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The Combined-Unified Hybrid Censored Samples from Pareto Distribution: Estimation and Properties

Khalaf S. Sultan *,† and Walid Emam †

Department of Statistics and Operations Research, College of Science, King Saud University, P.O. Box 2455, Riyadh 11451, Saudi Arabia; wemam.c@ksu.edu.sa
* Correspondence: ksultan@ksu.edu.sa
† The authors contributed equally in this work.

Abstract: In this paper, we use the combined-unified hybrid censoring samples to obtain the maximum likelihood estimates of the unknown parameters, survival, and hazard functions of Pareto distribution. Next, we discuss some efficiency criteria of the maximum likelihood estimators, including; the unbiasedness, consistency, and sufficiency. Additionally, we use MCMC to obtain the Bayesian estimates of the unknown parameters. In addition, we calculate the intervals estimation of the unknown parameters. Finally, we analyze a set of real data in view of the theoretical findings of the paper.

Keywords: maximum likelihood estimates; unbiased estimator; interval estimation; minimum variance bound and relative efficiency; combined hybrid censored and unified hybrid censored samples

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1. Introduction

Pareto distribution was introduced by Pareto [1] for the distribution of income. The importance of Pareto distribution lies in its applications in economics and reliability studies. Arnold [2] has given a wide historical aacount of Pareto distribution and its applications. Estimation and characteristics of Pareto distribution were investigated by many authors, among researchers, see for examples, Malik [3], Arnold and Press [4], Tiwari, Yang and Zalkikar [5], Abdel-Ghaly, Attia and Aly [6], Hossain and Zimmer [7], and Soliman [8]. Saldaña-Zepeda et al. [9] have proposed a goodness-of-fit test for Pareto distribution when the observations are Type-II right censoring. Wu [10] has constructed an interval estimation for Pareto distribution using a doubly Type-II censored sample. Recently, Han [11] has investigated the expected Bayesian estimation and its expected mean square error of Pareto distribution parameter under different loss functions and Poudyal [12] has investigated the truncated, censored, and actuarial payment-type moments of the robust fitting of a single parameter Pareto distribution.

A random variable X follows Pareto distribution $P(k, \alpha)$ if its probability density function (pdf) is given by

$$f(x) = \alpha k^{\alpha} x^{-\alpha-1}, \;\; \alpha > 0, k > 0, \; x \geq k, \tag{1}$$

with the corresponding cumulative distribution function (cdf) is given by

$$F(x) = 1 - \left(\frac{k}{x}\right)^{\alpha}, \;\; x \geq k. \tag{2}$$

The reliability function $R(t)$ and hazard function $H(t)$ are given, respectively, as

$$R(t) = \left(\frac{k}{t}\right)^{\alpha} \text{ and } H(t) = \frac{\alpha}{t}. \tag{3}$$

Quite a few techniques exist to estimate the shape parameter of Pareto distribution. Both of Type-I and Type-II censored are extensively used in practice. Type-I and Type-II censoring schemes can be merged to get the hybrid censoring scheme which was first introduced by Epstein [13]. The hybrid censoring scheme becomes quite important in the reliability and life testing problems, see Fairbanks et al. [14], Draper and Guttman [15], Chen and Bhattacharya [16], Jeong et al. [17], Childs et al. [18], and Gupta and Kundu [19]. Balakrishnan and Kundu [20] have discussed theinferences based on Type-I and Type-II hybrid censoring schemes. Next, they have discussed some details on the generalized hybrid censoring and unified hybrid censoring schemes. Additionally, they have shown the adaption of the hybrid censoring schemes in competing risks set-up and in step-stress modeling. Jeon and Kang [21] have discussed the parameter estimation from half-logistic distribution by using multiply Type-II hybrid censoring. Nassar and Dobbah [22] have investigated the reliability characteristics of bathtub-shaped distribution under adaptive Type-I progressive hybrid censoring. Algarni, Almarashi, and Abd-Elmougoud [23] have discussion the joint Type-I generalized hybrid censoring for estimating the two Weibull distributions. Mohie El-Din, et al. [24] have distressed the estimation and prediction for Pareto distribution under Type-II progressive hybrid censoring scheme, while Çetinkaya [25] has drawn inference based on Type-II hybrid censored data from a Pareto distribution. Huang and Yang [26] have suggested a combined hybrid censoring sampling scheme (CHCS) as follows: Assume n experimental units are placed under a certain experiment and let $X_{m:n}$ and $X_{l:n}$ denote the failure time of the mth and lth units, respectively, such that $(m, l) \in \{1, 2, \ldots, n\}$, $(t_1, t_2) \in (0, \infty)$, $m < l$, $t_1 < t_2$ and let t denote the termination time of the experiment. If the mth failure occurs before time t_1, the experiment terminates at $\min\{X_{l:n}, t_1\}$, when the mth failure occurs in the interval (t_1, t_2), then the experiment is stopped at $X_{m:n}$ and finally when the mth failure occurs after time t_2, the experiment is stooped at t_2. For our later convenience, we abbreviate this scheme as combined CHCS$(m, l; t_1, t_2)$. The system as devolved by Huang and Yang [26] includes six different cases, such that each case the data are unobservable as explained below:

$$t = \begin{cases} X_{m:n}, & 0 < t_1 < X_{m:n} < (t_2 < X_{l:n}), \\ X_{m:n}, & 0 < t_1 < X_{m:n} < (X_{l:n} < t_2), \\ t_2, & 0 < t_1 < t_2 < (X_{m:n} < X_{l:n}), \\ X_{l:n}, & 0 < X_{m:n} < X_{l:n} < (t_1 < t_2), \\ t_1, & 0 < X_{m:n} < t_1 < (X_{l:n} < t_2), \\ t_1, & 0 < X_{m:n} < t_1 < (t_2 < X_{l:n}), \end{cases} \tag{4}$$

where the unobservable data are marked by the parentheses.

