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Buckling and Free Vibrations of Nanoplates—Comparison of Nonlocal Strain and Stress Approaches

Małgorzata Chwał and Aleksander Muc *

Institute of Machine Design, Cracow University of Technology, ul. Warszawska 24, 31-155 Kraków, Poland; malgorzata.chwal@pk.edu.pl

* Correspondence: olekmuc@mech.pk.edu.pl

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Abstract: The buckling and free vibrations of rectangular nanoplates are considered in the present paper. The refined continuum transverse shear deformation theory (third and first order) is introduced to formulate the fundamental equations of the nanoplate. Besides, the analysis involve the nonlocal strain and stress theories of elasticity to take into account the small-scale effects encountered in nanostructures/nanocomposites. Hamilton's principle is used to establish the governing equations of the nanoplate. The Rayleigh-Ritz method is proposed to solve eigenvalue problems dealing with the buckling and free vibration analysis of the nanoplates considered. Some examples are presented to investigate and illustrate the effects of various formulations.

Keywords: nonlocal stress theory; nonlocal strain theory; buckling; free vibrations; nanoplates; higher order shear deformation theories

1. Introduction

Concerning nano-objects, it is observed that classical (local) continuum theories presenting scale-free relations are inadequate for the mechanical analysis because of the absence of any material length scale parameters, see e.g., works by Peddieson et al. [1], Lu et al. [2], Barretta et al. [3]. At the nanometer scale, the size effects often may become prominent. Some non-classical continuum approaches have been applied to describe the mechanical behavior of small-sized structures, such as couple-stress theory, e.g., papers by Chen and Li [4], Wang [5], Ma et al. [6], Kim et al. [7], nonlocal elasticity theory, e.g., works by Barretta et al. [8], Hu et al. [9], Arefi et al. [10], Arash and Wang [11], strain gradient theories, e.g., papers by Askes and Aifantis [12], Akgoz and Civalek [13], Lim et al. [14], Barretta et al. [15].

A continuum is described as classical if the displacement field completely defines the deformation process. So, every point of a classical continuum (or Cauchy-type) has three displacement degrees of freedom. In the classical theory of elasticity, the symmetric tensor of stress and strain exists. The classical continuum mechanics is widely used for many problems. However, discrepancies between theory and experiment were observed around holes and notches when the stress gradient is present [16]. The classical elasticity also fails in the vibration analysis, wave propagation etc., where the influence of the material microstructure is dominant. The same problem is connected with the mechanical analysis of granulated materials, thermosetting polymers, and nanomaterials.

Firstly, the non-classical continuum was discussed at the end of the nineteenth century, e.g., by Voigt (1887) [17]. At the beginning of the twenty century, the Cosserat brothers (1909) [18] formulated the couple-stress continuum theory to overcome the insufficiency of the classical approach. The Cosserat continuum model besides the ordinary (force-stresses) considers the couple-stresses.

Hence, the displacement degrees of freedom and independent, rotational degrees of freedom are assigned to every particle of the Cosserat continuum. The stress tensor in Cosserat medium is non-symmetrical. The Cosserat theory in the primary form did not involve the broad public. It was excessively complex to be applied because of its generality and geometrical non-linearity [19]. In the following years, the activities were arising on an extension of the couple-stress theory, e.g., by Toupin [20] and Mindlin and Tiersten [21]. In the 1970s and 1980s, Eringen and Eringen and Edelen [22–24] derived stress gradient theory, and then Aifantis [25,26] elaborated strain gradient theory. These gradient theories have a reduced amount of higher order terms [7,27], are much simpler than those proposed in the 1960s. Mindlin and co-workers [21] made some assumptions to formulate a much simpler version of the Cosserat theory. They significantly enriched the Cosserat theory by introducing the linearised form of the couple-stress theory for perfectly elastic, centrosymmetric-isotropic materials. They introduced an internal length parameter which brings equations with and without couple-stresses. The parameter is small in comparison with body dimensions, but its influence may become important as dimensions of a body or phenomenon are in the order of the length parameter. Mindlin's theory of elasticity incorporates kinematic quantities at macro-scale and micro-scale so it is a multi-scale description [28]. Mindlin and co-workers adopted the couple-stress theory to solve problems of wave propagation, vibration, stress-concentration and nuclei of strain. In the following years, the gradient theories of elasticity were developed having the additional higher order spatial derivatives of stress, strain or acceleration. New generations of gradient theories are focused on minimisation of the number of additional constitutive parameters [12].

Eringen and co-workers [22–24] introduced the stress gradient theory from the integral nonlocal theories. They formulated that the stress state at a given point x depends not only on the local stress at that point but is the weighted average of the local stress of all points in the neighbourhood of x. A nonlocal stress σ_{ij}^{NLoc} is defined as [24]:

$$\sigma_{ij}^{NLoc}(x) = \int \alpha(|x' - x|, \tau)\sigma_{ij}(x')dV(x') \tag{1}$$

where σ_{ij}^{NLoc} is nonlocal stress tensor at point x, σ_{ij} is classical (local) stress tensor at x' within domain V, α is nonlocal modulus that depends on the distance $|x' - x|$ and on the dimensionless length scale τ expressed as $\tau = e_0\theta/L$ with $e_0\theta$ nonlocal parameter, e_0 nonlocal material constant, θ and L internal and external characteristic lengths, respectively.

