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A Study on a Mechanism of Lateral Pedestrian-Footbridge Interaction

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Abstract: Based on the pedestrian lateral force hybrid Van der Pol/Rayleigh model, this study investigates the interaction dynamic model of a pedestrian-flexible footbridge lateral coupling system. A multi scale method is adopted to decouple the equation. The paper also studies the nonlinear dynamic response of the pedestrian-footbridge coupling system as well as the relationship between the lateral displacement of pedestrians and flexible footbridges, and the lateral interaction of the two variables. The results show that with the same frequency tuning parameters, when the mass ratio of pedestrians and footbridges is very small, the larger the mass ratio is, the larger the lateral response amplitude of pedestrians becomes. Conversely, when the mass ratio of pedestrians and footbridges is much larger, the larger the mass ratio is, the smaller the response amplitude becomes. When the natural frequency of a footbridge is larger, its Phase Angle becomes larger. As the lateral amplitude of pedestrians increases, the Phase Angle approaches zero. Moreover, regarding the variation of the Phase Angle between the interaction force and footbridge lateral vibration speed based on the lateral relative displacement of pedestrians, of which the variation range is $(0, \pi)$, as the pedestrians' lateral amplitude increases, the Phase Angle approaches $-\pi/2$. The dynamic load coefficient varies linearly with the lateral amplitude of pedestrian vibrations.

Keywords: footbridge; lateral pedestrian-footbridge interaction; hybrid Van der Pol/Rayleigh model; mechanism

1. Introduction

The vertical vibrations of a pedestrian-footbridge coupling have attracted considerable and increasing attention [1–5]. A new single DOF (degree of freedom) model of a bipedal pedestrian is proposed to study the dynamic interaction between a footbridge and pedestrians [6]. Li Q et al. [7] simulated crowd-footbridge interactions based on a multi free mass-damping coupling system. In line with the work of Federica Tubino [8], the probability assessment method is applied to study the interaction between the vertical vibrations of multiple pedestrians and a footbridge. In a study by Li Shoutao [9], a three-degree-of-freedom human-structure interaction model was established to study the system resonance frequency. Comparing the existing experimental results, it was found that the tandem three-degree-of-freedom human model is closer to the experimental results. A new biodynamic synchronized coupled model is based on a measured footbridge on a comparative basis [10]. In a study by Zhang M et al. [11], pedestrian excitation was considered to model the structure-human body model-damping of Phase Angle interactions based on biomechanics. Hamill J et al. [12] established a model of pedestrians' movements based on biomechanics. Zhou Chen et al. [13] studied the parameter

vibration responses while considering the time-lag effect by taking the Millennium Bridge in London as an example, which shows that the time-lag has no effect on the amplitude of the parameter vibration response. Belykh, Igor et al. [14] discussed parameter variation trends that were used to obtain coupled pedestrian-bridge oscillations, notably the influence of the mass ratio of pedestrians to the bridge mass in the mode considered, and the phase variation with the amplitude.

However, there are few studies that have been conducted on lateral coupled pedestrian-footbridge vibrations. Macdonald, SP Carrol, and M. Bocian considered pedestrians using a linear inverted pendulum model and studied the changes in parameters such as mass, damping, and force in the case of pedestrian-footbridge laterally coupled vibrations [15–17]. Among these studies, most researchers did not highlight the importance of pedestrians’ vibration responses, but simply focused on the vibration responses of the footbridge. Some scholars did not analyze the decoupling work of pedestrian-footbridge coupling systems when studying pedestrian-footbridge coupling vibrations [18]. Instead, the vibration of the footbridge was regarded as a vibration table in their studies. Its vibration amplitude and frequency were assumed to be a constant value [18]. In this case, this assumption is not reasonable because the lateral vibration response of pedestrians and footbridges interacts mutually. However, if the coupling is weak, it is reasonable to assume that the amplitude and frequency of the footbridge are constant. When the lateral swaying frequency of pedestrians and lateral vibration of the footbridge are the same or similar, the coupled vibration between pedestrians and the footbridge is strongly coupled. Moreover, both have a major influence on each other. The pedestrian-footbridge system resonance effect cannot be presented because the vibration amplitude and frequency of the footbridge are simply considered as being constant. Silvano Erlicher et al. [19,20] adopted a modified hybrid Van der Pol/Rayleigh model to simulate the lateral forces of pedestrians on periodic moving floors and rigid floors, respectively.

Therefore, based on a lateral force model of the hybrid Van der Pol/Rayleigh model proposed in their paper, the lateral interaction dynamic model of pedestrian-flexible footbridge coupling system was established. By decoupling the coupling equation, it derives the amplitude-frequency curve of pedestrians and pedestrian bridges. The paper also studies the lateral displacement and lateral force between pedestrians and footbridges. Finally, the study obtains a dynamic response of interactions between pedestrians and footbridges through an example.

2. A Dynamic Model of Lateral Pedestrian-Footbridge Interaction

A pedestrian is considered using an inverted pendulum model with the dynamic feature of the hybrid Van der Pol/Rayleigh model, where the coupling pedestrian-footbridge vibration is presented in Figure 1. When pedestrians walk on the footbridge, the footbridge is vibrated by the pedestrians, generating vibrations. The footbridge responds correspondingly according to the pedestrians’ vibrations. In addition, the pedestrians will make timely adjustments according to the vibrations of the footbridge. Therefore, the movement of the pedestrian consists of two parts: one is the lateral movement $u_{yp}(t)$ of the pedestrians relative to the footbridge, and the other is the lateral movement $u_{ys}(t)$ of the footbridge. The lateral movement of pedestrians is the sum of the two parts $u_{yp}(t) + u_{ys}(t)$. Thus, when considering the pedestrian-footbridge interaction, its lateral dynamic equation is as follows:

$$m\ddot{u}_{yp}(t) + F_y(u_{yp}(t), \dot{u}_{yp}(t)) + m\ddot{u}_{ys}(t) = 0. \tag{1}$$

The external force induced by walking pedestrians is loaded on the footbridge. As a result, the lateral dynamic equation of the footbridge is:

$$\ddot{u}_{ys}(t) + 2\zeta_s\omega_s\dot{u}_{ys}(t) + \omega_s^2u_{ys}(t) = \frac{F_y(u_{yp}(t), \dot{u}_{yp}(t))}{M} \tag{2}$$

$$F_y(t) = m\omega_0^2u_{yp}(t) - m\eta\dot{u}_{yp}(t) + m\rho\dot{u}_{yp}^3(t) + \xi m\dot{u}_{yp}(t)u_{yp}^2(t). \tag{3}$$

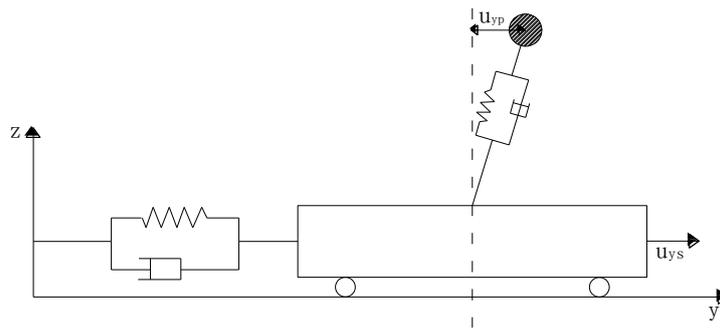


Figure 1. Lateral coupling of an individual pedestrian and a footbridge.

In the above equation, $u_{yp}(t)$ is the pedestrians' center of gravity relative to the lateral vibration displacement of the footbridge. $u_{ys}(t)$ is the lateral vibration displacement of the footbridge. M is the modal mass of the footbridge. m is the mass of a single pedestrian. $F_y(t)$ is the force loading on the footbridge structure when the pedestrians are walking. In general, Equations (1) and (2) are the dynamic equations of the pedestrian-footbridge interaction. They are coupled by the pedestrian-footbridge interaction force (related to the speed and displacement of pedestrians). It can be interpreted that Equations (1) and (2) are the lateral coupling dynamic equations of the pedestrian-bridge interaction. For the pedestrian system, the additional effect due to the movement of the footbridge is the external force. For the footbridge system, the excitation force generated by the movement of pedestrians is the external force.