Balakrishnan et al. [27] have proposed the unified hybrid censoring scheme (UHCS) that is for a certain $m, l \in \{1, 2, \ldots, n\}$, $(t_1, t_2) \in (0, \infty)$, $m < l$, $t_1 < t_2$ and t denote the experiment termination time. when the mth failure occurs before t_1, then the experiment is terminated at $min\{max\{X_{l:n}, t_1\}, t_2\}$, when mth failure occurs in the interval (t_1, t_2), then the experiment is terminated at $min\{X_{l:n}, t_2\}$ and when the mth failure occurs after time t_2, then the experiment is terminated at $X_{m:n}$. The symbol UHCS$(m, l; t_1, t_2)$ is used for such this scheme. Similarly, each type of these hybrid censored samples includes different six cases such that in each case some part of sample are unobservable as given below

$$t = \begin{cases} t_2, & 0 < t_1 < X_{m:n} < t_2 < (X_{l:n}), \\ X_{l:n}, & 0 < t_1 < X_{m:n} < X_{l:n} < (t_2), \\ X_{m:n}, & 0 < t_1 < t_2, < X_{m:n} < (X_{l:n}), \\ t_1, & 0 < X_{m:n} < X_{l:n} < t_1 < (t_2), \\ X_{l:n}, & 0 < X_{m:n} < t_1 < X_{l:n} < (t_2), \\ t_2, & 0 < X_{m:n} < t_1 < t_2 < (X_{l:n}), \end{cases} \tag{5}$$

where unobservable data are marked in the parentheses.

Emam and Sultan [28] have suggested a unified approach from CHCS$(m, l; t_1, t_2)$, and UHCS$(m, l; t_1, t_2)$ known as the combined-unified hybrid censored scheme (C-UHCS $(m, l; t_1, t_2)$). They have applied the proposed censoring sampling to derive the Bayesian and non-Bayesian estimates from Dagum distribution. We belief no attempt has been made for estimating of the parameters of the Pareto distribution by using CHCS$(m, l; t_1, t_2)$ or UHCS$(m, l; t_1, t_2)$, so, we apply C-UHCS$(m, l; t_1, t_2)$ to Pareto distribution. In this paper, we apply the combined-unified hybrid censored scheme to derive the estimates from Pareto distribution. We consider the maximum likelihood estimator of the parameters of Pareto distribution based on three cases: (i) the location parameter k when the shape parameter α is known; (ii) the shape parameters α when the location parameter k is unknown; and (iii) when the location and shape parameters are unknown. In addition, we state and prove four theorems discuss the efficiency of these estimators based on unbiasedness, consistency, and sufficiency. The remainder of this paper is structured as follows: in Section 2, we present the likelihood function of C-UHCS, in Section 3, we derive the maximum likelihood estimates of the unknown parameters in three different three cases and use them to construct the asymptotic confidence intervals (CI) for both of k and α. Next, in Section 4, we obtain the Bayes estimates of k and α under the squared error loss function using MCMC. In Section 5, we analyze a real dataset in using the theoretical findings of the paper. Finally, in Section 6, we draw a brief conclusion.

2. Likelihood Function of C-UHCS

Consider $X_{1:n}, X_{2:n}, \ldots X_{r:n}$ are the lifetimes of units that placed on a life-test, and let cumulative distribution distribution (cdf) $F(x)$ and probability density distribution (pdf) $f(x)$ and assume that, for any case, the experiment is terminated at t that may refer to time t_1, t_2, observation $x_{m:n}$ or observation $x_{l:n}$, and let r denote the maximum number of failures until t equal, respectively, D_1, D_2, m and l. Emam and Sultan [28] have proposed the likelihood function under the censoring samples C-UHCS$(m, l; t_1, t_2)$ under different choices of r, t and $\mathbf{x}_{r:n} = (x_{1:n}, x_{2:n}, ..., x_{r:n})$ as

$$L(\Omega|\mathbf{x}_{r:n}) = \frac{n!}{(n-r)!}[1 - F(t)]^{n-r} \prod_{i=1}^{r} f(x_{i:n}), \tag{6}$$

where r and t can be chosen in the different cases of censoring as:

	$L^{(C)}(\Omega\|\mathbf{x})$		$L^{(U)}(\Omega\|\mathbf{x})$	
Cases	r	t	r	t
$1:\ 0 < t_1 < X_{m:n} < t_2 < X_{l:n}$	m	$X_{m:n}$	D_2	t_2
$2:\ 0 < t_1 < X_{m:n} < X_{l:n} < t_2$	m	$X_{m:n}$	l	$X_{l:n}$
$3:\ 0 < t_1 < t_2 < X_{m:n} < X_{l:n}$	D_2	t_2	m	$X_{m:n}$
$4:\ 0 < X_{m:n} < X_{l:n} < t_1 < t_2$	l	$X_{l:n}$	D_1	t_1
$5:\ 0 < X_{m:n} < t_1 < X_{l:n} < t_2$	D_1	t_1	l	$X_{l:n}$
$6:\ 0 < X_{m:n} < t_1 < t_2 < X_{l:n}$	D_1	t_1	D_2	t_2

$L^{(C)}(\Omega|\mathbf{x})$ is the likelihood corresponding to CHCS$(m,l;t_1,t_2)$ given by Huang and Yang [26] as

$$L^{(C)}(\Omega|\mathbf{x}) = \begin{cases} \frac{n!}{(n-m)!}[1-F(x_{m:n})]^{n-m}\prod_{i=1}^{m} f(x_{i:n}); & D_1 = 0,\cdots,m-1, D_2 = m, \\ \frac{n!}{(n-D_2)!}[1-F(t_2)]^{n-D_2}\prod_{i=1}^{D_2} f(x_{i:n}); & D_1, D_2 = 0,\cdots,m-1, \\ \frac{n!}{(n-l)!}[1-F(x_{l:n})]^{n-l}\prod_{i=1}^{l} f(x_{i:n}); & D_1 = D_2 = l, \\ \frac{n!}{(n-D_1)!}[1-F(t_1)]^{n-D_1}\prod_{i=1}^{D_1} f(x_{i:n}); & D_1 = D_2 = m,\cdots,l-1, \end{cases} \tag{7}$$

and $L^{(U)}(\Omega|\mathbf{x})$ is the likelihood corresponding to UHCS$(m,l;t_1,t_2)$ given by Balakrishnan et al. [27] as