For particular nonlocal problems originally studied in [24], instead of the nonlocal integral Equation (1), Eringen formulated an equivalent differential law as:

$$\sigma_{ij}^{NLoc} - (e_0\theta)^2\, \nabla^2\sigma_{ij}^{NLoc} = \sigma_{ij} \tag{2}$$

where ∇^2 is the Laplace operator.

The problem of the equivalence between the original Eringen integral convolution Equation (1) and the differential form Equation (2) and associated mechanical paradoxes are discussed in the literature—see [1,29,30] for the detailed description.

In the 1984 Aifantis [31] formulated a simple model of strain gradient plasticity by introducing only one additional constitutive constant. Firstly, the strain gradient theory was applied to dislocation description. Next, in the 1990s, Aifantis and co-workers [12,25,26,32,33] presented another simple model for strain gradient elasticity having one additional constitutive constant based on the strain gradient plasticity. They proposed gradient elasticity theory for finite deformations and infinitesimal deformations and extended the linear elastic constitutive relations with the Laplacian of the strains as [12]:

$$\sigma_{ij} = C_{ijkl}(\varepsilon_{kl} - (e_0\theta)^2\nabla^2\varepsilon_{kl}) \tag{3}$$

Recent achievements in mechanics of generalized continuum inform that deformation measures describing the generalized continuum, such as Cosserat theories, are kinematically redundant.

The above original concept and the detailed discussion were first introduced in the paper by Romano et al. [34] and next referenced in [35–37].

Applications of nonlocal continuum mechanics in case of carbon nanostructures such as carbon nanotubes and graphene were studied by [38–55] among others. The nonlocal vibrations and buckling of graphene were anlyzed by Pradhan [39], Pradhan and co-workers [40,41] and Shakaee-Pour [42]. Aghababaei and Reddy [43] used nonlocal shear deformation theory to investigate the bending and vibration of plates. Fazelzadeh and Ghavanloo [45] applied the nonlocal anisotropic elastic shell model for vibrations of nanostructures. Nami and Janghorban [47] studied the resonance of functionally graded micro/nano plates involving nonlocal stress and strain gradient elasticity. Ansari et al. [48] applied the first order shear deformation theory and the modified couple-stress theory to analyze the vibration of microplates. Barretta and Marotti de Sciarra [49] presented the nonlocal elastostatic problem and pointed out the nonlocality effect. Sladek et al. [51] applied the nonlocal theories for a large deformation of piezoelectric nanoplates. Zenkour [53] analyzed the thermoelastic vibrations of nanoplates based on the nonlocal theory of elasticity. Currently, a monographic elaboration of nonlocal elasticity approaches is presented by Ghavanloo et al. [56]. The problem of mechanical properties of nanostructures and nanocomposites in local and nonlocal regime was also previously considered by authors [57–66].

The main aim of the current paper is to investigate the effect of various formulations on the description of buckling and free vibrations of a rectangular nanoplate. The nonlocal stress and strain theories are involved, and the influence of small-scale parameter on the mechanical behavior of the nanoplate is studied and discussed.

2. Formulation of the Problem

Let us consider a nanoplate of length L_x, width L_y and thickness t as shown in Figure 1. A Cartesian coordinate system *(x, y, z)* is used to label the material points of the nanoplate in the unstressed reference configuration. The transverse normal stress σ_{zz} is negligible in comparison with in-plane stresses σ_{xx}, σ_{yy} and transverse out of plane stresses σ_{xz}, σ_{yz}. In addition to the vibration analysis, the elastic buckling of nanoplate under uniform uniaxial compressive loadings as well as biaxial loadings (N_x^0, N_y^0) are considered in this work.

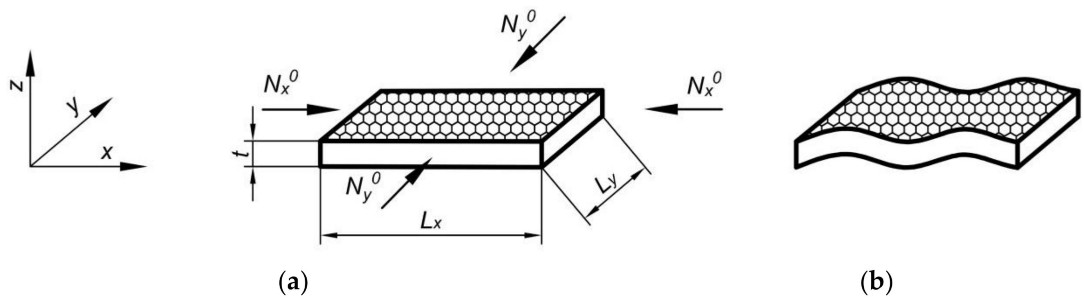

Figure 1. An analyzed continuum nanoplate: (**a**) Geometry and buckling loading, (**b**) Example of deformed shape.