3. Study on Dynamic Response of Pedestrian-Footbridge Interactions

3.1. Dynamic Coupling Equation Solution of the Lateral Interaction Pedestrian-Footbridge

The multiscale method is used to solve Equations (1)–(3). Assume that the nonlinear systems (1)–(3) are weak nonlinear systems. Then, a small parameter is added to the damping term, external excitation term, and nonlinear term in the system. Thus, Equations (1)–(3) can be written as:

$$\ddot{u}_{yp}(t) + \omega_0^2 u_{yp}(t) - \varepsilon \eta \dot{u}_{yp}(t) + \varepsilon \rho \dot{u}_{yp}^3(t) + \varepsilon \xi \dot{u}_{yp}(t) u_{yp}^2(t) + \varepsilon \ddot{u}_{ys}(t) = 0 \tag{4}$$

$$\ddot{u}_{ys}(t) + 2\varepsilon \zeta_s \omega_s \dot{u}_{ys}(t) + \omega_s^2 u_{ys}(t) = \frac{m \varepsilon \omega_0^2 u_{yp}(t) - m \varepsilon \eta \dot{u}_{yp}(t) + m \varepsilon \rho \dot{u}_{yp}^3(t) + m \varepsilon \xi \dot{u}_{yp}(t) u_{yp}^2(t)}{M} \tag{5}$$

It is assumed that the first-order approximate solutions of Equations (4) and (5) are:

$$u_{yp}(t) = u_0(T_0, T_1) + \varepsilon u_1(T_0, T_1) + \dots \tag{6}$$

$$u_{ys}(t) = U_0(T_0, T_1) + \varepsilon U_1(T_0, T_1) + \dots \tag{7}$$

of which $T_0 = \tau$ and $T_1 = \varepsilon \tau$.

Equations (6) and (7) are substituted into Equations (4) and (5) by adopting the multiscale method, aiming to achieve a situation where the coefficients of the small parameters ε^0 and ε^1 on both sides are equal. Then, the following partial differential equations can be defined as follows:

$$\varepsilon^0: \begin{cases} D_0^2 u_0 + \omega_p^2 u_0 = 0 \\ D_0^2 U_0 + \omega_s^2 U_0 = 0 \end{cases} \tag{8}$$

$$\varepsilon^1: \begin{cases} D_0^2 u_1 + \omega_p^2 u_1 = -2D_0 D_1 u_0 + \eta D_0 u_0 - \rho(D_0 u_0)^3 - \xi D_0 u_0 \cdot u_0^2 - D_0^2 u_0 \\ D_0^2 U_1 + \omega_s^2 U_1 = -2D_0 D_1 U_0 - 2\zeta_s \omega_s D_0 U_0 + \frac{m}{M} \omega_p^2 u_0 - \frac{m}{M} \eta D_0 u_0 \\ + \frac{m}{M} \rho(D_0 u_0)^3 + \frac{m}{M} \xi D_0 u_0 \cdot u_0^2 \end{cases} \quad (9)$$

Assume that the solution of Equation (8) is obtained as follows:

$$u_0 = A_p(T_1)e^{j\omega_p T_0} + cc \quad (10)$$

$$U_0 = A_s(T_1)e^{j\omega_s T_0} + cc. \quad (11)$$

With regard to Equations (10) and (11), cc is the conjugate complex number of the preceding expressions. Equations (10) and (11) are inserted into Equation (9) to derive:

$$\begin{aligned} D_0^2 u_1 + \omega_p^2 u_1 = & -2D_1 A_p(T_1)j\omega_p e^{j\omega_p T_0} + \eta A_p(T_1)j\omega_p e^{j\omega_p T_0} + 27\rho A_p^3(T_1) \\ & j\omega_p^3 e^{3j\omega_p T_0} - 3\rho A_p^2(T_1)\bar{A}_p(T_1)j\omega_p^3 e^{j\omega_p T_0} - \xi A_p^3(T_1)j\omega_p e^{3j\omega_p T_0} \\ & - \xi A_p^2(T_1)\bar{A}_p(T_1)j\omega_p e^{j\omega_p T_0} + A_s(T_1)\omega_s^2 e^{j\omega_s T_0} \end{aligned} \quad (12)$$

$$\begin{aligned} D_0^2 U_1 + \omega_s^2 U_1 = & -2D_1 A_s(T_1)j\omega_s e^{j\omega_s T_0} - 2\zeta_s \omega_s A_s j\omega_s e^{j\omega_s T_0} + \\ & \frac{m}{M} \omega_p^2 A_p(T_1)e^{j\omega_p T_0} - \frac{m}{M} \eta A_p(T_1)j\omega_p e^{j\omega_p T_0} - \frac{m}{M} \rho A_p^3(T_1)j\omega_p^3 e^{3j\omega_p T_0} \\ & + 3\frac{m}{M} \rho A_p^2(T_1)\bar{A}_p(T_1)j\omega_p^3 e^{j\omega_p T_0} + \frac{m}{M} \xi A_p^3(T_1)j\omega_p e^{3j\omega_p T_0} \\ & + \frac{m}{M} \xi A_p^2(T_1)\bar{A}_p(T_1)j\omega_p e^{j\omega_p T_0} \end{aligned} \quad (13)$$

It can be seen shown, according to the right-hand sides of Equations (12) and (13), that the terms containing $e^{j\omega_p T_0}$ and $e^{j\omega_s T_0}$ cause secular terms to occur in the solutions of Equations (12) and (13), respectively. Suppose that $\omega_p = \omega_s$; the solution of Equations (12) and (13) will generate secular terms, indicating that the internal resonance phenomenon is presented in the pedestrian-bridge bridging system in the case of $\omega_p = \omega_s$.

3.2. Internal Resonance Analysis of Pedestrian-Footbridge Lateral Coupling System

The pedestrians' lateral walking frequency and the natural frequency of the footbridge are assumed to satisfy:

$$\omega_s = \omega_p + \varepsilon\sigma, \quad \sigma = o(1). \quad (14)$$

Regarding Equation (14), where σ refers to the frequency tuning parameter, supposing the coefficient of the secular term is zero, associated with Equations (12) and (13), the following can be derived:

$$\begin{aligned} -2D_1 A_p(T_1)j\omega_p + \eta A_p(T_1)j\omega_p - 3\rho A_p^2(T_1)\bar{A}_p(T_1)j\omega_p^3 - \\ \xi A_p^2(T_1)\bar{A}_p(T_1)j\omega_p + A_s(T_1)\omega_s^2 e^{j\sigma T_1} = 0 \end{aligned} \quad (15)$$

$$\begin{aligned} -2D_1 A_s(T_1)j\omega_s - 2\zeta_s A_s(T_1)j\omega_s^2 + \frac{m}{M} \omega_p^2 A_p e^{-j\sigma T_1} - \eta \frac{m}{M} A_p j\omega_p e^{-j\sigma T_1} \\ + 3\frac{m}{M} \rho A_p^2(T_1)\bar{A}_p(T_1)j\omega_p^3 e^{-j\sigma T_1} + \frac{m}{M} \xi A_p^2(T_1)\bar{A}_p(T_1)j\omega_p e^{-j\sigma T_1} = 0 \end{aligned} \quad (16)$$

To facilitate the calculation, it is assumed that the phase of the footbridge is zero. Because the main focus here is on the Phase Angle between pedestrians and the footbridge, $A(T_1)$ is written in qn exponential form; hence, one can easily obtain:

$$A_p(T_1) = \frac{a(T_1)}{2} e^{j\theta_p} \quad (17)$$

$$A_s(T_1) = \frac{a(T_1)}{2} e^{j\theta_s} = \frac{a(T_1)}{2}. \quad (18)$$

Equations (17) and (18) are substituted into Equations (15) and (16); thus, Equations (15) and (16) become:

$$\begin{aligned}
 & -j\omega_p D_1 a_p(T_1) e^{j\theta_p} + \omega_p a_p(T_1) D_1 \theta_p(T_1) e^{j\theta_p} + \frac{1}{2} j \eta a_p(T_1) \omega_p e^{j\theta_p} \\
 & - \frac{3}{8} j \rho a_p^3(T_1) \omega_p^3 e^{j\theta_p} - \frac{1}{8} j \xi a_p^3(T_1) \omega_p e^{j\theta_p} + \frac{a_s(T_1)}{2} \omega_s^2 e^{j\theta_p} \cos(\sigma T_1 \\
 & - \theta_p) + \frac{a_s(T_1)}{2} j \omega_s^2 e^{j\theta_p} \sin(\sigma T_1 - \theta_p) = 0
 \end{aligned} \tag{19}$$

$$\begin{aligned}
 & -2j\omega_s D_1 a_s - \zeta_s a_s j \omega_s^2 + \frac{m}{2M} \omega_p^2 a_p \cos(\theta_p - \sigma T_1) + j \frac{m}{2M} \omega_p^2 a_p \sin(\theta_p \\
 & - \sigma T_1) - \eta \frac{m}{2M} a_p j \omega_p \cos(\theta_p - \sigma T_1) + \eta \frac{m}{2M} a_p \omega_p \sin(\theta_p - \sigma T_1) + \\
 & \frac{3m}{8M} \rho a_p^3 j \omega_p^3 \cos(\theta_p - \sigma T_1) - \frac{3m}{8M} \rho a_p^3 \omega_p^3 \sin(\theta_p - \sigma T_1) + \\
 & \frac{m}{8M} \xi a_p^3 j \omega_p \cos(\theta_p - \sigma T_1) - \frac{m}{8M} \xi a_p^3 \omega_p \sin(\theta_p - \sigma T_1) = 0
 \end{aligned} \tag{20}$$

