmatrix} 0.16737 & 0.25935 \\ 0.16737 & 0.25935 \\ -0.41404 & -1.6679 \\ -0.41404 & -1.6679 \end{bmatrix} \\
C_{c1} &= \begin{bmatrix} 2.2119 & 2.2119 & 1.586 & 1.586 \end{bmatrix} \; D_{c1} = \begin{bmatrix} -1.0595 & 1.526 \end{bmatrix} \\
A_{c2} &= \begin{bmatrix} 2.5788 & 2.164 & 2.0741 & 2.0741 \\ 2.164 & 2.5788 & 2.0741 & 2.0741 \\ -5.6261 & -5.6261 & -4.0998 & -6.0428 \\ -5.6261 & -5.6261 & -6.0428 & -4.0998 \end{bmatrix} \\
B_{c2} &= \begin{bmatrix} 0.32796 & 1.4251 \\ 0.32796 & 1.4251 \\ -0.80981 & -4.1123 \\ -0.80981 & -4.1123 \end{bmatrix} \\
C_{c2} &= \begin{bmatrix} 5.5029 & 5.5029 & 4.8801 & 4.8801 \end{bmatrix} \; D_{c2} = \begin{bmatrix} -0.5025 & 3.9468 \end{bmatrix} \\
A_{c3} &= \begin{bmatrix} 38.655 & 38.276 & 34.306 & 34.306 \\ 38.276 & 38.655 & 34.306 & 34.306 \\ -96.682 & -96.682 & -85.499 & -87.442 \\ -96.682 & -96.682 & -87.442 & -85.499 \end{bmatrix} \\
B_{c3} &= \begin{bmatrix} 1.6005 & 26.897 \\ 1.6005 & 26.897 \\ -4.1152 & -68.421 \\ -4.1152 & -68.421 \end{bmatrix} \\
C_{c3} &= \begin{bmatrix} 118.65 & 118.65 & 106.13 & 106.13 \end{bmatrix} \; D_{c3} = \begin{bmatrix} 3.7245 & 83.774 \end{bmatrix}
\end{aligned} \tag{52}$$

It is also worth pointing out that the associated minimum H_∞ performance index is computed as $\gamma_{min} = 0.1$.

Under the previous failure scenario, the proposed control law is implemented and the numerical simulations are plotted in Figures 13–16 in the context of $w(k) = \dfrac{0.8sin(5k)}{k+1}$ and initial condition $x(0) = \begin{bmatrix} -1 & 0.1 & 0 & 0 \end{bmatrix}^T$.

From this simulation, it is evident that the system is stabilized using the proposed controller. Moreover, the closed-loop system continues to have acceptable performances regardless of uncertainties, actuator faults, and sensor saturations.

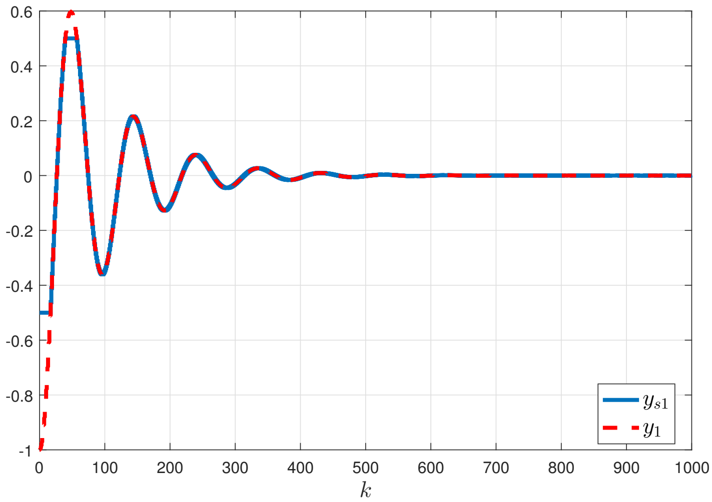

Figure 13. Actual and ideal measurements of y_1.