2.1. Kinematic Relations

The 3-D displacements U_1, U_2 and U_3 at any point of the nanoplate in the x, y and z directions (see also Figure 1), respectively, are expressed in the following form:

$$
\begin{aligned}
U_1(x,y,z) &= u(x,y) - z\frac{\partial w}{\partial x} + F(z)\psi_1(x,y) \\
U_2(x,y,z) &= v(x,y) - z\frac{\partial w}{\partial y} + F(z)\psi_2(x,y) \\
U_3(x,y,z) &= w(x,y)
\end{aligned} \tag{4}
$$

where u, v, w are the displacements of a generic point on the reference mid-surface, ψ_1, ψ_2 are the rotations of normal to the mid-surface about the y- and x-axes, respectively. The explicit form of the function $F(z)$ is presented in Table 1 and Figure 2.

Table 1. Function $F(z)$ for variants of plate theories.

	Classical Plate Theory (CPT)	First Order Shear Deformation Theory (FSDT)	Higher Order Shear Deformation Theory (HSDT)–Reddy [67]	Higher Order Shear Deformation Theory (HSDT)–Hosseini-Hashemi et al. [68]
$F(z)$	0	z; the second term in Equation (10) is eliminated	$z\left(1-\frac{4}{3}\frac{z^2}{t^2}\right)$	$z\exp\left(-\frac{2z^2}{t^2}\right)$

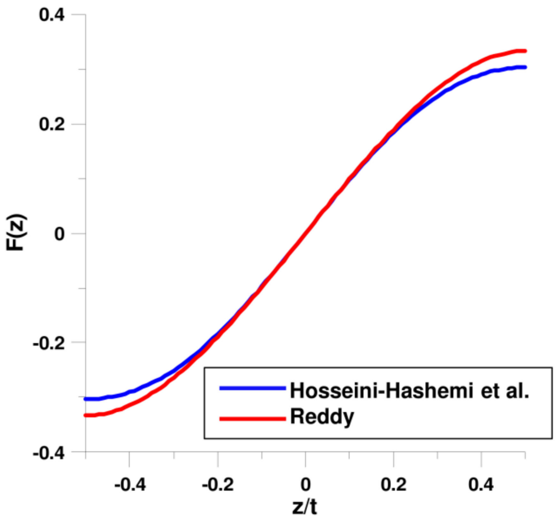

Figure 2. Variations of the function F(z) through the plate thickness t.

The distributions of the transverse shear corrections functions $F(z)$ used in the literature [67,68] are demonstrated in Figure 2 for third order transverse shear deformation theory (TSDT). The differences in the values of the functions are negligibly small. In HSDT there is no need to use shear correction factors since the form of the functions leads to the parabolic distributions of transverse shear stresses.

Based on the above expressions, the total, linear 3-D strain tensor can be written as follows:

$$\varepsilon_{ij}(x,y,z) = \frac{1}{2}\left(\frac{\partial U_i}{\partial x_j} + \frac{\partial U_j}{\partial x_i}\right), x_1 = x, x_2 = y, x_3 = z, i,j = 1,2,3 \tag{5}$$

2.2. Constitutive Nonlocal Equations and Variational Formulation

The brief overview of nonlocal theories has been presented in the Introduction. We have focused on the classical (local) and modified couple-stress formulations as well as the stress and strain gradient theories. Now, on the base of the local continuum, the constitutive nonlocal equations for the nanoplate are presented. Generally, both strain and stress gradients can influence the constitutive response of materials.

The local (classical) constitutive relations for a shear deformable nanoplate (orthotropic) are in the following form:

- local stress theory

$$\begin{Bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{12} \\ \sigma_{23} \\ \sigma_{13} \end{Bmatrix} = \begin{bmatrix} C_{11} & C_{12} & 0 & 0 & 0 \\ C_{12} & C_{22} & 0 & 0 & 0 \\ 0 & 0 & C_{66} & 0 & 0 \\ 0 & 0 & 0 & C_{44} & 0 \\ 0 & 0 & 0 & 0 & C_{55} \end{bmatrix} \times \begin{Bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{12} \\ \varepsilon_{23} \\ \varepsilon_{13} \end{Bmatrix} \quad (6)$$

- local strain theory

$$\begin{Bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{12} \\ \sigma_{23} \\ \sigma_{13} \end{Bmatrix} = \begin{bmatrix} C_{11} & C_{12} & 0 & 0 & 0 \\ C_{12} & C_{22} & 0 & 0 & 0 \\ 0 & 0 & C_{66} & 0 & 0 \\ 0 & 0 & 0 & C_{44} & 0 \\ 0 & 0 & 0 & 0 & C_{55} \end{bmatrix} \times \left(1 - (e_0\theta)^2\,\nabla^2\right) \begin{Bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{12} \\ \varepsilon_{23} \\ \varepsilon_{13} \end{Bmatrix} \quad (7)$$

∇^2 represents the classical 2-D Laplacian operator.