If the coefficients of the real part and imaginary part of Equations (19) and (20) are zero, respectively, then in this case, one obtains:

$$-\omega_p D_1 a_p + \frac{1}{2} \eta a_p \omega_p - \frac{3}{8} \rho \omega_p^3 a_p^3 - \frac{1}{8} \xi \omega_p a_p^3 + \frac{a_s}{2} \omega_s^2 \sin(\sigma T_1 - \theta_0) = 0 \tag{21}$$

$$a_p \omega_p D_1 \theta_p + \frac{a_s}{2} \omega_s^2 \cos(\sigma T_1 - \theta_0) = 0 \tag{22}$$

$$\begin{aligned}
 & -2\omega_s D_1 a_s - \zeta_s a_s \omega_s^2 + \frac{m}{2M} \omega_p^2 a_p \sin(\theta_p - \sigma T_1) - \eta \frac{m}{2M} a_p \omega_p \cos(\theta_p - \sigma T_1) \\
 & + \frac{3m}{8M} \rho a_p^3 \omega_p^3 \cos(\theta_p - \sigma T_1) + \frac{m}{8M} \xi a_p^3 \omega_p \cos(\theta_p - \sigma T_1) = 0
 \end{aligned} \tag{23}$$

$$\begin{aligned}
 & \frac{m}{2M} \omega_p^2 a_p \cos(\theta_p - \sigma T_1) + \eta \frac{m}{2M} a_p \omega_p \sin(\theta_p - \sigma T_1) - \frac{3m}{8M} \rho a_p^3 \omega_p^3 \sin(\theta_p \\
 & - \sigma T_1) - \frac{m}{8M} \xi a_p^3 \omega_p \sin(\theta_p - \sigma T_1) = 0
 \end{aligned} \tag{24}$$

Hence, Equations (21)–(24) can be rewritten as follows in the assumption of $\varphi = \sigma T_1 - \theta_p$:

$$\omega_p D_1 a_p = \frac{1}{2} \eta a_p \omega_p - \frac{3}{8} \rho \omega_p^3 a_p^3 - \frac{1}{8} \xi \omega_p a_p^3 + \frac{a_s}{2} \omega_s^2 \sin \varphi \tag{25}$$

$$a_p \omega_p D_1 \varphi = a_p \omega_p \sigma + \frac{a_s}{2} \omega_s^2 \cos \varphi \tag{26}$$

$$\begin{aligned}
 2\omega_s D_1 a_s = & -\zeta_s a_s \omega_s^2 - \frac{m}{2M} \omega_p^2 a_p \sin \varphi - \eta \frac{m}{2M} a_p \omega_p \cos \varphi + \\
 & \frac{3m}{8M} \rho a_p^3 \omega_p^3 \cos \varphi + \frac{m}{8M} \xi a_p^3 \omega_p \cos \varphi
 \end{aligned} \tag{27}$$

$$\begin{aligned}
 \frac{m}{2M} \omega_p^2 a_p \cos \varphi - \eta \frac{m}{2M} a_p \omega_p \sin \varphi + \frac{3m}{8M} \rho a_p^3 \omega_p^3 \sin \varphi + \\
 \frac{m}{8M} \xi a_p^3 \omega_p \sin \varphi = 0
 \end{aligned} \tag{28}$$

To obtain the amplitude frequency equation of pedestrians and the footbridge structure as well as the Phase Angle of displacements for pedestrians and the footbridge, it is assumed that $D_1 a_p = 0$, $D_1 \varphi = 0$, and $D_1 a_s = 0$ are the equations for which the amplitude and Phase Angle are satisfied. Subsequently, one obtains:

$$\frac{1}{2} \eta a_p \omega_p - \frac{3}{8} \rho \omega_p^3 a_p^3 - \frac{1}{8} \xi \omega_p a_p^3 = -\frac{a_s}{2} \omega_s^2 \sin \varphi \tag{29}$$

$$a_p \omega_p \sigma = -\frac{a_s}{2} \omega_s^2 \cos \varphi \tag{30}$$

$$\begin{aligned}
 \zeta_s a_s \omega_s^2 = & -\frac{m}{2M} \omega_p^2 a_p \sin \varphi - \eta \frac{m}{2M} a_p \omega_p \cos \varphi + \frac{3m}{8M} \rho a_p^3 \omega_p^3 \cos \varphi \\
 & + \frac{m}{8M} \xi a_p^3 \omega_p \cos \varphi
 \end{aligned} \tag{31}$$

$$\begin{aligned}
 \frac{m}{2M} \omega_p^2 a_p \cos \varphi - \eta \frac{m}{2M} a_p \omega_p \sin \varphi + \frac{3m}{8M} \rho a_p^3 \omega_p^3 \sin \varphi \\
 + \frac{m}{8M} \xi a_p^3 \omega_p \sin \varphi = 0
 \end{aligned} \tag{32}$$

Equations (29)–(32) can be rewritten as follows. If $\zeta = \rho\omega_p^2 + \eta$, one obtains the following equations:

$$\frac{1}{2}\eta a_p \omega_p - \frac{1}{8}\omega_p a_p^3 \zeta = -\frac{a_s}{2}\omega_s^2 \sin \varphi \tag{33}$$

$$a_p \omega_p \sigma = -\frac{a_s}{2}\omega_s^2 \cos \varphi \tag{34}$$

$$\zeta_s a_s \omega_s^2 = -\frac{m}{2M}\omega_p^2 a_p \sin \varphi - \eta \frac{m}{2M} a_p \omega_p \cos \varphi + \frac{m}{8M} a_p^3 \omega_p \zeta \cos \varphi \tag{35}$$

$$\frac{m}{2M}\omega_p^2 a_p \cos \varphi - \eta \frac{m}{2M} a_p \omega_p \sin \varphi + \frac{m}{8M} a_p^3 \omega_p \zeta \sin \varphi = 0. \tag{36}$$

Multiplying Equations (33) and (34) and then adding them both together leads to:

$$\left(\frac{1}{2}\eta a_p \omega_p - \frac{1}{8}\omega_p a_p^3 \zeta\right)^2 + \omega_p^2 a_p^2 \sigma^2 = \frac{a_s^2}{4}\omega_s^4. \tag{37}$$

Similarly, Equations (35) and (36) are multiplied, and then both of them are added together. The resulting equation can be defined as follows:

$$\zeta_s^2 \omega_s^4 a_s^2 = \frac{m^2}{4M^2}\omega_p^4 a_p^2 + \frac{m^2}{4M^2}\eta^2 \omega_p^2 a_p^2 - \frac{m^2}{8M^2}\eta \zeta \omega_p^2 a_p^4 + \frac{m^2}{64M^2}\omega_p^2 a_p^6 \zeta^2. \tag{38}$$

Equations (37) and (38) are jointly connected to decouple the lateral amplitude of pedestrians and footbridges, for which the amplitude frequency equation of pedestrians’ lateral vibration can be derived as:

$$\left(\frac{1}{16} - \frac{m^2}{64M^2\zeta_s^2}\right)\zeta^2 a_p^6 + \left(\frac{m^2}{8M^2\zeta_s^2} - \frac{1}{2}\right)\eta \zeta a_p^4 + (\eta^2 + 4\sigma^2 - \frac{m^2}{4M^2\zeta_s^2}\omega_p^2 - \frac{m^2}{4M^2\zeta_s^2}\eta^2)a_p^2 = 0 \tag{39}$$

With respect to Equations (33) and (34), the Phase Angle between the lateral relative pedestrians’ displacement and lateral vibration displacement of the footbridge is derived as:

$$\tan \varphi = \frac{4\eta - a_p^2 \zeta}{8\sigma}. \tag{40}$$

It can be concluded from Equation (40) that when the pedestrians’ lateral walking frequency is consistent with the vibration frequency of the footbridge, that is, when the resonance occurs, the Phase Angle between the pedestrians’ relative displacement and the vibration displacement of the footbridge tends to $\pm 90^\circ$.