$$L^{(U)}(\Omega|\mathbf{x}) = \begin{cases} \frac{n!}{(n-D)!}[1-F(t_1)]^{n-D}\prod_{i=1}^{m} f(x_{i:n}); & D_1 = D_2 = D = l,\cdots,n, \\ \frac{n!}{(n-l)!}[1-F(x_{l:n})]^{n-r}\prod_{i=1}^{l} f(x_{i:n}); & D_1 = m,\cdots,l-1, D_2 = l, \\ \frac{n!}{(n-D_2)!}[1-F(t_2)]^{n-D_2}\prod_{i=1}^{D_2} f(x_i); & D_1, D_2 = m,\cdots,l-1, \\ \frac{n!}{(n-l)!}[1-F(x_{l:n})]^{n-l}\prod_{i=1}^{l} f(x_{i:n}); & D_1 = 0,\cdots,m-1, D_2 = l, \\ \frac{n!}{(n-D_2)!}[1-F(t_2)]^{n-D_2}\prod_{i=1}^{D_2} f(x_{i:n}); D_1 = 0..m-1, D_2 = m...l-1, \\ \frac{n!}{(n-m)!}[1-F(x_{m:n})]^{n-m}\prod_{i=1}^{m} f(x_{i:n}); D_1, D_2 = 0,\cdots,m-1. \end{cases} \tag{8}$$

3. The Maximum Likelihood Estimates

Let $X_{1:n}, X_{2:n}, \ldots X_{r:n}$ be the C-UHCS$(m,l;t_1,t_2)$ from the Pareto distribution given in (1). The likelihood function given in (8) in this case may written as

$$L = \frac{n!}{(n-r)!}\left[\left(\frac{k}{t}\right)^{\alpha}\right]^{n-r}\prod_{i=1}^{r}\frac{\alpha}{x_{i:n}}\left(\frac{k}{x_{i:n}}\right)^{\alpha}, \; k \le x_{1:n} \le x_{2:n} \le \cdots \le t, \tag{9}$$

and hence

$$\log L = \log\left[\frac{n!}{(n-r)!\prod_{i=1}^{r} x_{i:n}}\right] + r\log\alpha + n\alpha\log k - \alpha\log\left(t^{n-r}\prod_{i=1}^{r} x_{i:n}\right). \tag{10}$$

From (9), we consider the following cases:

3.1. Case 1: α Is Known

The maximum likelihood estimate (MLE) of the parameter k is given by

$$\tilde{k} = x_{1:n}. \tag{11}$$

From (11), it is easy to show that the $X_{1:n} \sim P(k, n\alpha)$, hence

$$E(\tilde{k}) = \frac{n\alpha}{n\alpha-1}k, \;\; Var(\tilde{k}) = \frac{n\alpha k^2}{(n\alpha-2)(n\alpha-1)^2}, \text{ and } MSE(\tilde{k}) = \frac{2k^2}{(n\alpha-2)(n\alpha-1)}, \tag{12}$$

this shows that, the estimator in (11) is consistent sufficient estimator of k, while the estimator

$$\hat{k} = \frac{n\alpha - 1}{n\alpha} x_{1:n}, \tag{13}$$

is consistent sufficient unbiased estimate of k and more efficient than $\tilde{k}$, since

$$Var(\hat{k}) = \frac{k^2}{n\alpha(n\alpha - 2)} < Var(\tilde{k}),\ n\alpha > 2. \tag{14}$$

The corresponding MLEs of the reliability and hazard functions are given, respectively, by

$$\hat{R}(t) = \left(\frac{n\alpha - 1}{n\alpha t} x_{1:n}\right)^{\alpha} \text{ and } \hat{H}(t) = \frac{\alpha}{t}. \tag{15}$$

In order to construct a confidence estimation for k in this case, we consider the pivotal quantity

$$W_1 = \frac{\hat{k} - \mu_{\hat{k}}}{\sigma_{\hat{k}}} = \sqrt{n\alpha(n\alpha - 2)}\left(\frac{n\alpha - 1}{n\alpha k} x_{1:n} - 1\right),\ n\alpha > 2. \tag{16}$$

Again, it is easy to show that the distribution of W_1 given in (16) follows Pareto distribution of the second kind with shape parameter $\alpha^* = n\alpha$, location parameter $\theta^* = -\sqrt{n\alpha(n\alpha - 2)} + \left((n\alpha - 1)\sqrt{\frac{n\alpha - 2}{n\alpha}}\right)^{n\alpha}$, and scale parameter $k^* = \left((n\alpha - 1)\sqrt{\frac{n\alpha - 2}{n\alpha}}\right)^{n\alpha}$. Thus $F_{w_1}(w)$ can be written as

$$F_{W_1}(w) = 1 - \left(\frac{(n\alpha - 1)\sqrt{\frac{n\alpha - 2}{n\alpha}}}{w + \sqrt{n\alpha(n\alpha - 2)}}\right)^{n\alpha},\ w > -\sqrt{n\alpha(n\alpha - 2)} + \left((n\alpha - 1)\sqrt{\frac{n\alpha - 2}{n\alpha}}\right)^{n\alpha},$$

and then the $(1 - \gamma)100\%$ confidence interval for k is constructed by

$$\left(\left(\frac{\gamma}{2}\right)^{\frac{1}{n\alpha}} X_{1:n}\ ,\ \left(1 - \frac{\gamma}{2}\right)^{\frac{1}{n\alpha}} X_{1:n}\right). \tag{17}$$

The mean length of the confidence interval given in (17) is $\frac{n\alpha k}{n\alpha - 1}\left[(1 - \gamma/2)^{\frac{1}{n\alpha}} - (\gamma/2)^{\frac{1}{n\alpha}}\right]$ which approaches to zero as $n \to \infty$.

3.2. Case 2: k Is Known

The MLE of α can be obtained from (10) as

$$\tilde{\alpha} = \frac{r}{\log[y_1]}, \tag{18}$$

where $y_1 = \left(\frac{t}{k}\right)^{n-r} \prod_{i=1}^{r} \left(\frac{x_{i:n}}{k}\right)$. From (18), we can see that $\tilde{\alpha}$ is sufficient statistics of α. The mean and variance of the MLE of α and $\frac{1}{\alpha}$ can be derived in Theorem 1 below.