In the Cartesian system of coordinates the nonlocal functional of the strain energy of orthotropic plates becomes:

$$U_L = \frac{1}{2}\int\limits_{\Omega}\int\limits_{-t/2}^{t/2} \left\{\sigma^{NLoc}\right\}\{\varepsilon\}dxdydz \quad (8)$$

where Ω denotes the space occupied by the nanoplate, $\{\varepsilon\}$ is the local strain tensor and $\{\sigma^{NLoc}\}$ is the nonlocal stress tensor. The nonlocal stress tensor $\{\sigma^{NLoc}\}$ is defined as follows:

- the nonlocal stress theory

$$\{\sigma\} = [C]\{\varepsilon\} = (1 - (e_0\theta)^2\,\nabla^2)\left\{\sigma^{NLoc}\right\} = \Re\left\{\sigma^{NLoc}\right\} \quad (9)$$

- the nonlocal strain theory

$$\left\{\sigma^{NLoc}\right\} = [C](1 - (e_0\theta)^2\,\nabla^2)\{\varepsilon\} \quad (10)$$

where $\{\sigma\}$ is the local stress tensor, $[C]$ is the stiffness matrix and $\Re$ is an operator.

The components of the strain tensor are functions of the kinematic variables (4), (5) so that inserting the definitions (9) or (10) in Equation (8) one can find that the functional U_L becomes a function of the kinematic variables u, v, w, ψ_1, ψ_2 in the right-hand side of (4) only.

Besides, the work done by the exerted loads on the nanoplate is as follows (see Figure 1):

$$W = -\frac{1}{2}\int\limits_{\Omega}\left[N_x^0\left[\frac{\partial w}{\partial x}\right]^2 + N_y^0\left[\frac{\partial w}{\partial y}\right]^2\right]dxdy \quad (11)$$

Finally, the functional of the total potential energy in the Lagrange form can be written in the following form:

$$\Pi_L = U_L + W \quad (12)$$

The kinetic energy T of the analyzed nanoplate can be expressed as:

$$T = \frac{1}{2}\int\limits_{\Omega}\int\limits_{-t/2}^{t/2} \rho\left(\frac{\partial \widetilde{U}_i}{\partial \tau}\right)^2 dxdydz \quad (13)$$

where ρ and τ denotes the nanoplate density and time, respectively, and i = 1, 2, 3.

When the nanoplate is oscillating in a normal mode, the motion is harmonic so that the solutions may be taken in the form of a harmonic function of time τ with a natural circular frequency ω_{mn} (m, n denote the modes of vibration), such as

$$\{\widetilde{U}_i\}(x,y,z,\tau) = \{U_i\}(x,y,z)(cos\omega_{mn}\tau) \tag{14}$$

where $\{U_i\}$ and $\{\widetilde{U}_i\}$ are the components of displacements dependent and independent on time τ, respectively.

The Hamilton principle is applied to derive governing equations which is defined in a classical way as energy functional:

$$H = U_L + W + T \tag{15}$$

The Hamilton's principle states that

$$\delta \int_{\tau_1}^{\tau_2} H d\tau = 0 \tag{16}$$

where δ denotes the variation and τ_1 and τ_2 denotes the prescribed time range.

For the simplicity of the further considerations, the Hamilton function is replaced by the more convenient form in the case of the nonlocal stress theory, i.e.,:

$$\widetilde{H} = \Re H, \text{ where } \Re = 1 - e_0^2\theta^2\ \nabla^2 \tag{17}$$

using the definition (9) and assuming that $\Re^{-1}\ \Re$ is equal to the unity operator.

As it may be seen for CPT and FSDT/HSDT the governing relations of motion derived from the first variations of the Hamilton functional $\widetilde{H}$ are reduced to the addition of the linear operator $\Re$ acting on the right side of equilibrium equations. The stress resultants defined for the local stress components are written as:

$$N_{ij} = \int_{-t/2}^{t/2} \sigma_{ij} z dz,\ Q_{i3} = \frac{5}{6}\sigma_{i3} t,\ M_{ij} = \int_{-t/2}^{t/2} \sigma_{ij} z^2 dz,\ i,j = x,y \tag{18}$$

where N_{ij}, M_{ij} and Q_{i3} are the resultant forces, moments and shear forces.

The sense of the transformation (17) is obvious by analysis of the relations written in Table 2.

Table 2. The governing equations of motion for nanoplates – the nonlocal stress formulation.

