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Youla–Kučera Parametrization with no Coprime Factorization—Single-Input Single-Output Case

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Abstract: We present a generalization of the Youla—Kučera parametrization to obtain all stabilizing controllers for single-input and single-output plants. This uses three parameters and can be applied to plants that may not admit coprime factorizations. In this generalization, at most two rational expressions of plants are required, while the Youla–Kučera parametrization requires precisely one rational expression.

Keywords: Youla–Kučera parametrization; factorization approach; coprime factorization; linear systems

1. Introduction

So far, the coprime factorization has played a central role to obtain stabilizing controllers in the factorization approach [1]. The factorization approach to control systems has the advantage that it includes, within a single framework, numerous linear systems such as continuous-time as well as discrete-time systems, lumped as well as distributed systems, one-dimensional as well as multidimensional systems, etc. [1,2]. A transfer function of this approach is considered as the ratio of two stable causal transfer functions. One of the attractive points of the factorization approach is the fact that all stabilizing controllers can be obtained by the Youla–Kučera parametrization with coprime factorization [3–5]. This Youla–Kučera parametrization has been used in a wide variety of applications for a long time (e.g., [6–10]).

Unfortunately, the Youla–Kučera parametrization cannot be applied to the plants that do not admit coprime factorizations. Mori, so far, gave the method to obtain part of stabilizing controllers by some different factorizations [11,12]. The objective of this paper is to generalize the Youla–Kučera parametrization to be applicable even for single-input single-output plants that may not admit the coprime factorization. This generalization employs three parameters and requires at most two rational expressions of plant, while the Youla–Kučera parametrization requires only one parameter and one rational expression of plant. The generalization will be expressed with an extension of Bézout identity. We will show that this generalization is equivalent to the parameterization method of [13], which does not require coprime factorization.

This paper is started with preliminaries from Section 2 to recall the notion of the factorization approach. We next state the main results of this paper, generalization of the Bézout identity and the Youla–Kučera parametrization, in Section 3. Then we review, in Section 4, the parametrization of stabilizing controllers of plants which may not admit coprime factorizations [13]. The proofs of the main results are given in Section 5. In Section 6, we give examples for the main results of Section 3. First example will be the plants that admit coprime factorizations. The next one will be Anantharam's example [14]. Third one will be the discrete-time systems without the unit-delay element.

2. Preliminaries

The stabilization problem considered in this paper follows that of [15,16], shown in Figure 1. In the figure, u_1 and u_2 are inputs, y_1 and y_2 outputs, and e_1 and e_2 errors. We employ the symbols used in [13,15] in general. For further details, the reader is referred to [1–13].

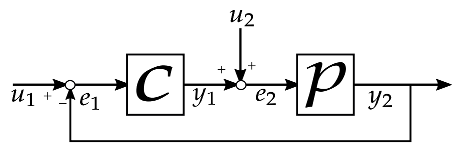

Figure 1. Feedback System.

We consider that the set of stable causal transfer functions is an integral domain with identity, denoted by $\mathcal{A}$. The total field of fractions of $\mathcal{A}$ is denoted by $\mathcal{F}$; that is, $\mathcal{F} = \{n/d \mid n, d \in \mathcal{A},\ d \neq 0\}$. This $\mathcal{F}$ is considered as the set of all possible transfer functions. Let $\mathcal{Z}$ be a prime ideal of $\mathcal{A}$ with $\mathcal{Z} \neq \mathcal{A}$. Define the subsets $\mathcal{P}$ and $\mathcal{P}_\mathrm{s}$ of $\mathcal{F}$ as follows: $\mathcal{P} = \{a/b \in \mathcal{F} \mid a \in \mathcal{A},\ b \in \mathcal{A}\backslash\mathcal{Z}\}$, $\mathcal{P}_\mathrm{s} = \{a/b \in \mathcal{F} \mid a \in \mathcal{Z},\ b \in \mathcal{A}\backslash\mathcal{Z}\}$. Then, a transfer function in $\mathcal{P}$ ($\mathcal{P}_\mathrm{s}$) is called *causal* (*strictly causal*).

Throughout the paper, the plant we consider has single-input and single-output, and its transfer function, which is also called a *plant* itself simply, is denoted by p and belongs to $\mathcal{P}$ (that is, p is causal).