Theorem 1. *If $X_{1:n}, X_{2:n}, \ldots X_{r:n}$ be the C-UHCS$(m, l; t_1, t_2)$ from Pareto distribution given in (1), then the consistent sufficient MVUE of $\frac{1}{\alpha}$ is $\frac{1}{\tilde{\alpha}}$, where $\tilde{\alpha}$ is a biased consistent sufficient statistic of α and for $r > 2$, we have*

$$E(\tilde{\alpha}) = \frac{r}{r - 1}\alpha,\ \ Var(\tilde{\alpha}) = \frac{r^2\alpha^2}{(r - 2)(r - 1)^2} \text{ and } MSE(\tilde{\alpha}) = \frac{(r + 2)\alpha^2}{(r - 1)(r - 2)}, \tag{19}$$

and

$$E(\frac{1}{\tilde{\alpha}}) = \frac{1}{\alpha}, \; Var(\frac{1}{\tilde{\alpha}}) = \frac{1}{r\alpha^2}. \tag{20}$$

Proof. See Appendix A. □

The unbiased estimate of α is given by

$$\hat{\alpha} = \frac{r-1}{r}\tilde{\alpha} = \frac{r-1}{\log\left[\left(\frac{t}{k}\right)^{n-r}\prod_{i=1}^{r}\left(\frac{x_{i:n}}{k}\right)\right]},$$

and hence

$$Var(\hat{\alpha}) = \frac{\alpha^2}{r-2} < Var(\tilde{\alpha}), \; r > 2,$$

this shows that $\hat{\alpha}$ is an unbiased consistent sufficient estimate for α.

3.3. Case 3: k and α Are Unknown

The MLEs of $k, \alpha, R(t)$ and $H(t)$ can be derived, respectively, by

$$\tilde{k} = x_{1:n}, \; \tilde{\alpha} = \frac{r}{\log(y_2)}, \; \tilde{R}(t) = exp\left(\frac{r\log\left(\frac{x_{1:n}}{t}\right)}{\log[y_2]}\right) \text{ and } \tilde{H}(t) = \frac{r}{t\log[y_2]}, \tag{21}$$

where

$$y_2 = \left(\frac{t}{x_{1:n}}\right)^{n-r}\prod_{i=2}^{r} x_{i:n}. \tag{22}$$

From (21), we see that $(\tilde{k}, \tilde{\alpha})$ are jointly sufficient statistics for (k, α). The following theorem states the mean and variance of the MLE of $\frac{1}{\tilde{\alpha}}$ in this case.

Theorem 2. *If $X_{1:n}, X_{2:n}, \ldots X_{r:n}$ be the C-UHCS$(m, l; t_1, t_2)$ from Pareto distribution given in (1). For the biased estimate of $\frac{1}{\alpha}$, we have*

$$E\left(\frac{1}{\tilde{\alpha}}\right) = \frac{r-1}{r\alpha}, \; Var(\frac{1}{\tilde{\alpha}}) = \frac{r-1}{r^2\alpha^2}, \text{ and } MSE(\frac{1}{\tilde{\alpha}}) = \frac{1}{r\alpha^2}, \tag{23}$$

while $\log[y_2]/(r-1)$ is an unbiased estimate of $1/\alpha$ with variance $1/((r-1)\alpha^2)$ and relative efficiency $1 - 1/r$.

Proof. See Appendix B. □

Theorem 3. *If $X_{1:n}, X_{2:n}, \ldots X_{r:n}$ be the C-UHCS$(m, l; t_1, t_2)$ from Pareto distribution given in (1). For the biased estimate of α, we have*

$$E(\tilde{\alpha}) = \frac{r}{r-2}\alpha, \; r > 2, \tag{24}$$

$$Var(\tilde{\alpha}) = \frac{r^2\alpha^2}{(r-3)(r-2)^2}, \; r > 3, \tag{25}$$

$$MSE(\tilde{\alpha}) = \frac{r^2 + 2r - 6}{(r-3)(r-2)^2}\alpha^2, \; r > 3,. \tag{26}$$

and for the unbiased estimate of α, we have

$$\hat{\alpha} = \frac{r-2}{r}\tilde{\alpha} = \frac{r-2}{\log y_2}, \; Var(\hat{\alpha}) = \frac{\alpha^2}{r-3}, r > 3 \text{ and } RE(\hat{\alpha}) = \frac{r-3}{r}. \tag{27}$$

Proof. See Appendix C. □

Theorem 4. *If $X_{1:n}, X_{2:n}, \dots X_{r:n}$ be the C-UHCS$(m,l;t_1,t_2)$ from Pareto distribution given in (1). Then the unbiased MLE of k of Pareto distribution is*

$$\left(1 - \frac{\log y_2}{n(r-1)}\right) X_{1:n}. \tag{28}$$

Proof. See Appendix D. □

Remark 1. *The MLEs of Pareto parameters and their properties based on complete sample given in Baxter [29] can be easily derived from our results in the cases 1, 2, and 3 by setting $r = n$.*

Now, we apply the normality appromimation of the MLEs to obtain the appromimate confidence intervals for k and α. The variance-covariance matrix of the parameters $\hat{V} = [\sigma_{i,j}], i, j = 1, 2$ can be witten as

$$V(\hat{k}, \hat{\alpha}) = -\begin{bmatrix} \dfrac{\partial^2 \log[L]}{\partial k^2} & \dfrac{\partial^2 \log[L]}{\partial k \partial \alpha} \\ \dfrac{\partial^2 \log[L]}{\partial \alpha \partial k} & \dfrac{\partial^2 \log[L]}{\partial \alpha^2} \end{bmatrix}^{-1}_{(k=\hat{k}, \alpha=\hat{\alpha})}, \tag{29}$$

where the elements of the observed variance-covariance matrix can be derived from (10) as

$$\frac{\partial^2 \log[L]}{\partial k^2} = \frac{-n\alpha}{k^2}, \qquad \frac{\partial^2 \log[L]}{\partial \alpha^2} = \frac{-r}{\alpha^2}, \qquad \frac{\partial^2 \log[L]}{\partial \alpha \partial k} = \frac{n}{k}, \tag{30}$$

and hence the minimum variance bound of the MLEs of α and $\frac{1}{\alpha}$ are given, respectively, by $\frac{\alpha^2}{r}$ and $\frac{1}{r\alpha^2}$. $V(k, \alpha)$ takes the form