	CPT	**FSDT/HSDT**
Equations of equilibrium	$\frac{\partial N_{xx}}{\partial x} + \frac{\partial N_{xy}}{\partial y} = \Re\rho t\ddot{u}$ $\frac{\partial N_{xy}}{\partial x} + \frac{\partial N_{yy}}{\partial y} = \Re\rho t\ddot{v}$ $\frac{\partial^2 M_{xx}}{\partial x^2} + 2\frac{\partial^2 M_{xy}}{\partial x\partial y} + \frac{\partial^2 M_{yy}}{\partial y^2} = \Re\big(\rho t\ddot{w} +$ $N_x^0\frac{\partial^2 w}{\partial x^2} + N_y^0\frac{\partial^2 w}{\partial y^2}\big)$	$\frac{\partial N_{xx}}{\partial x} + \frac{\partial N_{xy}}{\partial y} = \Re\rho t\ddot{u}$ $\frac{\partial N_{xy}}{\partial x} + \frac{\partial N_{yy}}{\partial y} = \Re\rho t\ddot{v}$ $\frac{\partial Q_{x3}}{\partial x} + \frac{\partial Q_{y3}}{\partial y} =$ $\Re(\rho t\ddot{w} + N_x^0\frac{\partial^2 w}{\partial x^2} + N_y^0\frac{\partial^2 w}{\partial y^2})$ $\frac{\partial M_{xx}}{\partial x} + \frac{\partial M_{xy}}{\partial y} - Q_{x3} = \Re\rho t^3\ddot{\psi}_1/12$ $\frac{\partial M_{xy}}{\partial x} + \frac{\partial M_{yy}}{\partial y} - Q_{y3} = \Re\rho t^3\ddot{\psi}_2/12$
Kinematical variables	u, v, w	u, v, w, ψ_1, ψ_2
Stress resultants	N_{xx}, N_{xy}, N_{yy}, M_{xx}, M_{xy}, M_{yy}	N_{xx}, N_{xy}, N_{yy}, M_{xx}, M_{xy}, M_{yy}, Q_{x3}, Q_{y3}

3. The Rayleigh-Ritz Method

The Rayleigh quotient is involved in the min-max theorem to obtain exact values of all eigenvalues (see the analysis presented by Muc et al. [69,70]. It can also be applied in eigenvalue algorithms to get an eigenvalue approximation from an eigenvector approximation. Exactly, this is the base for Rayleigh quotient iteration. A careful selection for the trial function that fulfills boundary conditions and depends on the set of variational parameters must be given in advance. Therefore, the Rayleigh-Ritz variational principle is a powerful tool method for the approximate solution of eigenvalue problems where a trial function (or functions) is introduced.

For simply-supported boundary conditions the displacement trial functions $u(x,y)$, $v(x,y)$, $w(x,y)$, $\psi_1(x,y)$ and $\psi_2(x,y)$ (Equation (4)) are expressed in terms of Fourier functions that satisfy the geometric and stresses boundary conditions. Thus the essential boundary conditions are satisfied, and the Fourier polynomials provide a complete and orthogonal set leading to a relatively fast convergence. The trial functions are defined in the whole space Ω occupied by the nanoplate.

Assuming the buckling (free vibrations) displacements of the form:

$$\begin{aligned} u(x,y) &= \sum_{m,n=1}^{\infty} U_{mn}\cos(\alpha_m x) sin(\beta_n y) \\ v(x,y) &= \sum_{m,n=1}^{\infty} V_{mn}\sin(\alpha_m x)\cos(\beta_n y) \\ w(x,y) &= \sum_{m,n=1}^{\infty} W_{mn}\sin(\alpha_m x)\cos(\beta_n y) \\ \psi_1(x,y) &= \sum_{m,n=1}^{\infty} \Psi_{1mn}\cos(\alpha_m x)\sin(\beta_n y) \\ \psi_2(x,y) &= \sum_{m,n=1}^{\infty} \Psi_{2mn}\sin(\alpha_m x)\cos(\beta_n y) \end{aligned} \tag{19}$$

where $\alpha_m = \frac{m\pi}{L_x}$, $\beta_n = \frac{m\pi}{L_y}$, $m,\ n$natural wavenumbers.

The explicit form of the Rayleigh-Ritz quotient is given below:

- the nonlocal stress theory

$$\lambda = \frac{\iint_\Omega \int_{-\frac{t}{2}}^{\frac{t}{2}} dxdydz(\sigma_{11}\varepsilon_{11} + \sigma_{12}\varepsilon_{12} + \sigma_{22}\varepsilon_{22} + \sigma_{13}\varepsilon_{13} + \sigma_{23}\varepsilon_{23})}{\Re(W+T)},\ \{\sigma\} = [C]\{\varepsilon\} \tag{20}$$

- the nonlocal strain theory

$$\begin{gathered} \lambda = \frac{\iint_\Omega \int_{-\frac{t}{2}}^{\frac{t}{2}} dxdydz(\sigma_{11}\varepsilon_{11}+\sigma_{12}\varepsilon_{12}+\sigma_{22}\varepsilon_{22}+\sigma_{13}\varepsilon_{13}+\sigma_{23}\varepsilon_{23})}{(W+T)}, \\ \{\sigma\} = [C](1-(e_0\theta)^2\,\nabla^2)\{\varepsilon\} \end{gathered} \tag{21}$$

where λ is the Rayleigh quotient.