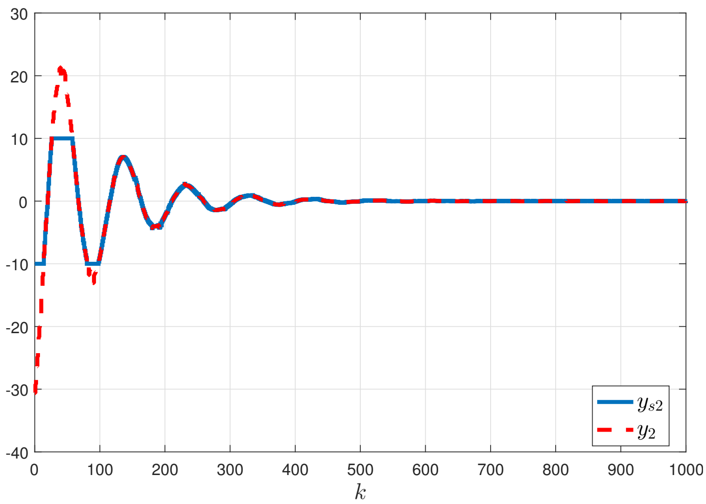

Figure 14. Actual and ideal measurements of y_2.

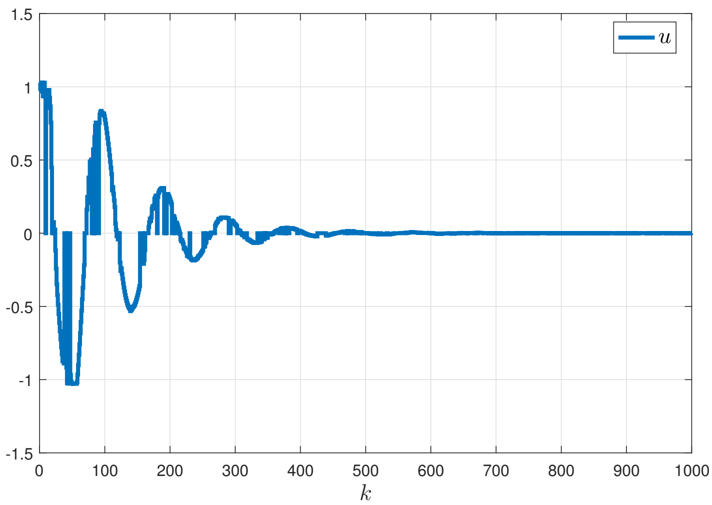

Figure 15. Control response $u(k)$.

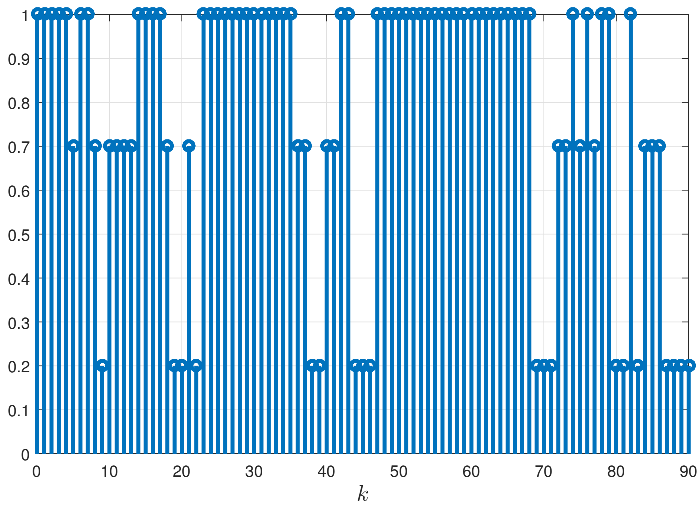

Figure 16. Failure modes.

6. Conclusions

In this study, we have investigated the output-feedback control problem for a class of linear discrete-time singular plants with unmeasured states and in the presence of actuator faults and sensor saturations. A reliable (DOF) controller has been designed to guarantee the stability of the closed-loop system and eliminate the negative effect caused by actuator faults and sensor saturations. The key point of the designed control scheme lies in the establishment of a set of feasible LMI-based constraints so that the closed-loop system is stochastically admissible with H_∞ performance. Three practical examples have been provided to validate the theoretical approach.