$$V(k, \alpha) = \frac{k^2 \alpha}{n(r - n\alpha)} \begin{bmatrix} \dfrac{r}{\alpha^2} & \dfrac{k}{n} \\ \dfrac{k}{n} & \dfrac{n\alpha}{k^2} \end{bmatrix}, \hat{k} = \left(1 - \frac{\log y_2}{n(r-1)}\right) x_{1:n} \text{ and } \hat{\alpha} = \frac{r-2}{\log y_2}, \tag{31}$$

then the $100(1-\tau)\%$ confidence intervals for the parameters k and α are given by

$$\left(\hat{k} - z_{\tau/2}\sqrt{V(\hat{k})}, \hat{k} + z_{\tau/2}\sqrt{V(\hat{k})}\right) \text{ and } \left(\hat{\alpha} - z_{\tau/2}\sqrt{V(\hat{\alpha})}, \hat{\alpha} + z_{\tau/2}\sqrt{V(\hat{\alpha})}\right), \tag{32}$$

where $V(\hat{k})$ and $V(\hat{\alpha})$ are the estimated variances of $\hat{k}$ and $\hat{\alpha}$, which are given by the diagonal elements of $V(\hat{k}, \hat{\alpha})$, and $z_{\tau/2}$ is the upper $(\tau/2)$ percentile of standard normal distribution, where $\tau/2 = \int_{z_{\tau/2}}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-z^2/2} dz$. The delta method was used for derive approximate confidence intervals for $R(t)$ and $H(t)$ as

$$\Psi_1 = \begin{bmatrix} \frac{\partial R(t)}{\partial k} & \frac{\partial R(t)}{\partial \alpha} \end{bmatrix}_{(k=\hat{k}, \alpha=\hat{\alpha})}, \text{ and } \Psi_2 = \begin{bmatrix} \frac{\partial H(t)}{\partial k} & \frac{\partial H(t)}{\partial \alpha} \end{bmatrix}_{(k=\hat{k}, \alpha=\hat{\alpha})}, \tag{33}$$

where

$$\frac{\partial R(t)}{\partial k} = \left(\frac{k}{t}\right)^{\alpha-1} \left(\frac{\alpha}{t}\right), \frac{\partial R(t)}{\partial \alpha} = \left(\frac{k}{t}\right)^{\alpha} \log\left(\frac{k}{t}\right), \frac{\partial H(t)}{\partial k} = \frac{1}{t}, \text{ and } \frac{\partial H(t)}{\partial \alpha} = 0. \tag{34}$$

Then, the approximate estimates of $V(\hat{R}(t))$ and $V(\hat{H}(t))$ are given, respectively, by

$$V(\hat{R}(t)) \simeq \Psi_1 V\left(\hat{k}, \hat{\alpha}\right) \Psi_1^t, \text{ and } V\left(\hat{H}(t)\right) \simeq \Psi_2 V\left(\hat{k}, \hat{\alpha}\right) \Psi_2^t, \tag{35}$$

where Ψ^t is the transpose of Ψ and

$$\hat{R}(t) \pm z_{\tau/2} \sqrt{V(\hat{R}(t))} \text{ and } \hat{H}(t) \pm z_{\tau/2} \sqrt{V(\hat{H}(t)}. \tag{36}$$

4. Bayesian Estimation: MCMC Method

In the Bayesian approach, the risk functions are chosen depending on how one measures the distance between the estimate and the unknown parameter. To perform the Bayesian analysis, usually we use loss the squared error (SE) loss function as

$$L_{SE}(g(\varphi), \hat{g}(\varphi)) = (g(\varphi) - \hat{g}(\varphi))^2, \tag{37}$$

where $\hat{g}(\varphi)$ is an estimate of $g(\varphi)$ and the Bayes estimate of $g(\varphi)$ using the SE loss function is given by

$$\hat{g}_{SE}(\varphi) = E_{\varphi}[g(\varphi)|\mathbf{x}_{r:n}]. \tag{38}$$

In this section, we use the Metropolis Hastings algorithm within Gibbs sampling approach for generating random samples from the conditional densities of the parameters and use them to get the Bayian estimates and interval (HPD credible intervals) estimates of the unknown parameters. The unknown parameters k and α are assigned independent gamma distributions. Then, the joint prior distribution for k and α, is given by

$$\pi(k, \alpha) = \frac{b^a}{\Gamma(a)} k^{a-1} \exp[-bk] * \frac{d^c}{\Gamma(c)} \alpha^{c-1} \exp[-d\alpha], \quad a, b, c, d > 0. \tag{39}$$

Then, the posterior distributrion of k and α, is given by

$$\begin{aligned} \pi^*(k, \alpha) &= \frac{L\pi(k, \alpha)}{\int_\alpha \int_k L\pi(k, \alpha) \partial k \partial \alpha} \\ &= \frac{\alpha^r k^{n\alpha} \left(T^{n-r} \prod_{i=1}^r x_{i:n}\right)^{-\alpha} k^{a-1} \exp[-bk] \alpha^{c-1} \exp[-d\alpha]}{\int_\alpha \int_k \alpha^r k^{n\alpha} \left(T^{n-r} \prod_{i=1}^r x_{i:n}\right)^{-\alpha} k^{a-1} \exp[-bk] \alpha^{c-1} \exp[-d\alpha] \partial k \partial \alpha}. \end{aligned} \tag{40}$$

In the following algorithm, we apply Metropolis Hastings (M-H) technique with normal proposal distribution for generainge samples from these distributions.

1. Start with initial values of the parameters $(k^{(0)}, \alpha^{(0)})$. Then, set $i = 1$;
2. Generate $k^{(*)}, \alpha^{(*)}$ using the proposal distributions $N(k^{(i-1)}, Var(\hat{k}))$ and $N(\alpha^{(i-1)}$, $Var(\hat{\alpha}))$, respectively, such that $Var(\hat{k})$ and $Var(\hat{\alpha})$ are given in (3.21);
3. Compute the acceptance probability $r = min\left(1, \frac{\pi^*(k^{(*)}, \alpha^{(*)})}{\pi^*(k^{(i-1)}, \alpha^{(i-1)})}\right)$;
4. Generate U from uniform (0, 1);
5. Accept the proposal distribution and set $\left(k^{(i)}, \alpha^{(i)}\right) = \left(k^{(*)}, \alpha^{(*)}\right)$ if $U < r$. Otherwise, reject the proposal distribution and set $\left(k^{(i)}, \alpha^{(i)}\right) = \left(k^{(i-1)}, \alpha^{(i-1)}\right)$;
6. Set $i = i + 1$;
7. Repeat Steps 2–6, M times, and obtain $k^{(i)}$ and $\alpha^{(i)}$ for $i = 1, ..., M$.