In detail, the amplitude of a pedestrian’s lateral vibration is derived according to Equation (39), and then the lateral vibration amplitude of the pedestrian bridge is determined based on Equation (38). Assuming:

$$A_p = a_p^2, k_1 = \left(\frac{1}{16} - \frac{m^2}{64M^2\zeta_s^2}\right)\zeta^2, k_2 = \left(\frac{m^2}{8M^2\zeta_s^2} - \frac{1}{2}\right)\eta \zeta, k_3 = \eta^2 + 4\sigma^2 - \frac{m^2}{4M^2\zeta_s^2}\omega_p^2 - \frac{m^2}{4M^2\zeta_s^2}\eta^2 \tag{41}$$

Equation (39) can be simplified as:

$$k_1 A_p^3 + k_2 A_p^2 + k_3 A_p = 0. \tag{42}$$

4. Amplitude Frequency Curve

4.1. Amplitude Frequency Curve of Pedestrians

Equation (42) is related to the unary cubic equation A_p , which has a zero solution $A_p = 0$ only, while the rest are related to the unary quadratic equation A_p . Hence, its discriminant can be rewritten as:

$$\Delta_p = k_2^2 - 4k_1k_3 = \left[\left(\frac{m^2}{8M^2\zeta_s^2} - \frac{1}{2} \right) \eta \zeta \right]^2 - 4 \left(\frac{1}{16} - \frac{m^2}{64M^2\zeta_s^2} \right) \zeta^2 (\eta^2 + 4\sigma^2 - \frac{m^2\omega_p^2}{4M^2\zeta_s^2} - \frac{m^2\eta^2}{4M^2\zeta_s^2}) = \left(\frac{m^2}{4M^2\zeta_s^2} - 1 \right) \zeta^2 \left(\sigma^2 - \frac{m^2\omega_p^2}{16M^2\zeta_s^2} \right) \tag{43}$$

For convenience of the following analysis, the modal mass ratio parameters of pedestrians and the footbridge are defined as:

$$m_r = \frac{m}{M} \tag{44}$$

If $\Delta_p = 0$, then $m_r = 2\zeta_s$, or if $\sigma = \frac{m_r\omega_p}{4\zeta_s}$, Equation (42) has a zero solution and two equal real roots, namely, $A_{p1} = 0$ and $A_{p2,3} = \frac{-k_2}{2k_1}$. Therefore, the pedestrians' amplitude frequency curve equation becomes:

$$a_{p1} = 0 \tag{45}$$

$$a_{p2,3} = \sqrt{\frac{-k_2}{2k_1}} = 2 \sqrt{\frac{\eta}{\xi + \rho\omega_p^2}} \tag{46}$$

If $\Delta_p > 0$, then $m_r > 2\zeta_s$, and if $\sigma > \frac{m_r\omega_p}{4\zeta_s}$, $\sigma < -\frac{m_r\omega_p}{4\zeta_s}$ or if $0 < m_r < 2\zeta_s$ or even $-\frac{m_r\omega_p}{4\zeta_s} < \sigma < \frac{m_r\omega_p}{4\zeta_s}$, then Equation (42) has a zero solution and two equal real roots, namely, $A_{p1} = 0$ and $A_{p2,3} = \frac{-k_2 \pm \sqrt{k_2^2 - 4k_1k_3}}{2k_1}$. Thus, the pedestrians' amplitude frequency curve equation can be derived:

$$a_{p1} = 0 \tag{47}$$

$$a_{p2,3} = 2 \sqrt{\frac{(4\zeta_s^2 - m_r^2)\eta \pm \sqrt{(m_r^2 - 4\zeta_s^2)(16\zeta_s^2\sigma^2 - m_r^2\omega_p^2)}}{(\xi + \rho\omega_p^2)(4\zeta_s^2 - m_r^2)}} \tag{48}$$

Equations (45) and (47) are trivial solutions for the pedestrians' lateral vibration, while Equations (46) and (48) are nontrivial solutions for the pedestrians' lateral vibration. Equation (46) implies that in the case where $m_r = 2\zeta_s$ or $\sigma = \frac{m_r\omega_p}{4\zeta_s}$, the pedestrians' lateral vibration is related only to the pedestrians' parameters. Moreover, according to Equation (48), it is indicated that the pedestrians' lateral amplitude is related to the modal mass ratio, the pedestrians' parameters, and the damping of the footbridge.

4.2. Amplitude Frequency Curve of Footbridge

According to Equation (38), if $a_p = 0$, $a_s = 0$, then it can be illustrated that the lateral vibration displacement of pedestrians and the lateral vibration displacement of the footbridge are both 0, which is the trivial solution of the footbridge.

Equation $A_{p2,3} = \frac{-k_2}{2k_1}$ is inserted into Equation (38) with the assumption that $\Delta_p = 0$; as a result, one gets

$$a_s = \sqrt{-\frac{64m_r^2\omega_p^4k_1^2k_2 + 64m_r^2\eta^2\omega_p^2k_1^2k_2 + 16m_r^2\eta\zeta\omega_p^2k_1k_2^2 + m_r^2\zeta^2\omega_p^2k_2^3}{512k_1^3\zeta_s^2\omega_s^4}} \tag{49}$$

When equation $A_{p2,3} = \frac{-k_2 \pm \sqrt{k_2^2 - 4k_1k_3}}{2k_1}$ is inserted into Equation (38) in the case of $\Delta_p > 0$, we can obtain:

$$a_{s2} = \sqrt{\frac{-64m_r^2\omega_p^4k_1^2k_2 + 64m_r^2\omega_p^4k_1^2\sqrt{k_2^2 - 4k_1k_3} - 64m_r^2\eta^2\omega_p^2k_1^2k_2 + 64m_r^2\eta^2\omega_p^2k_1^2\sqrt{k_2^2 - 4k_1k_3}}{512k_1^3\zeta_s^2\omega_s^4} - \frac{16m_r^2\eta\zeta\omega_p^2k_1k_2^2 - 32m_r^2\eta\zeta\omega_p^2k_1k_2\sqrt{k_2^2 - 4k_1k_3} - 16m_r^2\eta\zeta\omega_p^2k_1(k_2^2 - 4k_1k_3) + m_r^2\zeta^2\omega_p^2k_2^3}{512k_1^3\zeta_s^2\omega_s^4} + \frac{m_r^2\zeta^2\omega_p^2(k_2^2 - 4k_1k_3)^{\frac{3}{2}} - 3m_r^2\zeta^2\omega_p^2(k_2^2 - 4k_1k_3)k_2 + 3m_r^2\zeta^2\omega_p^2k_2^2\sqrt{k_2^2 - 4k_1k_3}}{512k_1^3\zeta_s^2\omega_s^4}} \quad (50)$$

$$a_{s3} = \sqrt{\frac{64m_r^2\omega_p^4k_1^2k_2 + 64m_r^2\omega_p^4k_1^2\sqrt{k_2^2 - 4k_1k_3} + 64m_r^2\eta^2\omega_p^2k_1^2k_2 + 64m_r^2\eta^2\omega_p^2k_1^2\sqrt{k_2^2 - 4k_1k_3}}{512k_1^3\zeta_s^2\omega_s^4} - \frac{16m_r^2\eta\zeta\omega_p^2k_1k_2^2 + 32m_r^2\eta\zeta\omega_p^2k_1k_2\sqrt{k_2^2 - 4k_1k_3} + 16m_r^2\eta\zeta\omega_p^2k_1(k_2^2 - 4k_1k_3) + m_r^2\zeta^2\omega_p^2k_2^3}{512k_1^3\zeta_s^2\omega_s^4} + \frac{m_r^2\zeta^2\omega_p^2(k_2^2 - 4k_1k_3)^{\frac{3}{2}} + 3m_r^2\zeta^2\omega_p^2(k_2^2 - 4k_1k_3)k_2 + 3m_r^2\zeta^2\omega_p^2k_2^2\sqrt{k_2^2 - 4k_1k_3}}{512M^2k_1^3\zeta_s^2\omega_s^4}} \quad (51)$$

Here, Equation (49) can be regarded as a trivial solution for the lateral vibration of the footbridge, and Equations (50) and (51) are nontrivial solutions for the lateral vibration of the footbridge.

Therefore, the first approximate solution of the lateral steady-state response of the pedestrians and footbridge is written as:

$$U_s = a_s(T_1) \cos(\omega_s t) \quad (52)$$

$$u_p = a_p(T_1) \cos[\omega_p t + \theta_p(T_1)]. \quad (53)$$

5. Study on Pedestrian-Footbridge Lateral Displacement and Lateral Interaction

5.1. The Phase Angle between the Lateral Relative Displacement of Pedestrians and Lateral Displacement of the Footbridge

In Equation (40), θ_p is the Phase Angle between the lateral relative displacement of pedestrians and the lateral displacement of the footbridge. Assuming $\theta_p < 0$, the lateral absolute displacement of pedestrians lags behind the lateral displacement of the footbridge; if $\theta_p > 0$, the lateral absolute displacement of pedestrians is ahead of the lateral displacement of the footbridge; the amount of time lagging behind or in advance of is shown in the equation $\tau = |\varphi|/\omega_p$. The relationship between Phase Angle φ and frequency tuning parameters σ can be derived as follows based on Equation (34):

$$\cos \varphi = -\frac{2a_p\omega_p\sigma}{a_s\omega_s^2}. \quad (54)$$

The phase angle is located at $[-\pi, \pi]$. Figure 2 presents the difference between pedestrians' lateral walking frequency and the footbridge fundamental frequency (frequency tuning parameters). It also shows the impact of the difference on the Phase Angle between pedestrians and the footbridge in the case of different amplitude ratios. According to Figure 3, when pedestrians' lateral walking frequency tends towards the fundamental frequency of the footbridge, the corresponding Phase Angle approaches $\pi/2$. Taking Newland's study into account, it can be concluded that the range of Phase Angles between pedestrians and footbridge motion is limited to $[0, \pi/2]$ degree. In addition, the phase angle at resonance is $\pi/2$. The results deduced in the paper are in accordance with the results based on the Newland model [18]. In Abrams' study of the Millennium Bridge, it is concluded that the Phase Angle of pedestrians with regard to the footbridge is $[-\pi, \pi]$, which is similar to the results found in this paper [21].