The Equations (20) and (21) are written in the Lagrange form since the strains are functions of the kinematical variables u, v, w, ψ_1, ψ_2 reduced to three for CPT.

4. Numerical Results and Discussion

We consider the structural setup appearing in Figure 1. We likewise investigate how different constitutive formulations and the introduced kinematical hypotheses for the nanoplate (here a graphene sheet) affect the solutions. The properties of the graphene sheets are assumed as follows: the thickness t = 0.34 nm, the length of the nanoplate $L_x = L_y$ = 10 nm, Young's modulus E = 1.06TPa and Poisson's ratio ν = 0.25. However there are many articles that deal with the quantitative estimation of characteristics of nanoscale materials using local/nonlocal continuum mechanics with well-established parameters fitted to experimental results—see, e.g., papers by Shi et al. [71–73].

For the simplicity of writing the results in a more compact manner let us assume that:

$$C_{11} = C_{22} = \frac{E}{1-\nu^2}, C_{12} = \nu C_{11}, C_{44} = C_{55} = C_{66} = \frac{1-\nu}{2} C_{11} \tag{22}$$

The nanoplates are subjected to uni-axial uniform compression loading (see Figure 1).

The numerical analysis of buckling loads and natural frequencies is conducted with the use of the symbolic package Mathematica. The program allows us to integrate symbolically overall geometrical variables x, y and z appearing in the functionals (20) and (21). Since for CPT and FSDT the equilibrium equations are presented in the explicit form in Table 2, the appropriate buckling loads and natural frequencies can be evaluated almost automatically inserting the Fourier expansions (19) to Equations (20) and (21). Let us also note that in plate equations (Equation (4)) the terms including in-plane displacements, the normal deflection and two rotations are uncoupled. Therefore, the governing relations are a set of three equations only—see Table 2.

4.1. Buckling Loads–Uniform, Uni-Axial Compression along the x-axis ($N_y^0 = 0$)

4.1.1. Nonlocal Stress Theory

Taking into account the uniform, uni-axial compression loading along the x-axis (assuming $N_y^0 = 0$) the following relation for the nonlocal stress theory are considered:

- CPT:

$$N_x^0 = \frac{D\left(\alpha_m^2 + \beta_n^2\right)^2}{\alpha_m^2\left[1 + e_0^2\theta^2(\alpha_m^2 + \beta_n^2)\right]} \tag{23}$$

- FSDT and HSDT: The determinant of the coefficient matrix derived from Equation (20) must be equal to zero, i.e.,:

$$\det\begin{vmatrix} K_{33} & K_{33} & K_{33} \\ K_{43} & K_{44} & K_{45} \\ K_{53} & K_{54} & K_{55} \end{vmatrix} = 0 \tag{24}$$

The determinant is cut off to three coupled displacement terms (3–w, 4–ψ_1, 5–ψ_2) and in this way, the coefficients vary in the range 3–5.

$$\begin{gathered}
K_{33}^{FSDT} = \left(\alpha_m^2 + \beta_n^2\right) - \frac{N_x^0}{tC_{44}}\alpha_m^2\left[1 + e_0^2\theta^2\left(\alpha_m^2 + \beta_n^2\right)\right],\ K_{34}^{FSDT} = -\alpha_m,\ K_{35}^{FSDT} = -\beta_n \\
K_{43}^{FSDT} = -K_{34}^{FSDT}, K_{44}^{FSDT} = 1 - \frac{D}{tC_{44}}\alpha_m^2\left[1 + \frac{[1-\nu]}{2}\right], K_{45}^{FSDT} = -\frac{D}{tC_{44}}\alpha_m\beta_n\frac{(1+\nu)}{2}, \\
K_{53}^{FSDT} = K_{35}^{FSDT}, K_{54}^{FSDT} = K_{45}^{FSDT}, K_{55}^{FSDT} = 1 - \frac{D}{tC_{44}}\beta_n^2\left[1 + \frac{[1-\nu]}{2}\right], \\
K_{33}^{HSDT} = D\left(\alpha_m^2 + \beta_n^2\right)^2 - N_x^0\alpha_m^2\left[1 + e_0^2\theta^2\left(\alpha_m^2 + \beta_n^2\right)\right] \\
K_{34}^{HSDT} = -\xi_1\alpha_m\left(\alpha_m^2 + \beta_n^2\right), K_{35}^{HSDT} = -\xi_1\beta_n\left(\alpha_m^2 + \beta_n^2\right), \\
K_{43}^{HSDT} = K_{34}^{HSDT}, K_{44}^{HSDT} = \xi_2\left(\alpha_m^2 + \frac{1-\nu}{2}\beta_n^2\right) + \xi_3, K_{45}^{HSDT} = \xi_2\frac{1-\nu}{2}\alpha_m\beta_n, \\
K_{53}^{HSDT} = K_{35}^{HSDT}, K_{54}^{HSDT} = K_{45}^{HSDT}, K_{55}^{HSDT} = \xi_2\left(\frac{1-\nu}{2}\alpha_m^2 + \beta_n^2\right) + \xi_3
\end{gathered} \tag{25}$$