By using the generated random samples from the Gibbs sampling procedure with N unburn units, then Bayes estimate of the parameters using the squared error loss functions are

$$\hat{k} = \frac{1}{M-N}\sum_{i=N+1}^{M} k^{(i)}, \qquad \hat{\alpha} = \frac{1}{M-N}\sum_{i=N+1}^{M} \alpha^{(i)},$$

$$\hat{R}(t) = \frac{1}{M-N}\sum_{i=N+1}^{M} \left(\frac{k^{(i)}}{t}\right)^{\alpha^{(i)}}, \text{ and } \hat{H}(t) = \frac{1}{M-N}\sum_{i=N+1}^{M} \frac{\alpha^{(i)}}{t}.$$

MCMC HPD credible interval Algorithm:

1. Arrange the values of $k^{(*)}, \alpha^{(*)}, R^{(*)}$ and $H^{(*)}$ in increasing magnitude;
2. Find the positions of the lower bounds which is $(M-N) * q/2$, where q is the significance, then determine the lower bounds of k, α, R and H;
3. Find the positions of the upper bounds which is $(M-N) * (1-q/2)$, then determine the upper bounds of k, α, R and H;
4. Repeat the above steps M times. Find the average value of the lower and upper bounds MCMC HPD credible interval of k, α, R and H.

5. Data Analysis

In this section, we apply the proposed MLEs and the Bayesian estimates to analyze a set of real data distributed as Pareto distribution given by Nigm and Hamdy [30] and Wong [31]. The data represents the first 10 observations of sample size n = 15 businesses as: 1.01, 1.05, 1.08, 1.14, 1.28, 1.30, 1.33, 1.43, 1.59, 1.62.

The calculations are carried out through the steps below:

1. Assume different censoring schemes from our model C-UHCS$(m, l; t_1, t_2)$.
2. Calculate the MLEs estimations of $k, \alpha, R(t)$ and $H(t)$ at the termination time T.
3. Calculate the Bayesian estimations of $k, \alpha, R(t)$ and $H(t)$ at the termination time T by MCMC (with 100,000 repetitions and 20,000 burns).
4. For the Bayesian analysis, we select the values of the hyper-parameter a, b, c, and d as:
   (a) Calculate the MLEs of k and α using (3.23) when $n = 15$, $r = 10$ and $t = x_{10:15} = 1.62$ as $\hat{k} = 0.983433$ and $\hat{\alpha} = 3.80173$.
   (b) Assume $k \sim gamma(a, b)$ and $\alpha \sim gamma(c, d)$, hence solve the two equations (the mean and variance of gamma distribution) $ab = \hat{k}$ and $ab^2 = 0.01$ to get $a = 96$ and $b = 0.01$
   (c) Repeat step (b) for c and d to get $c = 1445$ and $d = 0.0026$.
5. The corresponding variances of the point estimates are calculated.
6. The 95% and 90% interval estimation using of the unknown parameters, as well as the reliability and hazard function, are calculated.
7. The numerical results are displayed in Table 1.

From Table 1, we see that

(i) The estimates of Type-I and Type-II censoring are very close as both of T and $X_{r:n}$ become very closed to each other;
(ii) In the most cases, the standard deviation of the Bayesian estimate is smaller than the MLE;
(iii) In the most cases, the interval width of the Bayesian estimate is shorter than the MLE at the some confidence level;
(iv) In general the model C-UHCS enables us to have flexible way for selecting the censoring schemes.

Table 1. Estimation of the unknown parameters, reliability and hazard functions under different censoring choices.