There are many interesting studies that should be carried out for Markovian jump singular systems, taking into account the phenomena that can be faced in real systems, such as sensor and actuator saturations, time-varying delay, and unknown transition probabilities.

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References

1. Cao, Y.Y.; Lin, Z.; Chen, B. An Output Feedback H_∞ Controller Design for Linear Systems Subject to Sensor Nonlinearities. *IEEE Trans. Circuits Syst. Fundam. Theory Appl.* **2003**, *50–57*, 914–921.
2. Zuo, Z.; Wang, J.; Huang, L. Output feedback H_∞ controller design for linear discrete-time systems with sensor nonlinearities. *IEE Proc. Control Theory Appl.* **2005**, *152*, 19–26. [CrossRef]
3. Rubio, J.D.J.; Lughofer, E.; Pieper, J.; Cruz, P.; Martinez, D.I.; Ochoa, G.; Islas, M.A.; Garcia, E. Adapting H_∞ controller for the desired reference tracking of the sphere position in the maglev process. *Inf. Sci.* **2021**, *569*, 669–686. [CrossRef]
4. Martinez, D.I.; Rubio, J.D.; Garcia, V.; Vargas, T.M.; Islas, M.A.; Pacheco, J.; Gutierrez, G.J.; Meda-Campaña, J.A.; Mujica-Vargas, D.; Aguilar-Ibañez, C. Transformed Structural Properties Method to Determine the Controllability and Observability of Robots. *Appl. Sci.* **2021**, *11*, 3082. [CrossRef]
5. Escobedo-Alva, J.O.; García-Estrada, E.C.; Páramo-Carranza, L.A.; Meda-Campaña, J.A.; Tapia-Herrera, R. Theoretical Application of a Hybrid Observer on Altitude Tracking of Quadrotor Losing GPS Signal. *IEEE Access* **2018**, *6*, 76900–76908. [CrossRef]
6. Soriano, L.A.; Zamora, E.; Vazquez-Nicolas, J.M.; Hernández, G.; Barraza Madrigal, J.A.; Balderas, D. PD Control Compensation Based on a Cascade Neural Network Applied to a Robot Manipulator. *Front. Neurorobot.* **2020**, *14*, 78. [CrossRef]