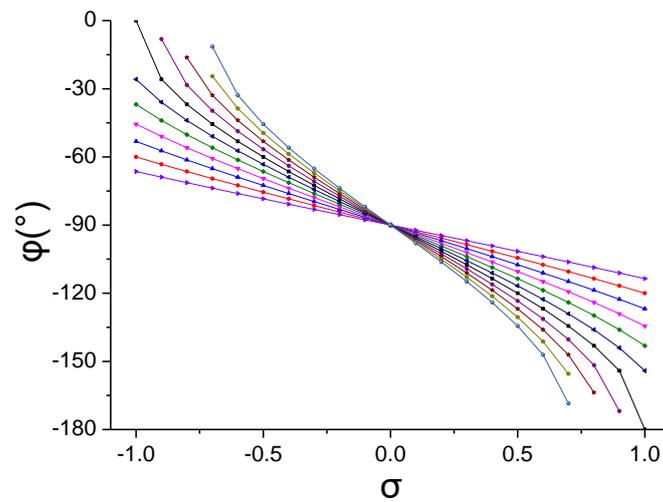


Figure 2. Effect of frequency tuning parameters on the Phase Angle between pedestrians and footbridges under different amplitude ratios.

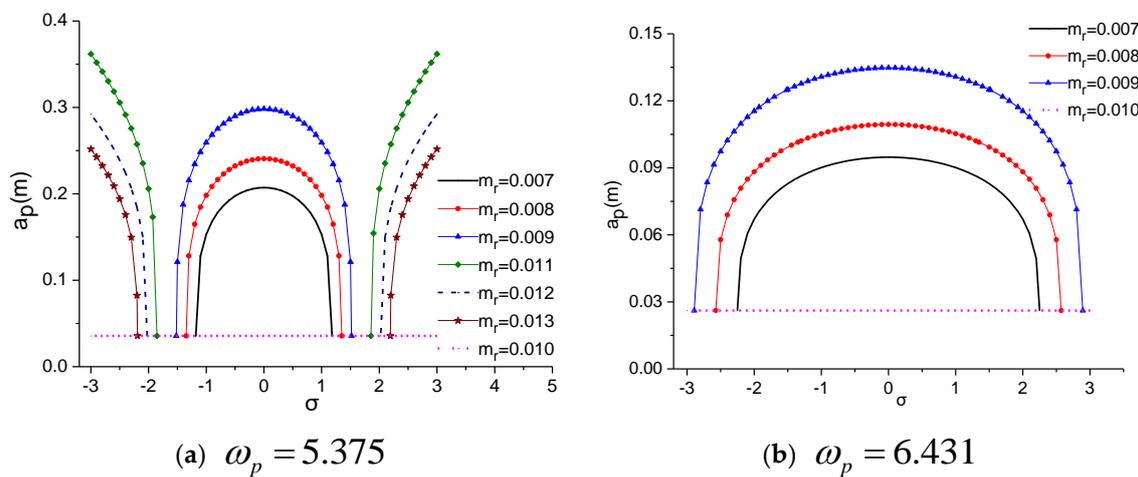


Figure 3. Amplitude frequency curve of pedestrians’ lateral vibration.

First, in the case where pedestrians’ lateral walking frequency is consistent with the lateral fundamental frequency of the footbridge, that is, in the case of $\sigma = 0$, while considering Equation (54), it can be proven that $\sigma = 0$ and $\cos \varphi = 0$. Thus, $\cos \varphi = 0$ is inserted into Equation (35), after which we can obtain:

$$\sin \varphi = -\frac{2M\zeta_s\omega_s^2 a_s}{m\omega_p^2 a_p} \tag{55}$$

Taking Equation (54) into consideration, it can be derived that $\sin \varphi = -1$; thus, $\varphi = -\frac{\pi}{2} + 2k\pi$, $k = 0, \pm 1, \pm 2, \pm 3, \dots$, which is consistent with the above results based on the relationship between the Phase Angle and frequency tuning parameters. Therefore, if $\sigma = 0$, then $\theta_p = -\varphi$; if $\theta_p > 0$, then the pedestrians’ vibration response is ahead of the footbridge response, with the amount of time in advance of the footbridge response shown as $\tau = \theta_p / \omega_s$. When the pedestrians walk along the footbridge, the amount of time before the footbridge’s response is represented as $\tau = (\pi - 4k\pi) / 2\omega_s$. For $\theta_p < 0$, the pedestrians’ vibration response lags behind the footbridge’s response, with the lag time $\tau = -\theta_p / \omega_s$; when the pedestrians walk on the footbridge, the lag time becomes $\tau = (-\pi + 4k\pi) / 2\omega_s$.

5.2. Phase Angle between the Lateral Absolute Displacement of Pedestrians and the Lateral Displacement of the Footbridge

With regard to resonance, the vibration frequency caused by pedestrians is the same as that of footbridges but differs in phase. The absolute displacement of pedestrians is the sum of the pedestrians' relative displacement and the footbridge structure displacement, which results in an absolute displacement of pedestrians of:

$$\begin{aligned}
 u_{pa} &= U_s + u_p = a_s \cos(\omega_s t) + a_p \cos(\omega_p t + \theta_p) \\
 &= a_s \cos(\omega_s t) + a_p \cos(\omega_s t - \varepsilon \sigma t + \theta_p) \\
 &= \sqrt{(a_s + a_p \cos \varphi)^2 + (a_p \sin \varphi)^2} \cos(\omega_s t - \phi)
 \end{aligned}
 \tag{56}$$

Therefore, the absolute displacement amplitude is defined as:

$$A = \sqrt{(a_s + a_p \cos \varphi)^2 + (a_p \sin \varphi)^2}
 \tag{57}$$

After considering $\cos \varphi = -\frac{2a_p \omega_p \sigma}{a_s \omega_s^2}$ and then $\sin \varphi = \pm \frac{\sqrt{a_s^2 \omega_s^4 - 4a_p^2 \omega_p^2 \sigma^2}}{a_s \omega_s^2}$, we can differentiate the Equation in (58):

$$A = \frac{a_p \sqrt{(\eta \omega_p - \frac{1}{4} \xi \omega_p a_p^2 - \frac{1}{4} \rho \omega_p^3 a_p^2)^2 + 4\omega_p^2 \sigma^2 - 4\omega_s^2 \omega_p \sigma + \omega_s^2}}{\omega_s^2}
 \tag{58}$$

The Phase Angle between the absolute displacement of pedestrians and the footbridge displacement is expressed in Equation (59):

$$\tan \psi = \frac{a_p \sin \varphi}{a_s + a_p \cos \varphi}
 \tag{59}$$

Equations (37), (54), and (55) are inserted into Equation (59). This leads to:

$$\tan \psi = \frac{(\eta - \frac{1}{4} \xi a_p^2 - \frac{1}{4} \rho \omega_p^2 a_p^2) \omega_s^2}{\omega_p (\eta - \frac{1}{4} \xi a_p^2 - \frac{1}{4} \rho \omega_p^2 a_p^2)^2 + 4\omega_p \sigma^2 - 2\omega_s^2 \sigma}
 \tag{60}$$

From Equation (58), it can be known that the absolute displacement amplitude of pedestrians is related to the following parameters: lateral response amplitude of the footbridge, relative displacement of pedestrians, Phase Angle, frequency difference, and the time of footbridge displacement, which in turn has nothing to do with the mass of the footbridge and pedestrians. Consider that the parameters are constant, including the lateral response amplitude of the footbridge, the relative displacement of pedestrians and the Phase Angle, frequency difference, and time of footbridge displacement. In this case, the absolute displacement amplitude of pedestrians is linear with respect to the response amplitude. In addition, the larger the amplitude of the footbridge response is, the larger the absolute displacement amplitude of pedestrians becomes.