T		**Point Estimation**		**Interval Estimation**			
		MLE	**Bayes**	**MLE**		**Bayes**	
				95%	**90%**	**95%**	**90%**
$X_{m=5,15} = 1.28$	k	0.976	1.042	0.818 1.134	0.843 1.109	0.963 1.075	0.980 1.074
		0.081	0.066	0.317	0.267	0.112	0.094
	α	2.962	2.963	1.792 4.132	1.977 3.947	2.867 3.057	2.872 3.052
		0.597	0.841	2.340	1.970	0.190	0.180
	$R(.)$	0.223	0.271	0.000 0.485	0.002 0.444	0.213 0.303	0.225 0.301
		0.134	0.841	0.485	0.337	0.090	0.076
	$H(.)$	1.828	1.829	0.845 2.812	1.001 2.656	1.770 1.887	1.773 1.884
		0.502	0.841	1.444	1.216	0.117	0.111
$X_{l=7,15} = 1.33$	k	0.983	1.056	0.831 1.135	0.855 1.111	0.991 1.082	1.007 1.082
		0.077	0.077	0.303	0.255	0.091	0.075
	α	3.730	3.732	2.402 5.058	2.612 4.848	3.635 3.825	3.640 3.820
		0.678	0.091	2.656	2.236	0.190	0.180
	$R(.)$	0.155	0.203	0.025 0.285	0.046 0.265	0.159 0.227	0.169 0.225
		0.066	0.091	0.309	0.260	0.068	0.056
	$H(.)$	2.302	2.303	1.175 3.430	1.353 3.251	2.244 2.361	2.247 2.358
		0.575	0.091	1.640	1.380	0.117	0.111
$X_{l=9,15} = 1.59$	k	0.984	1.059	0.813 1.155	0.840 1.128	0.998 1.083	1.012 1.083
		0.087	0.079	0.341	0.287	0.085	0.071
	α	3.869	3.872	2.477 5.260	2.697 5.040	3.774 3.964	3.780 3.959
		0.710	0.091	2.783	2.343	0.190	0.180
	$R(.)$	0.145	0.193	0.042 0.249	0.058 0.233	0.153 0.215	0.161 0.213
		0.053	0.091	0.291	0.245	0.062	0.052
	$H(.)$	2.388	2.390	1.175 3.601	1.367 3.410	2.330 2.447	2.333 2.444
		0.619	0.091	1.718	1.446	0.117	0.111
$t_1 = 1.3, D = 6$	k	0.980	1.050	0.827 1.134	0.851 1.109	0.979 1.079	0.996 1.078
		0.078	0.072	0.307	0.259	0.100	0.083
	α	3.379	3.380	2.123 4.635	2.321 4.437	3.285 3.474	3.290 3.469
		0.641	0.426	2.513	2.115	0.189	0.179
	$R(.)$	0.183	0.232	0.007 0.359	0.035 0.331	0.181 0.259	0.192 0.257
		0.089	0.426	0.349	0.294	0.078	0.064
	$H(.)$	2.086	2.086	1.025 3.146	1.193 2.979	2.028 2.144	2.031 2.141
		0.541	0.426	1.551	1.306	0.117	0.111
$t_2 = 1.4, D = 8$	k	0.983	1.058	0.822 1.145	0.848 1.119	0.995 1.083	1.010 1.082
		0.082	0.078	0.323	0.272	0.087	0.072
	α	3.802	3.805	2.442 5.162	2.657 4.947	3.707 3.897	3.713 3.892
		0.694	0.057	2.720	2.290	0.190	0.180
	$R(.)$	0.150	0.198	0.035 0.265	0.053 0.247	0.156 0.221	0.165 0.219
		0.059	0.057	0.299	0.252	0.065	0.054
	$H(.)$	2.347	2.349	1.177 3.516	1.362 3.331	2.288 2.406	2.292 2.403
		0.597	0.057	1.679	1.414	0.117	0.111
$t_2 = 1.7, D = 10$	k	0.986	1.064	0.822 1.151	0.848 1.125	1.007 1.086	1.022 1.085
		0.084	0.083	0.329	0.277	0.078	0.063
	α	4.246	4.251	2.786 5.707	3.017 5.476	4.152 4.342	4.158 4.337
		0.745	0.452	2.922	2.459	0.189	0.179
	$R(.)$	0.122	0.168	0.041 0.202	0.054 0.189	0.132 0.187	0.140 0.185
		0.041	0.452	0.254	0.214	0.054	0.044
	$H(.)$	2.621	2.624	1.345 3.897	1.547 3.695	2.563 2.680	2.567 2.677
		0.651	0.452	1.803	1.518	0.117	0.110

The second rows represent the standard deviation of the point estimates and width of the interval estimates.

6. Conclusions

In this paper, MLEs and the Bayesian estimates of the unknown parameters from Pareto distribution under C-UHCS$(m, l; t_1, t_2)$ model are obtained. Some efficiency properties of the MLEs estimate are discussed, including, the unbiasedness, minimum variance unbiased estimate and sufficiency. MCMC technique is used to carry out the Bayesian estimates. The interval estimates are developed using the observed Fisher information matrix and MCMC. The theoretical findings of the paper are applied to analyze a real dataset under different choices of the censoring. The numerical results show the efficiency performance of the proposed model. Finally, the proposed censoring model has flexible futures to switch between Type-I and Type-II censoring according the experiments need.

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Appendix A. Proof of Theorem 1

Proof. By using the well known relation

$$1 = \int_{k}^{x_1}\int_{x_1}^{x_2}\cdots\int_{x_{r-1}}^{\infty} L dx = \int\cdots\int A\alpha^r y_1^{-\alpha} dx, \tag{A1}$$

where

$$A = \frac{n!}{(n-r)!}\frac{1}{\prod_{i=1}^{r} x_{i:n}}.$$

From (A1), we may write

$$\int\cdots\int A y_1^{-\alpha} dx = \alpha^{-r}, \tag{A2}$$

differentiating both sides of (A2) with respect to α, we have

$$\int\cdots\int A y_1^{-\alpha}\log y_1 dx = r\alpha^{-r-1}, \tag{A3}$$

$$\int\cdots\int A y_1^{-\alpha}(\log y_1)^2 dx = r(r+1)\alpha^{-r-2}. \tag{A4}$$

Integrating both sides of (A2) with respect to α, we have

$$\int\cdots\int A y_1^{-\alpha}\frac{1}{\log y_1} dx = \frac{\alpha^{-r+1}}{r-1} + c_1,\ r > 1, \tag{A5}$$

$$\int\cdots\int A y_1^{-\alpha}\frac{1}{(\log y_1)^2} dx = \frac{\alpha^{-r+2}}{(r-1)(r-2)} + c_1\alpha + c_2,\ r > 2. \tag{A6}$$

It is easy to show form (A5) and (A6) that

$$\lim_{\alpha\to\infty} y_1^{-\alpha} = \lim_{\alpha\to\infty}\alpha^{-r+1} = \lim_{\alpha\to\infty}\alpha^{-r+2} = 0, \text{ then } c_1 = c_2 = 0.$$

Now, from (A3), we have

$$\int\cdots\int A\alpha^r y_1^{-\alpha}\log y_1 dx = E(\log y_1) = \frac{r}{\alpha}.$$

Similarly, from (A4)–(A6), we have

$$E(\log y_1)^2 = \frac{r(r+1)}{\alpha^2},\ E\left(\frac{1}{\log y_1}\right) = \frac{\alpha}{(r-1)} \text{ and } E\left(\frac{1}{(\log y_1)^2}\right) = \frac{\alpha}{(r-1)(r-2)}, \quad \text{(A7)}$$

and hence Theorem 1 is proved. □

Appendix B. Proof of Theorem 2

Proof. Let $Y = \log\left(\frac{X_{1:n}}{k}\right)^n \sim \text{Exponential}(\frac{1}{\alpha})$, then

$$\begin{aligned} E(Y^m) &= \frac{m!}{\alpha^m}, && \text{(A8)} \\ E(\log(y_2)) &= E[\log(y_1) - Y] = \frac{r-1}{\alpha}, && \text{(A9)} \\ E[\log(y_2)]^2 &= E[\log(y_1)]^2 + E[Y^2] - 2E[Y\log(y_1)] \\ &= \frac{r(r+1)}{\alpha^2} + \frac{2}{\alpha^2} - 2E[\log Y \log y_1], && \text{(A10)} \end{aligned}$$

and let

$$E[Y \log y_1] = g(\alpha) \text{ then } \alpha^{-r} g(\alpha) = \int \cdots \int A y_1^{-\alpha} y \log y_1 dx,$$

by integrating over α, we get

$$\alpha^r \int \alpha^{-r} g(\alpha) d\alpha = -E[Y] = \frac{-1}{\alpha},$$

differentiating with respect to α, we have

$$g(\alpha) = \frac{r+1}{\alpha^2}, \quad \text{(A11)}$$

hence

$$E[\log(y_2)]^2 = \frac{r(r-1)}{\alpha^2}, \text{ and } Var(\log y_2)) = \frac{r-1}{\alpha^2}. \quad \text{(A12)}$$