7. Zhang, D.; Zhou, Z.; Jia, X. Networked fuzzy output feedback control for discrete-time Takagi-Sugeno fuzzy systems with sensor saturation and measurement noise. *Inf. Sci.* **2018**, *457–458*, 182–194. [CrossRef]
8. Duan, R.; Li, J. Finite-time distributed H_∞ filtering for Takagi-Sugeno fuzzy system with uncertain probability sensor saturation under switching network topology: Non-PDC approach. *Appl. Math. Comput.* **2020**, *371*, 1–21. [CrossRef]
9. Liu, Y.; Ma, Y.; Wang, Y. Reliable sliding mode finite-time control for discrete-time singular Markovian jump systems with sensor fault and randomly occurring nonlinearities. *Int. J. Robust Nonlinear Control* **2018**, *28*, 381–402. [CrossRef]
10. Tao, Y.; Shen, D.; Wang, Y.; Ye, Y. Reliable H_∞ control for uncertain nonlinear discrete-time systems subject to multiple intermittent faultsin sensors and/or actuators. *J. Frankl. Inst.* **2015**, *352*, 4721–4740. [CrossRef]
11. Qiu, J.; Wei, Y.; Karimi, H.R.; Gao, H. Reliable control of discrete-time piecewise-affine time-delay systems via output feedback. *IEEE Trans. Reliab.* **2018**, *67*, 79–91. [CrossRef]
12. Wang, Y.; Shi, P.; Yan, H. Reliable control of fuzzy singularly perturbed systems and its application to electronic circuits. *IEEE Trans. Circuits Syst. I* **2018**, *65*, 3519–3528. [CrossRef]
13. Wang, Y.; Yang, X.; Yan, H. Reliable Fuzzy Tracking Control of Near-Space Hypersonic Vehicle Using Aperiodic Measurement Information. *IEEE Trans. Ind. Electron.* **2019**, *66*, 9439–9447. [CrossRef]
14. Xu, S.; Lam, J.; Zou, Y. H_∞ filtering for singular systems. *IEEE Trans. Autom. Control* **2003**, *48*, 2217–2222.
15. Feng, J.; Cui, P.; Cheng, Z. H_∞ output feedback control for descriptor systems with delayed states. *J. Control Theory Appl.* **2005**, *4*, 342–347. [CrossRef]
16. Boukas, E. Singular linear systems with delay: H_∞ stabilization. *Optim. Control Appl. Methods* **2007**, *28*, 259–274. [CrossRef]
17. Dai, L. Singular Control Systems. In *Lecture Notes in Control and Information Sciences*; Springer: New York, NY, USA, 1989; Volume 118.
18. Duan, G. *Analysis and Design of Descriptor Linear Systems*; Springer: New York, NY, USA, 2010.
19. Kchaou, M. Robust observer-based sliding mode control for nonlinear uncertain singular systems with time-varying delay and input non-linearity. *Eur. J. Control* **2019**, *49*, 15–25. [CrossRef]
20. Liu, Y.; Ma, Y.; Wang, Y. Reliable finite-time sliding-mode control for singular time-delay system with sensor faults and randomly occurring nonlinearities. *Appl. Math. Comput.* **2018**, *320*, 341–357. [CrossRef]
21. Li, M.; Chen, X.; Liu, M.; Zhang, Y.; Zhang, H. Asynchronous Adaptive Fault-Tolerant Sliding-Mode Control for T-S Fuzzy Singular Markovian Jump Systems with Uncertain Transition Rates. *IEEE Trans. Cybern.* **2020**. [CrossRef]
22. Jun, H.; Cui, Y.; Lv, C.; Chen, D.; Zhang, H. Robust adaptive sliding mode control for discrete singular systems with randomly occurring mixed time-delays under uncertain occurrence probabilities. *Int. J. Syst. Sci.* **2020**, *51*, 987–1006.
23. Ma, S.; Cheng, Z.; Zhang, C. Delay-dependent robust stability and stabilisation for uncertain discrete singular systems with time-varying delays. *IET Control Theory Appl.* **2007**, *1*, 1086–1095. [CrossRef]
24. Fang, M. Delay-dependent stability analysis for discrete singular systems with time-varying delays. *Acta Autom. Sin.* **2010**, *36*, 751–755. [CrossRef]
25. Feng, Z.; Li, W.; Lam, J. New admissibility analysis for discrete singular systems with time-varying delay. *Appl. Math. Comput.* **2015**, *265*, 1058–1066. [CrossRef]
26. Regaieg, M.A.; Kchaou, M.; El Hajjaji, A.; Chaabane, M. Robust H_∞ guaranteed cost control for discrete-time switched singular systems with time-varying delay. *Optim. Control Appl. Methods* **2019**, *40*, 119–140. [CrossRef]
27. Ma, Y.; Jia, X.; Zhang, Q. Robust finite-time non-fragile memory H_∞ control for discrete-time singular Markovian jumping systems subject to actuator saturation. *J. Frankl. Inst.* **2017**, *354*, 8256–8282. [CrossRef]
28. Li, F.; Xu, S.; Zhang, B. Resilient Asynchronous H_∞ Control for Discrete-Time Markov Jump Singularly Perturbed Systems Based on Hidden Markov Model. *IEEE Trans. Syst. Man Cybern. Syst.* **2020**, *50*, 2860–2869. [CrossRef]
29. Kchaou, M.; Jerbi, H.; Abassi, R.; VijiPriya, J.; Hmidi, F.; Kouzou, A. Passivity-based asynchronous fault-tolerant control for nonlinear discrete-time singular markovian jump systems: A sliding-mode approach. *Eur. J. Control* **2021**, *60*, 95–113. [CrossRef]
30. Kwon, N.K.; Park, I.S.; Park, P.; Park, C. Dynamic Output-Feedback Control for Singular Markovian Jump System: LMI Approach. *IEEE Trans. Autom. Control* **2017**, *62*, 5396–5400. [CrossRef]
31. Long, S.; Zhong, S. H_∞ control for a class of discrete-time singular systems via dynamic feedback controller. *Appl. Math. Lett.* **2016**, *58*, 110–118. [CrossRef]
32. Zhang, J.; Ma, S. Robust H_∞ indefinite guaranteed cost dynamic output feedback control of singular semi-Markov jump systems. *J. Frankl. Inst.* **2021**. [CrossRef]
33. Park, I.S.; Park, C.; Kwon, N.K.; Park, P. Dynamic output-feedback control for singular interval-valued fuzzy systems: Linear matrix inequality approach. *Inf. Sci.* **2021**, *576*, 393–406. [CrossRef]
34. Wu, Z.G.; Park, J.; Su, H.; Chu, J. Stochastic stability analysis for discrete-time singular Markov jump systems with time-varying delay and piecewise-constant transition probabilities. *J. Frankl. Inst.* **2012**, *349*, 2889–2902. [CrossRef]
35. Ma, Y.; Jia, X.; Liu, D. Finite-time dissipative control for singular discrete-time Markovian jump systems with actuator saturation and partly unknown transition rates. *Appl. Math. Model.* **2018**, *53*, 49–70. [CrossRef]
36. Wang, Z.; Ho, Z.W.; Dong, H.; Gao, H. Robust H_∞Finite-Horizon Control for a Class of Stochastic Nonlinear Time-Varying Systems Subject to Sensor and Actuator Saturations. *IEEE Trans. Autom. Control* **2010**, *55*, 1716–1723.
37. Hu, J.; Wang, Z.; Gao, H.; Stergioulas, L.K. Probability-guaranteed H_∞ finite-horizon filtering for a class of nonlinear time-varying systems with sensor saturations. *Syst. Control Lett.* **2012**, *64*, 477–484. [CrossRef]