When $\phi < 0$, the absolute displacement of pedestrians lags behind the displacement of the footbridge. Conversely, in the case of $\phi > 0$, the absolute displacement of pedestrians is ahead of the footbridge displacement.

5.3. Phase Angle between Lateral Interaction Force and Lateral Excitation Speed

When the pedestrians' lateral walking frequency is not equal to the fundamental frequency of the footbridge, the interaction force can be presented as:

$$\begin{aligned}
 F &= m\left(-\frac{d^2U_y}{dt^2} - \frac{d^2u_y}{dt^2}\right) = m\omega_s^2 a_s \cos(\omega_s t) + m\omega_p^2 a_p \cos(\omega_p t + \theta_p) \\
 &= m\sqrt{(\omega_s^2 a_s + \omega_p^2 a_p \cos \varphi)^2 + (\omega_p^2 a_p \sin \varphi)^2} \cos(\omega_s t - \phi)
 \end{aligned}
 \tag{61}$$

Because $\cos \varphi = -\frac{2a_p \omega_p \sigma}{a_s \omega_s^2}$, in this case $\sin \varphi = \pm \frac{\sqrt{a_s^2 \omega_s^4 - 4a_p^2 \omega_p^2 \sigma^2}}{a_s \omega_s^2}$; hence, Equation (61) can be written as:

$$\begin{aligned}
 F &= m\left(-\frac{d^2U_y}{dt^2} - \frac{d^2u_y}{dt^2}\right) \\
 &= m\sqrt{\left[\omega_s^2 a_s - \omega_p^2 a_p \frac{2a_p \omega_p \sigma}{a_s \omega_s^2}\right]^2 + \left[\pm \omega_p^2 a_p \frac{\sqrt{a_s^2 \omega_s^4 - 4a_p^2 \omega_p^2 \sigma^2}}{a_s \omega_s^2}\right]^2} \cos(\omega_s t - \phi) \\
 &= m\sqrt{\omega_s^4 a_s^2 - 4\omega_p^3 a_p^2 \sigma + \omega_p^4 a_p^2} \cos(\omega_s t - \phi)
 \end{aligned}
 \tag{62}$$

The lateral dynamic amplitude of pedestrians on the footbridge is defined as:

$$F_{\max} = m\sqrt{\omega_s^4 a_s^2 - 4\omega_p^3 a_p^2 \sigma + \omega_p^4 a_p^2}
 \tag{63}$$

Equation (37) is inserted in Equation (63) to easily obtain:

$$\begin{aligned}
 F_{\max} &= m\omega_p a_p \sqrt{\left(\eta - \frac{1}{4}a_p^2 \zeta\right)^2 + 4\sigma^2 - 4\omega_p \sigma + \omega_p^2} \\
 &= m\omega_p a_p \sqrt{\left(\eta - \frac{1}{4}\xi a_p^2 - \frac{1}{4}\rho \omega_p^2 a_p^2\right)^2 + 4\sigma^2 - 4\omega_p \sigma + \omega_p^2}
 \end{aligned}
 \tag{64}$$

As a result, the lateral dynamic coefficient is presented as:

$$\alpha = \frac{\omega_p a_p}{g} \sqrt{\left(\eta - \frac{1}{4}\xi a_p^2 - \frac{1}{4}\rho \omega_p^2 a_p^2\right)^2 + 4\sigma^2 - 4\omega_p \sigma + \omega_p^2}
 \tag{65}$$

The phase angle between the interaction force and vibration displacement of the footbridge can be computed as:

$$\tan \phi = \frac{m\omega_p^2 a_p \sin \varphi}{m\omega_s^2 a_s + m\omega_p^2 a_p \cos \varphi}
 \tag{66}$$

Equations (37), (54), and (55) are substituted into Equation (66), and the following equation holds:

$$\tan \phi = \frac{\omega_p \left(\eta - \frac{1}{4}\xi a_p^2 - \frac{1}{4}\rho \omega_p^2 a_p^2\right)}{\left(\eta - \frac{1}{4}\xi a_p^2 - \frac{1}{4}\rho \omega_p^2 a_p^2\right)^2 + 4\sigma^2 - 2\omega_p \sigma}
 \tag{67}$$

Since the lateral vibration displacement of the footbridge is expressed as $U_s = a_s(T_1) \cos(\omega_s t)$, the lateral vibration speed of the footbridge becomes:

$$\dot{U}_s = -a_s(T_1)\omega_s \sin(\omega_s t) = a_s(T_1)\omega_s \cos\left(\omega_s t + \frac{\pi}{2}\right)
 \tag{68}$$

Hence, the Phase Angle between the pedestrian-footbridge interaction force and the lateral vibration speed of the footbridge is:

$$\phi' = \phi - \frac{\pi}{2}
 \tag{69}$$

Therefore, it can be proven that the lateral interaction force amplitude and the dynamic load coefficient are related to the relative lateral response amplitude of pedestrians, pedestrians' parameters, walking frequency, and frequency differences but have nothing to do with the mass of the footbridge. When these parameters are considered to be constant, in this case, the dynamic load coefficient is linear with respect to the amplitude of the pedestrian lateral response. Based on this, the larger the response amplitude is, the larger the dynamic load coefficient becomes, which is proven to be the main principle of pedestrian-footbridge interactions.

It is assumed that when $\phi < 0$, the lateral interaction force lags behind the response of the footbridge; conversely, when $\phi > 0$, the lateral interaction force is ahead of the pedestrians' response.

6. Case Analysis

The amplitude-frequency curve and Phase Angle of pedestrians are calculated separately in different situations. It is assumed that the damping ratio of the footbridge structure is 0.005. In the reference [22], the parameters of the hybrid Van der Pol/Rayleigh model are obtained according to the experimental values and the least squares method, and the parameters are verified. Therefore, the parameters of pedestrians are taken as two groups: the first group includes $\omega_0 = 5.375$, $\eta = 0.101$, $\xi = 0.144$, and $\rho = 27.899$; the second group includes $\omega_0 = 6.431$, $\eta = 0.519$, $\xi = 0.273$, and $\rho = 73.439$ [22].

6.1. Amplitude Frequency Curve

Figure 3, Figure 4 show that the lateral response amplitude frequency curves of pedestrians and footbridges will vary due to the different walking frequencies of pedestrians. In detail, assuming $m_r < 0.01$, with the same frequency tuning parameters, the larger the mass ratio is, the larger the pedestrian lateral response amplitude becomes. Moreover, the closer the pedestrians' frequency is to the natural frequency of the footbridge, the larger the response amplitude. When $m_r > 0.01$, under the same frequency tuning parameters, the larger the mass ratio is, the smaller the response amplitude is. In the case of $m_r = 0.01$, its response amplitude is the smallest. In addition, the lateral response amplitude frequency curve of the footbridge tends to shift to the left side, which presents features of a soft spring. This is because the pedestrian-footbridge interaction is taken into consideration, which has altered the dynamic features of the pedestrian-footbridge coupling system.

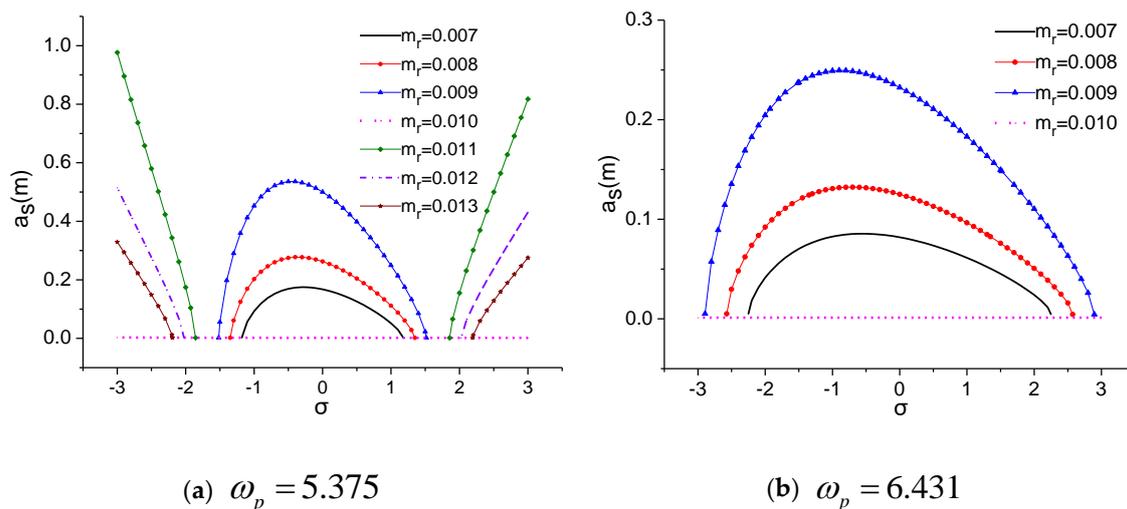


Figure 4. Amplitude frequency curve of lateral vibration of the footbridge.