For $p \in \mathcal{P}$ and $c \in \mathcal{F}$, a matrix $H(p,c) \in \mathcal{F}^{2\times 2}$ is defined as

$$H(p,c) = \begin{bmatrix} (1+pc)^{-1} & -p(1+pc)^{-1} \\ c(1+pc)^{-1} & (1+pc)^{-1} \end{bmatrix} \tag{1}$$

provided that $1 + pc$ is nonzero. This $H(p,c)$ is the transfer matrix from $[\, u_1\ u_2 \,]^t$ to $[\, e_1\ e_2 \,]^t$ of the feedback system of Figure 1. If $1 + pc$ is nonzero and $H(p,c) \in \mathcal{A}^{2\times 2}$, then we say that the plant p is *stabilizable*, p is *stabilized* by c, and c is a *stabilizing controller* of p. In the definition above, we do not mention the causality of the stabilizing controller. Even so, it is known that if a causal plant is stabilizable, there always exists a causal stabilizing controller of the plant, and further if a strictly causal plant is stabilizable, any stabilizing controller of the plant is causal [16] [Propositions 6.1 and 6.2].

We denote by $\mathcal{S}(p)$ the set of all stabilizing controllers of the plant p, and by $\mathcal{H}(p)$ the set of $H(p,c)$'s with all stabilizing controllers c of p. The relationship between $\mathcal{S}(p)$ and $\mathcal{H}(p)$ is as follows [17]:

$$\mathcal{H}(p) = \{H(p,c) \in \mathcal{A}^{2\times 2} \mid c \in \mathcal{S}(p)\}, \tag{2}$$

$$\mathcal{S}(p) = \left\{ h_{11}^{-1} h_{21} \in \mathcal{F} \;\middle|\; \begin{bmatrix} h_{11} & h_{12} \\ h_{21} & h_{11} \end{bmatrix} \in \mathcal{H}(p) \right\}. \tag{3}$$

Thus, obtaining $\mathcal{S}(p)$ and obtaining $\mathcal{H}(p)$ are equivalent to each other.

3. Main Results

We present three main results. The first one is a generalization of the notions of Bézout identity and coprime factorization. The others are generalizations of the Youla–Kučera parametrization that can be applied to stabilizable plants even with no coprime factorization.

Theorem 1. *Let p be a causal plant ($p \in \mathcal{P}$). Then p is stabilizable if and only if there exist n_1 and n_2 of $\mathcal{A}$, and d_1 and d_2 of $\mathcal{A} - \{0\}$ such that*

$$p = n_1/d_1 = n_2/d_2, \tag{4}$$

$$y_1 n_1 + x_1 d_1 + y_2 n_2 + x_2 d_2 = 1 \tag{5}$$

with y_1, x_1, y_2, x_2 of $\mathcal{A}$.

Theorem 2. *Let p be a stabilizable causal plant of $\mathcal{P}$ with symbols in Theorem 1 satisfying (4) and (5). Then the set $\mathcal{S}(p)$ of all stabilizing controllers of p is given as*

$$\mathcal{S}(p) = \left\{ \frac{y_1 d_1 + y_2 d_2 + r d_1^2 + s d_1 d_2 + t d_2^2}{x_1 d_1 + x_2 d_2 - r n_1 d_1 - s n_1 d_2 - t n_2 d_2} \;\middle|\; r,\ s,\ t \in \mathcal{A},\ (x_1 d_1 + x_2 d_2 - r n_1 d_1 - s n_1 d_2 - t n_2 d_2) \text{ is nonzero} \right\}. \tag{6}$$

Theorem 3. *Let p be a stabilizable causal plant of $\mathcal{P}$ and c its stabilizing controller. Denote*

$$\begin{bmatrix} h_{11} & h_{12} \\ h_{21} & h_{11} \end{bmatrix} = H(p,c).$$

Then the set $\mathcal{S}(p)$ of stabilizing controllers of p is given as

$$\mathcal{S}(p) = \left\{ \frac{h_{21} + h_{11}^2 r + h_{11} h_{21} s + h_{21}^2 t}{h_{11} + h_{11} h_{12} r + h_{12} h_{21} s - (1 - h_{11}) h_{21} t} \;\middle|\; r,\ s,\ t \in \mathcal{A},\ (h_{11} + h_{11} h_{12} r + h_{12} h_{21} s - (1 - h_{11}) h_{21} t) \text{ is nonzero} \right\}. \tag{7}$$

Remark 1. *The fraction in (6) can be rewritten as*

$$\frac{(y_1 + r d_1 + s d_2) d_1 + (y_2 + t d_2) d_2}{(x_1 - r n_1) d_1 + (x_2 - t n_2 - s n_1) d_2} \tag{8}$$

and, by noting that $n_1 d_2 = n_2 d_1$,

$$\frac{(y_1 + r d_1) d_1 + (y_2 + t d_2 + s d_1) d_2}{(x_1 - r n_1 - s n_2) d_1 + (x_2 - t n_2) d_2}. \tag{9}$$