$$D = \frac{Et^3}{12(1-\nu^2)}, \xi_1 = \frac{E}{1-\nu^2}\int_{-t/2}^{t/2} zF(z)dz, \xi_2 = \frac{E}{1-\nu^2}\int_{-t/2}^{t/2} F^2(z)dz, \xi_3 = \frac{E}{2(1+\nu)}\int_{-t/2}^{t/2}\left[\frac{dF(z)}{dz}\right]^2 dz \tag{26}$$

Computing the values of the parameters ξ_1, ξ_2, ξ_3 one can verify easily that for the third order polynomial and exponential approximations of the functions $F(z)$ (see Equation (4), Table 1 and Figure 2) the differences of their values are negligibly small.

In order to compare various effects of constitutive relation formulations (the nonlocal stress and strain theory) as well as of kinematical relations (Equation (4)) the buckling loads will be further referred to the value:

$$\overline{N} = \frac{D\left(\alpha_m^2 + \beta_n^2\right)^2}{\alpha_m^2} \tag{27}$$

It corresponds to the buckling loads evaluated with the use of the classical plate theory. In addition, the results of the analysis carried out in Refs [74–76] show that for the $L_x/L_y > 1$ the buckling mode $n = 1$ and $m \geq 1$. The presented plots are just restricted to the abovementioned limits.

Figure 3 demonstrates the classical results—the formulations of 2-D kinematical relations have a significant effect on the values of buckling loads which increase with the growth of the t/L_x ratio.

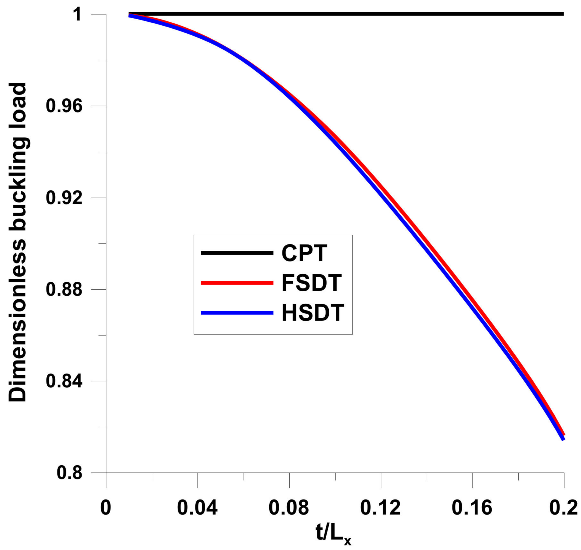

Figure 3. The effects of the accuracy of 2-D kinematical formulations on the buckling loads—a square nanoplate, $m = n = 1$.

The decrease of buckling loads for more accurate kinematical formulations in the thickness (z-direction) is generally accepted. However it is essential to point out that the percentage error between classical and higher order plate theories is also a function of mechanical properties of nanoplates, particularly in view of the values of G_{xz}/E and G_{yz}/E ratios, where G_{xz} and G_{yz} are the transverse (out-of-plane) shear moduli—see, e.g., Muc [76].

As it may be seen from Equations (23)–(26) the nonlocal stress theory formulation is strictly connected with the existence of the multiplication factor $1/\left[1 + e_0^2\theta^2\left(\alpha_m^2 + \beta_n^2\right)\right]$. The influence of the constitutive relations of nonlocal stress theory on the buckling loads is plotted in Figure 4.

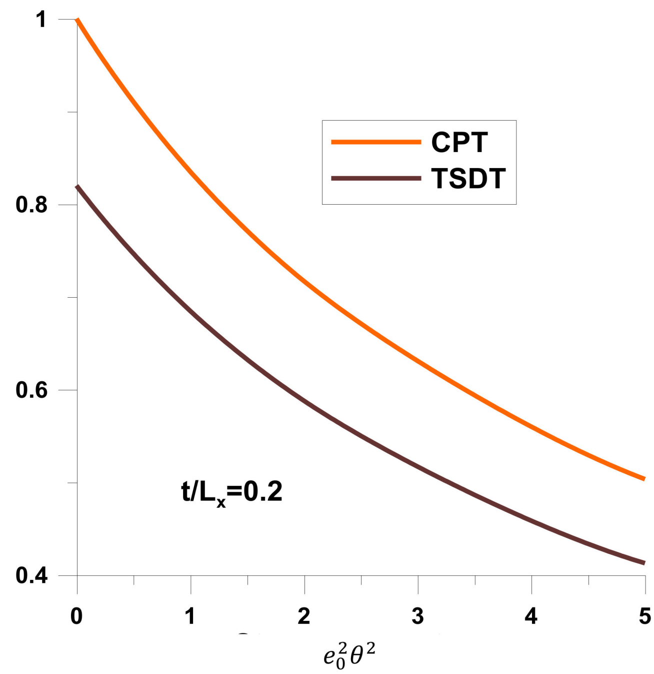

Figure 4. The influence of the constitutive relations (nonlocal stress theory) on the buckling loads.