6.2. Phase Frequency Curve

According to Figure 5, assuming that $\omega_p = 3.373$, when the frequency modulation parameter is confined to the range of [1, 2.45], the Phase Angle between pedestrians' absolute displacement and the footbridge displacement is positive, while the rest are negative; the variation range of the Phase Angle

is limited to $[-\pi/2, \pi/2]$. If $m_r < 0.01$, the Phase Angle has two solutions: for $m_r > 0.01$, the Phase Angle has three solutions; while for $m_r = 0.01$, the phase is zero. Assuming $\omega_p = 6.431$, the range of the frequency modulation parameter is limited to $[1.7, 2.9]$, resulting in a positive Phase Angle of the pedestrians' absolute displacement and footbridge displacement, while the rest are negative; the variation range of the Phase Angle is confined to $[-\pi/2, \pi/2]$. If $m_r < 0.01$, the Phase Angle has two solutions, while for $m_r = 0.01$, the phase is zero.

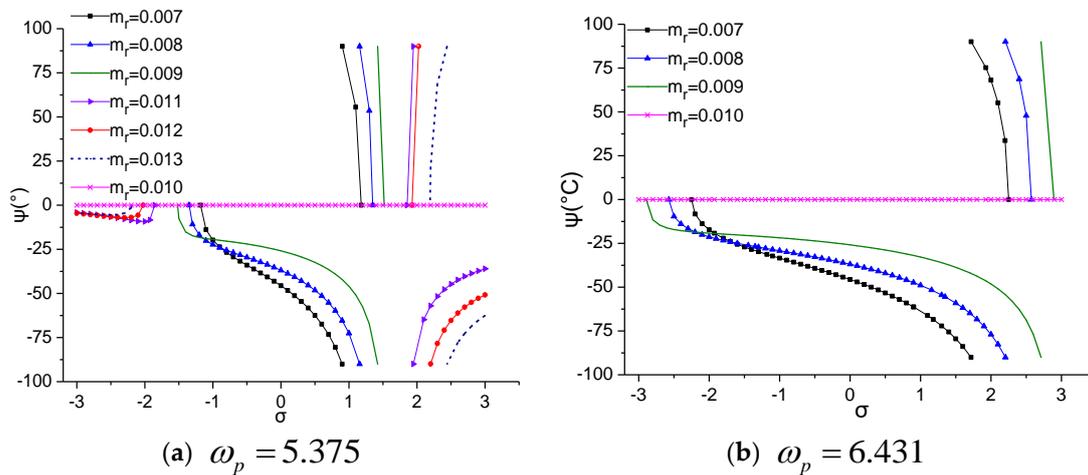


Figure 5. Phase frequency curve between pedestrians' absolute displacement and footbridge displacement.

Based on Figure 6, in the case where $\omega_p = 3.373$ and the frequency modulation parameter is limited to the range of $[1, 1.5]$, the Phase Angle of the interaction force and excitation speed is positive, while the rest are negative, and its variation range of the Phase Angle is confined to $[-\pi/2, \pi/2]$. For $m_r \neq 0.01$, correspondingly, its Phase Angle has two solutions; for $m_r = 0.01$, the phase is zero. However, it is assumed that $\omega_p = 6.431$ and that the range of the frequency modulation parameter is limited to $[1.7, 2.9]$. The Phase Angle between the interaction force and excitation speed is positive, while the rest are negative, and its variation of the Phase Angle is confined to the range of $[-\pi/2, \pi/2]$. For $m_r < 0.01$, the Phase Angle has two solutions, while for $m_r = 0.01$, the phase is zero.

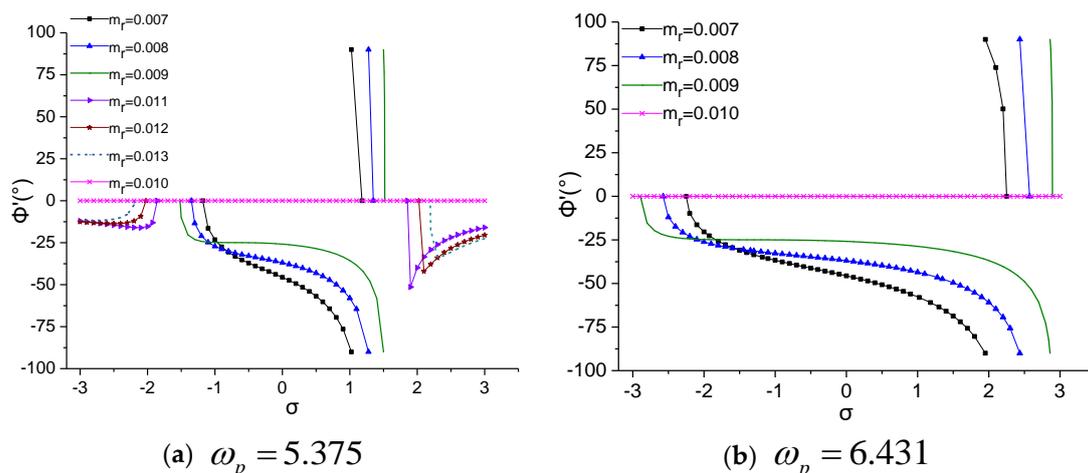


Figure 6. Phase Angle between interaction force and excitation speed.

Taking Figure 7 into account, it is proven that, based on different pedestrian lateral walking frequencies, the Phase Angle between pedestrians' lateral absolute displacement and the footbridge lateral vibration displacement varies according to the lateral relative displacement of pedestrians. The larger the natural frequency of the footbridge is, the larger the Phase Angle becomes. In addition,

the Phase Angle tends to be 0 with an increasing pedestrian lateral amplitude, which indicates that the Phase Angle is a constant value in the case of a considerably large vibration amplitude.

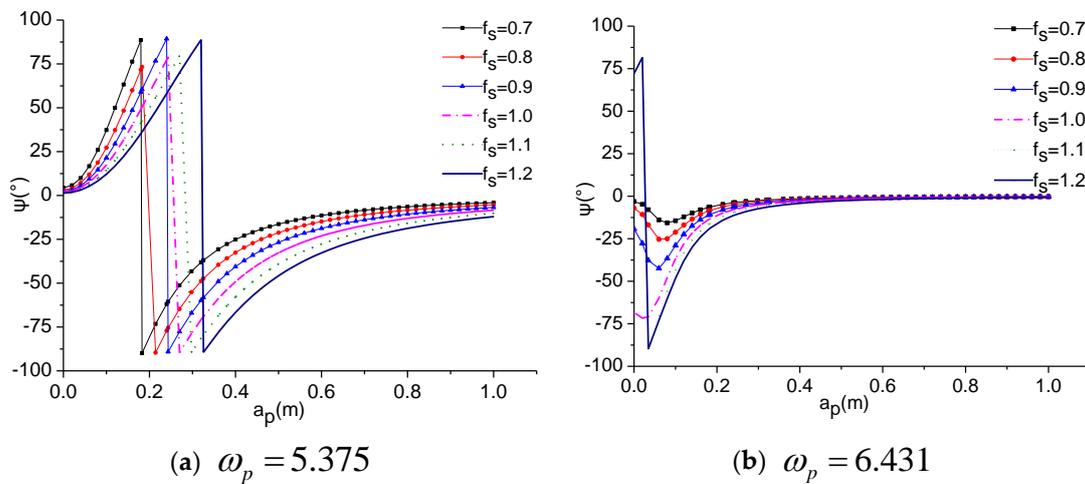


Figure 7. Variation of the Phase Angle between pedestrians’ lateral absolute displacement and footbridge lateral vibration displacement according to the lateral relative displacement of pedestrians.

Figure 8 shows that, based on different pedestrian lateral walking frequencies, the Phase Angle between the interaction force and footbridge lateral vibration speed differs according to the lateral relative displacement of pedestrians. In addition, the Phase Angle tends to -90° as pedestrians’ lateral amplitude increases. For this reason, it is concluded that the Phase Angle is a constant value due to the considerable large vibration amplitude.