Observing the fractions above, we might add new parameter s' of $\mathcal{A}$ as follows:

$$\frac{(y_1 + r d_1 + s d_2) d_1 + (y_2 + t d_2 + s' d_1) d_2}{(x_1 - r n_1 - s' n_2) d_1 + (x_2 - t n_2 - s n_1) d_2},$$

where s and s' are s of (8) and (9), respectively. Even so, rearranging the numerator and denominator, we have

$$\frac{y_1 d_1 + y_2 d_2 + r d_1^2 + (s + s') d_1 d_2 + t d_2^2}{x_1 d_1 + x_2 d_2 - r n_1 d_1 - (s + s') n_1 d_2 - t n_2 d_2}.$$

Since s and s' appear as in the form $s + s'$ only, we can remove the parameter s' and consider s only.

Remark 2. *Even without considering the coprimeness, any stabilizable plant must satisfy (4) and (5) of Theorem 1. Otherwise, the plant is not stabilizable.*

Remark 3. *An attractive point of Theorem 2 is that the set of stabilizing controllers can be obtained in the generalization form of the Youla–Kučera parametrization without the computation of coprime factorization, once the Equations (4) and (5) are obtained. On the other hand, Theorem 3 has the same attractive point once exactly one stabilizing controller is obtained.*

Remark 4. *Theorems 1 and 2 are generalizations of Corollary 3.1.11 and Theorem 3.1.13 of [1], respectively.*

Remark 5. *Suppose that a plant $p \in \mathcal{P}$ has two rational representations n_1/d_1 and n_2/d_2 with $n_1, n_2, d_1, d_2 \in \mathcal{A}$. Suppose further that we have found y_1 and x_2 of $\mathcal{A}$ such that*

$$y_1 n_1 + x_2 d_2 = 1. \tag{10}$$

In this case, we can apply Theorems 1 and 2 to the plant with $y_2 = x_1 = 0$ as special cases of Theorems 1 and 2, so that a stabilizing controller can be obtained and the set of stabilizing controllers can also be obtained. See Sections 6.2 and 6.3 for examples.

Note that, in (10), we consider only numerator and denominator of (possibly) different rational expressions. Also, (10) does not mean the coprimeness of the plant. Evan so, once we have (10), we can obtain all stabilizing controllers.

4. Parametrization without Coprime Factorizability

Here, we briefly review the parameterization method of [13], which does not require coprime factorization. This is used to give the proof of Theorem 2.

Theorem 4. *(Single-input single-output version of Theorem 4 and Corollary 1 of [13]) Let p be a stabilizable causal plant of $\mathcal{P}$. Let H_0 be $H(p,c) \in \mathcal{A}^{2\times 2}$, where c is a fixed stabilizing controller of p. Let $\Omega(Q)$ be a matrix defined as*

$$\Omega(Q) = \left(H_0 - \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \right) Q \left(H_0 - \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right) + H_0 \tag{11}$$

with a stable causal and square matrix Q in $\mathcal{A}^{2\times 2}$. Then we have the identity

$$\mathcal{H}(p) = \{\Omega(Q) \mid Q \in \mathcal{A}^{2\times 2} \text{ and } \det(\Omega(Q)) \text{ is nonzero}\}. \tag{12}$$

Then, any stabilizing controller has the form $\omega_{21}\omega_{22}^{-1}$, where ω_{21} and ω_{22} are the (2,1)- and (2,2)-entries of $\Omega(Q)$, provided that ω_{22} is nonzero.

This theorem gives the parameterization with a parameter matrix Q without coprime factorizability of the plant. The parameterization by $\Omega(Q)$ is independent of the choice of stabilizing controller c.

Decompose Q, $\Omega(Q)$, and $H(p,c)$ in Theorem 4 as follows:

$$\begin{bmatrix} q_{11} & q_{12} \\ q_{21} & q_{22} \end{bmatrix} = Q, \quad \begin{bmatrix} \omega_{11} & \omega_{12} \\ \omega_{21} & \omega_{22} \end{bmatrix} = \Omega(Q), \quad \begin{bmatrix} h_{11} & h_{12} \\ h_{21} & h_{11} \end{bmatrix} = H(p,c). \tag{13}$$

Then, by noting that $-h_{12}h_{21} = (1-h_{11})h_{11}$, we have

$$\begin{aligned} \omega_{11} = \omega_{22} &= h_{11} + (h_{11}-1)h_{11}(q_{11}+q_{22}) + (h_{11}-1)h_{21}q_{12} + h_{11}h_{12}q_{21}, \\ \omega_{12} &= h_{12} + (h_{11}-1)h_{12}(q_{11}+q_{22}) + (h_{11}-1)^2 q_{12} + h_{12}^2 q_{21}, \\ \omega_{21} &= h_{21} + h_{11}h_{21}(q_{11}+q_{22}) + h_{21}^2 q_{12} + h_{11}^2 q_{21}. \end{aligned}$$