Then

$$E\left(\frac{1}{\tilde{\alpha}}\right) = \frac{r-1}{r\alpha},\ Var\left(\frac{1}{\tilde{\alpha}}\right) = \frac{r-1}{r^2\alpha^2} \text{ and } MSE\left(\frac{1}{\tilde{\alpha}}\right) = \frac{1}{r\alpha^2}. \quad \text{(A13)}$$

Hence Theorem 2 is proved. □

Appendix C. Proof of Theorem 3

Proof. Consider the relation

$$\int \cdots \int A\alpha^r y_1^{-\alpha} y^j dx = E(Y^j) = \frac{j!}{\alpha^j},$$

then

$$\int \cdots \int A y_1^{-\alpha} y^j dx = j! \alpha^{-(r+j)}. \quad \text{(A14)}$$

Integrating both sides of (A14) $(j+1)$ times over α, we get

$$\int \cdots \int A y_1^{-\alpha} \frac{y^j}{(\log y_1)^j} dx = \frac{j! \alpha^{-(r-1)}}{(r-1)r(r+1)\ldots(r+j-1))},$$

then

$$\begin{aligned}\int\cdots\int A\alpha^r y_1^{-\alpha}\frac{y^j}{(\log y_1)^{j+1}}dx &= E\left(\frac{Y^j}{(\log y_1)^{j+1}}\right)=\frac{j!(r-2)!\alpha}{(r+j-1)!}\\ &= \alpha B(j+1,r-1),\end{aligned}\tag{A15}$$

where $B(j+1,r-1)$ is the beta function defined as $B(j+1,r-1)=\int_0^1 t^j(1-t)^{r-2}dt$.
Similarly, integrating both sides of (A14) $(j+2)$ times over α, we have

$$E\left(\frac{y^j}{(\log y_1)^{j+2}}\right)=B(j+2,r-2)\frac{\alpha^2}{j+1}.\tag{A16}$$

Now to find the mean and variance of $\tilde{\alpha}$, we get

$$E\left(\frac{1}{\log y_2}\right)=E\left(\frac{1}{\log y_1-Y}\right)=E\left[\frac{1}{\log y_1}\left(1-\frac{Y}{\log y_1}\right)^{-1}\right]=\alpha\sum_{i=0}^{\infty}B(j+1,r-1),\tag{A17}$$

using the the following relation $\sum_{k=0}^{\infty}B(x,y+k)=B(x-1,y)$, see Gradshtegn and Ryzhik [32], we have

$$E\left(\frac{1}{\log y_2}\right)=\alpha B(r-1,1)=\frac{\alpha}{r-2},\tag{A18}$$

and

$$E\left(\frac{1}{\log y_2}\right)^2=\sum_{j=0}^{\infty}(j+1)E\left(\frac{Y^j}{(\log y_1)^{j+1}}\right)=\alpha^2\sum_{j=0}^{\infty}B(j+2,r+2)=\frac{\alpha^2}{(r-2)(r-3)}.\tag{A19}$$

Then

$$\begin{aligned}E(\tilde{\alpha}) &= E\left(\frac{r}{\log y_2}\right)=\frac{r\alpha}{r-2},\\ Var(\tilde{\alpha}) &= \frac{r^2\alpha^2}{(r-3)(r-2)^2},\\ MSE(\tilde{\alpha}) &= \frac{(r^2+2r-6)\alpha^2}{(r-3)(r-2)^2}.\end{aligned}$$

For the unbiased estimator of α, we consider

$$\hat{\alpha}=\frac{r-2}{r}\tilde{\alpha}=\frac{r-2}{\log y_2},$$

then

$$Var(\hat{\alpha})=\frac{\alpha^2}{r-3}<V(\tilde{\alpha})\text{ and }RE(\tilde{\alpha})=\frac{r-3}{r}$$

□

Appendix D. Proof of Theorem 4

Proof. To find $E(X_{1:n}\log y_2)$, we use the relation

$$\int\cdots\int A\alpha^r y_1^{-\alpha}x_{1:n}dx_{1:n}=E(X_{1:n})=\frac{n\alpha}{n\alpha-1}k.$$

Then

$$\int \cdots \int A y_1^{-\alpha} x_{1:n} dx = \frac{n\alpha\alpha^{-r}}{n\alpha - 1} k. \tag{A20}$$

By differentiating both sides of (A20) with respect to α, we have

$$\int \cdots \int A y_1^{-\alpha} (\log y_1) x_{1:n} dx = -\frac{d}{d\alpha}\left(\alpha^{-r}\frac{n\alpha k}{n\alpha - 1}\right) = \frac{nk\alpha^{-r}(n\alpha r - r + 1)}{(n\alpha - 1)^2}, \tag{A21}$$

and

$$\begin{aligned} \int \cdots \int A\alpha^r y_1^{-\alpha} (\log y_2 + y) x_{1:n} dx &= \frac{nk(n\alpha r - r + 1)}{(n\alpha - 1)^2} = E(\log y_2 x_{1:n}) + E(Y x_{1:n}) \\ &= E(\log y_2 x_{1:n}) + \frac{nn\alpha^2}{(n\alpha - 1)^2}. \end{aligned} \tag{A22}$$

Then

$$E(\log y_2 x_{1:n}) = \frac{n(r-1)}{n\alpha - 1} k, \tag{A23}$$

hence

$$E\left[x_{1:n}\left(1 - \frac{\log y_2}{n(r-1)}\right)\right] = k, \tag{A24}$$

his shows that $x_{1:n}\left(1 - \frac{\log y_2}{n(r-1)}\right)$ is the unbiased estimate of k. □

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