38. Khalil, H. *Nonlinear Systems*; Prentice-Hall: Upper Saddle River, NJ, USA, 1996.
39. Li, Z.; Mobin, M.; Keyser, T. Multi-objective and multi-stage reliability growth planning in early product development stage. *IEEE Trans. Reliab.* **2015**, *65*, 769–781. [CrossRef]
40. Xu, S.; Lam, J. *Robust Control and Filtering of Singular Systems*; Springer: Berlin/Heidelberg, Germany, 2006.
41. Ma, S.; Boukas, E. Robust H_∞ filtering for uncertain discrete Markov jump singular systems with mode-dependent time delay. *IET Control Theory Appl.* **2009**, *3*, 351–361. [CrossRef]
42. Peterson, I. A stabilization algorithm for a class uncertain linear systems. *Syst. Control Lett.* **1987**, *8*, 351–357. [CrossRef]
43. Saadni, S.; Chaabane, M.; Mehdi, D. Robust stability and stabilization of a class of singular systems with multiple time-varying delays. *Asian J. Control* **2006**, *8*, 1–11. [CrossRef]
44. Chadli, M.; Darouach, M. Novel bounded real lemma for discrete-time descriptor systems: Application to H_∞ control design. *Automatica* **2012**, *2012*, 449–453. [CrossRef]
45. Chadli, M.; Darouach, M. Further enhancement on robust control design for discrete-time singular systems. *IEEE Trans. Autom. Control* **2014**, *59*, 494–500. [CrossRef]
46. Wang, W.; Ma, S.; Zhang, C. Stability and Static Output Feedback Stabilization for a Class of Nonlinear Discrete-Time Singular Switched Systems. *Int. J. Control Autom. Syst.* **2013**, *11*, 1138–1148. [CrossRef]
47. Ji, X.; Su, H.; Chu, J. Robust state feedback H_∞ control for uncertain linear discrete singular systems. *IET Control Theory Appl.* **2007**, *1*, 195–200. [CrossRef]
48. Belov, A.A.; Andrianova, O.G.; Kurdyukov, A. *Control of Discrete-Time Descriptor Systems: An Anisotropy-Based Approach*; Springer: Cham, Switzerland, 2018.