In the equations above, q_{11} and q_{22} appears always in the form $(q_{11}+q_{22})$. Thus, the parameter q_{22} can be removed with keeping the parameter q_{11} effective, so that ω_{11}, ω_{12}, ω_{21}, and ω_{22} can be given as

$$\omega_{11} = \omega_{22} = h_{11} + (h_{11}-1)h_{11}q_{11} + (h_{11}-1)h_{21}q_{12} + h_{11}h_{12}q_{21}, \tag{14}$$

$$\omega_{12} = h_{12} + (h_{11}-1)h_{12}q_{11} + (h_{11}-1)^2 q_{12} + h_{12}^2 q_{21}, \tag{15}$$

$$\omega_{21} = h_{21} + h_{11}h_{21}q_{11} + h_{21}^2 q_{12} + h_{11}^2 q_{21}. \tag{16}$$

Hence, noting that $\omega_{11} = \omega_{22}$, we can express $\mathcal{H}(p)$ and $\mathcal{S}(p)$ as follows:

$$\mathcal{H}(p) = \left\{ \begin{bmatrix} \omega_{11} & \omega_{12} \\ \omega_{21} & \omega_{11} \end{bmatrix} \;\middle|\; \omega_{11}, \omega_{12}, \omega_{21} \text{ are of (14), (15), (16), respectively, and } \omega_{11} \text{ is nonzero.} \right\}, \tag{17}$$

$$\mathcal{S}(p) = \left\{ \frac{\omega_{21}}{\omega_{11}} \;\middle|\; \omega_{11}, \omega_{21} \text{ are of (14), (16), respectively, and } \omega_{11} \text{ is nonzero.} \right\}, \tag{18}$$

in which the parameters are q_{11}, q_{12}, and q_{21}. In the proof of Theorem 2, we will show that the parametrization of (17) is equal to the set of $H(p,c)$'s based on the right-hand side of (6).

5. Proofs of Theorems 1, 2 and 3

Proof of Theorem 1. (Only If). Suppose that p is stabilizable. Then, there exists a stabilizing controller c. Thus, $H(p,c)$ of (1) is over $\mathcal{A}$. Let

$$\begin{bmatrix} h_{11} & h_{12} \\ h_{21} & h_{11} \end{bmatrix} := H(p,c) = \begin{bmatrix} (1+pc)^{-1} & -p(1+pc)^{-1} \\ c(1+pc)^{-1} & (1+pc)^{-1} \end{bmatrix} \tag{19}$$

((1,1)- and (2,2)-entries of $H(p,c)$ are identical).

If c is zero, then p is in $\mathcal{A}$ because of $-h_{12} = p$. Letting $n_1 = n_2 = p$, $y_1 = y_2 = x_2 = 0$, $d_1 = d_2 = x_1 = 1$, we have (4) and (5).

In the case where c is nonzero, both h_{11} and h_{21} are nonzero. Letting $n_1 = -h_{12}$, $d_1 = h_{11}$, $n_2 = ph_{21}$ $(= 1 - h_{11})$, $d_2 = h_{21}$, $x_1 = y_2 = 1$, $y_1 = x_2 = 0$, we have (4) and (5).

(If). Suppose that there exist $n_1, d_1, y_1, x_1, n_2, d_2, y_2, x_2$ of $\mathcal{A}$ with (4) and (5).

Consider the case where $x_1d_1 + x_2d_2$ is zero. This is equal to $1 - (y_1n_1 + y_2n_2)$, so that $y_1n_1 + y_2n_2 = 1$. It follows that at least one of y_1n_1 and y_2n_2 is nonzero. Assume, without loss of generality, that y_1n_1 is nonzero, which means that both y_1 and n_1 are nonzero. Because d_1 is nonzero, n_1d_1 is a nonzero. Thus, $x_1d_1 + x_2d_2 + n_1d_1$ is nonzero.

From the previous paragraph, we observe that the expression

$$x_1d_1 + x_2d_2 + r_0n_1d_1 + t_0n_2d_2 \tag{20}$$

is nonzero by appropriate choice of (r_0, t_0) from $(0,0)$, $(1,0)$, $(0,1)$. In the following, we suppose that (20) is nonzero with (r_0, t_0) being one of $(0,0)$, $(1,0)$, $(0,1)$.