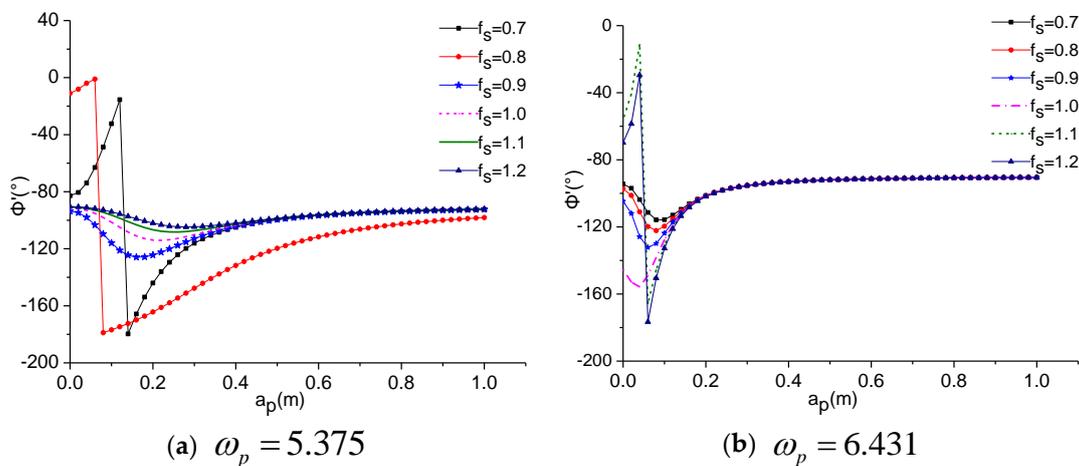


Figure 8. Variation of the Phase Angle between the interaction force and footbridge lateral vibration speed according to the lateral relative displacement of pedestrians.

Considering Figure 9, the pedestrian-footbridge interaction dynamic load factor α varies based on the lateral walking frequency of pedestrians. In fact, the dynamic load coefficient varies linearly with pedestrians’ lateral amplitude.

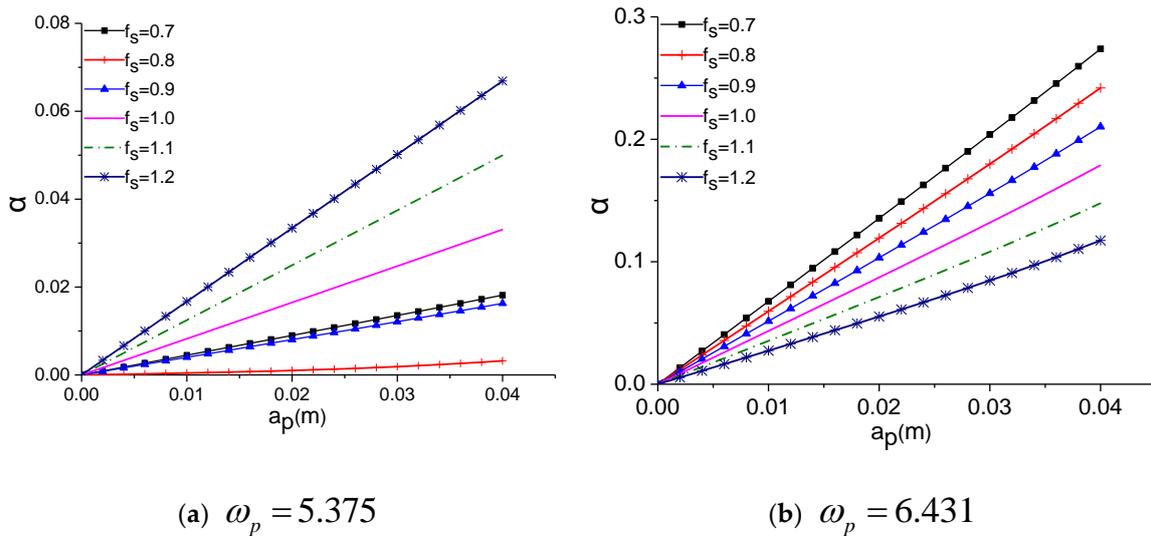


Figure 9. Variation of pedestrian-footbridge interaction dynamic load factor α according to pedestrians' lateral relative displacement.

Figure 10 indicates that the pedestrian-footbridge interaction dynamic load factor becomes different as the pedestrian lateral walking frequency changes. To be more specific, the smaller the natural frequency of the footbridge f_s is, the larger the A/a_p is; conversely, the larger the natural frequency of the footbridge f_s is, the smaller the A/a_p is, and A/a_p becomes larger due to the increasing lateral relative displacement of pedestrians.

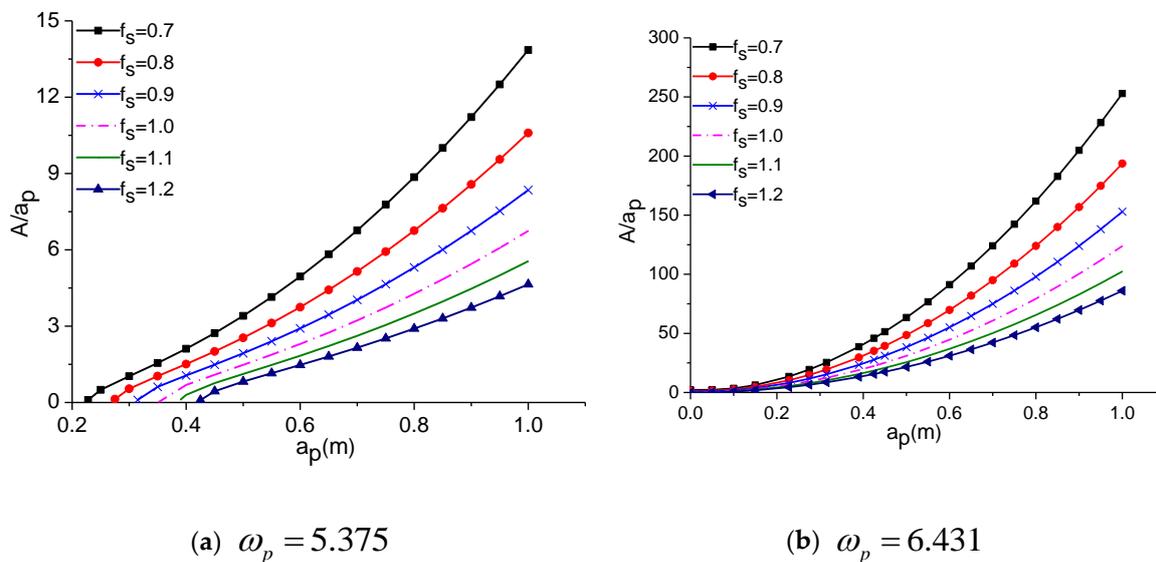


Figure 10. The impact of pedestrians' lateral relative displacement on the ratio of pedestrians' lateral absolute displacement to pedestrians' lateral relative displacement.

7. Conclusions

This paper presents research on a pedestrians' lateral force model of the hybrid Van der Pol/Rayleigh to model the interaction dynamic pedestrian-flexible footbridge lateral coupling system. In addition, the nonlinear dynamic response of the pedestrian-footbridge coupling system and the lateral interaction of pedestrians and flexible footbridges are investigated by adopting a multiscale method to decouple the equation.

(1) The lateral response amplitude frequency curves of pedestrians and footbridges vary for different pedestrian walking frequencies. If $m_r < 0.01$, under the same frequency tuning parameter,

the larger the mass ratio is, the larger the pedestrian lateral response amplitude is. If $m_r > 0.01$, the larger the mass ratio is, the smaller the response amplitude. The response amplitude is the smallest in the case of $m_r = 0.01$. Moreover, the lateral response amplitude frequency curve of the footbridge tends to shift to the left side, which shows features of a soft spring. The pedestrian-footbridge interaction is taken into consideration because the dynamic features of the pedestrian-footbridge coupling system are shifted.

(2) The range of the Phase Angle is confined to $[-\pi/2, \pi/2]$ between pedestrians' absolute displacement and the displacement of the footbridge, as well as between the interaction force and excitation speed.

(3) The Phase Angle between pedestrians' lateral absolute displacement and the footbridge lateral vibration displacement varies according to the lateral relative displacement of pedestrians. The range of its variation is limited to $[-\pi/2, \pi/2]$. The larger the natural frequency of the footbridge is, the larger the Phase Angle is. In addition, the Phase Angle approaches zero with an increasing pedestrian lateral amplitude. The Phase Angle between the interaction force and the footbridge lateral vibration speed changes according to the lateral relative displacement of pedestrians. Its variation range is $(0, \pi)$. As the lateral amplitude of pedestrians increases, the Phase Angle tends to $-\pi/2$.

(4) The dynamic load coefficient varies linearly in the case of different pedestrians' lateral amplitudes. The smaller the natural frequency of the footbridge is, the larger the ratio of A/a_p is. Furthermore, the larger the natural frequency of the footbridge is, the smaller the ratio of A/a_p is. In addition, A/a_p increases according to the increase in the lateral relative displacement of pedestrians.

Author Contributions: The work presented in this paper was conducted in collaboration with all the authors. Z.C.: Data curation, Formal analysis, Investigation, Methodology, Software, Visualization, Writing—original draft, Funding acquisition; S.C.: Data curation, Validation, Visualization. X.Y.: Funding acquisition, Methodology, Project administration, Validation, Writing—review & editing. Y.Z.: supervision.

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