4.1.2. Nonlocal Strain Theory

Taking into account as the above, uniform uni-axial compression loading along the *x*-axis, the following relation for the nonlocal strain theory is considered:

- CPT:

$$N_x^0 = \frac{D\left(\alpha_m^2 + \beta_n^2\right)^2 \left[1 + e_0^2\theta^2\left(\alpha_m^2 + \beta_n^2\right)\right]}{\alpha_m^2} \tag{28}$$

- FSDT and HSDT: The fundamental relations for FSDT or HSDT can be derived in a similar manner as in the previous section. Since they are more complicated as for the nonlocal stress theory, we do not demonstrate them in the explicit form. Instead of, we show the effects in the form of graphs plotted in Figure 5. In general, the nonlocal strain theory leads to the growth of buckling loads in comparison with the nonlocal stress theory (Figure 4). The increase factor is equal to $1 + e_0^2\theta^2\left(\alpha_m^2 + \beta_n^2\right)$. Those effects are identical to the reported in the literature by Lim et al. [14] for beams.

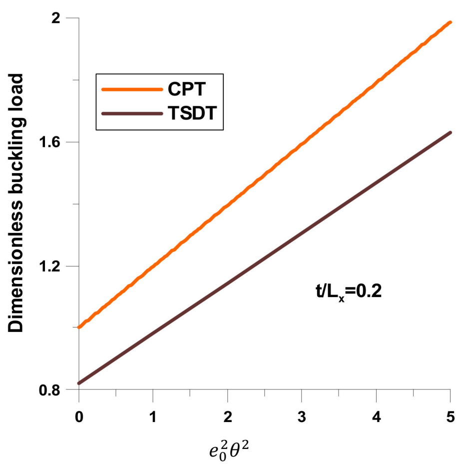

Figure 5. The influence of the constitutive relations (nonlocal strain theory) on the buckling loads.

4.2. Free Vibrations

Using the Rayleigh-Ritz method, it is possible to derive the natural frequencies easily for nanoplates. For the classical plate theory the frequencies are described by the simple equations having the following form:

- the nonlocal stress theory:

$$\omega_{mn}^2 \rho t = \frac{D\left(\alpha_m^2 + \beta_n^2\right)^2}{1 + e_0^2\theta^2\left(\alpha_m^2 + \beta_n^2\right)} \tag{29}$$

- the nonlocal strain theory:

$$\omega_{mn}^2 \rho t = D\left(\alpha_m^2 + \beta_n^2\right)^2\left[1 + e_0^2\theta^2\left(\alpha_m^2 + \beta_n^2\right)\right] \tag{30}$$

In the dimensionless form of free vibrations, the influence of the constitutive equations is identical to those plotted in Figures 4 and 5 for CPT.

Using the transverse shear deformation theory, the fundamental flexural frequencies are derived in a slightly different manner presented—see work [57] for the detailed description. In this case, the eigenproblem is transformed to (Table 2):

$$\det\left|K - \omega^2 M\right| = 0 \text{ where } M = \begin{bmatrix} \rho t & & \\ & \rho t^3/12 & \\ & & \rho t^3/12 \end{bmatrix} \tag{31}$$

The fundamental frequency is the lowest value obtained from Equation (31).

5. Conclusions and Future Works

The present work is devoted to the characterization of the formulation of basic equations on the eigenfrequencies and buckling loads for the nanoplates considered as a 2-D continuum. The analysis allows us to formulate the following conclusions:

1. The description of the displacements with the use of the third order polynomial or exponential approximations leads to the identical results.
2. The first order shear deformation theory seems to be a good approximation; the use of the higher order shear deformation theory can be necessary as transverse shear effects are significantly enhanced by the low values of the G_{xz}/E and G_{yz}/E ratios.
3. In the strain or stress nonlocal problems, the evaluation of eigenvalues is reduced to the operation of division/multiplication by the identical factor; therefore two different physical formulations have different effects on the searched values understood in the sense of weakening or enhancement the buckling/vibration effects in comparison with the classical global approach.

The proposed Rayleigh-Ritz method can be easily applied to the analysis of different variants of boundary conditions—the proposed method of solution is used, e.g., by Shi et al. [77].

In view of the opposite effects on the nonlocal strain and stress formulations, it seems to be reasonable to investigate also Mindlin's nonlocal theories.

In our opinion, there is an open question about the limits of the continuum theory application to the description of nanostructures. It is necessary to extend eigenproblems analysis of nanostructures/nanocomposites to the area of molecular dynamic problems—see, e.g., works by Muc and Barski [78] and Barski et al. [79].

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