From now, we show that the following c is a stabilizing controller of p:

$$c = \frac{y_1 d_1 + y_2 d_2 - r_0 d_1^2 - t_0 d_2^2}{x_1 d_1 + x_2 d_2 + r_0 n_1 d_1 + t_0 n_2 d_2}. \tag{21}$$

This is done by showing that $H(p,c)$ with c of (21) is over $\mathcal{A}$, which consists of h_{11}, h_{12}, h_{21} of (19). Observe that

$$\begin{aligned} h_{11} &= x_1 d_1 + x_2 d_2 + r_0 n_1 d_1 + t_0 n_2 d_2, \\ h_{12} &= -(x_1 n_1 + x_2 n_2 + r_0 n_1^2 + t_0 n_2^2), \\ h_{21} &= y_1 d_1 + y_2 d_2 - r_0 d_1^2 - t_0 d_2^2, \end{aligned}$$

which are all in $\mathcal{A}$, so that $H(p,c)$ is over $\mathcal{A}$. Further, $1 + pc$ is $1/(x_1 d_1 + x_2 d_2 + r_0 n_1 d_1 + t_0 n_2 d_2)$, which is nonzero. Hence, c is a stabilizing controller of p. Therefore, p is stabilizable. □

Remark 6. *Analogously to the construction of (20), we can make $x_1 d_1 + x_2 d_2 + r_0 n_1 d_1 + t_0 n_2 d_2$ not in $\mathcal{Z}$ with (r_0, t_0) being one of $(0,0)$, $(1,0)$, $(0,1)$, so that c of (21) is causal. This fact can be shown by discussion analogous to the proof above.*

Proof of Theorem 2. Suppose that p is stabilizable.

We denote by $\overline{\mathcal{S}}(p)$ the right-hand side of (6). We also introduce $\overline{\mathcal{H}}(p)$, by virtue of the relation (2), as follows:

$$\overline{\mathcal{H}}(p) = \{H(p,c) \mid c \in \overline{\mathcal{S}}(p)\}.$$

This $\overline{\mathcal{H}}(p)$ is expressed as follows:

$$\overline{\mathcal{H}}(p) = \left\{ \begin{bmatrix} x_1 d_1 + x_2 d_2 - r n_1 d_1 - s n_1 d_2 - t n_2 d_2 & -(x_1 n_1 + x_2 n_2 - r n_1^2 - s n_1 n_2 - t n_2^2) \\ y_1 d_1 + y_2 d_2 + r d_1^2 + s d_1 d_2 + t d_2^2 & x_1 d_1 + x_2 d_2 - r n_1 d_1 - s n_1 d_2 - t n_2 d_2 \end{bmatrix} \;\middle|\; r,\ s,\ t \in \mathcal{A},\ x_1 d_1 + x_2 d_2 - r n_1 d_1 - s n_1 d_2 - t n_2 d_2 \text{ is nonzero} \right\} \tag{22}$$

(The determinant of the matrix in (22) is equal to $x_1 d_1 + x_2 d_2 - r n_1 d_1 - s n_1 d_2 - t n_2 d_2$).

Thus, the proof of Theorem 2 is achieved by showing $\overline{\mathcal{H}}(p) = \mathcal{H}(p)$, which is done by showing $\overline{\mathcal{H}}(p) \supset \mathcal{H}(p)$ and $\overline{\mathcal{H}}(p) \subset \mathcal{H}(p)$.

In the following, based on the proof of Theorem 1, we assume, without loss of generality, that $x_1 d_1 + x_2 d_2 + r_0 n_1 d_1 + t_0 n_2 d_2$ is nonzero, where (r_0, t_0) is one of $(0,0)$, $(1,0)$, $(0,1)$.

($\overline{\mathcal{H}}(p) \supset \mathcal{H}(p)$). Let

$$c = \frac{y_1 d_1 + y_2 d_2 - r_0 d_1^2 - t_0 d_2^2}{x_1 d_1 + x_2 d_2 + r_0 n_1 d_1 + t_0 n_2 d_2},$$

which is a stabilizing controller of p. Then $H(p,c)$ is as follows:

$$H(p,c) = \begin{bmatrix} h_{11} & h_{12} \\ h_{21} & h_{11} \end{bmatrix} = \begin{bmatrix} x_1 d_1 + x_2 d_2 + r_0 n_1 d_1 + t_0 n_2 d_2 & -(x_1 n_1 + x_2 n_2 + r_0 n_1^2 + t_0 n_2^2) \\ y_1 d_1 + y_2 d_2 - r_0 d_1^2 - t_0 d_2^2 & x_1 d_1 + x_2 d_2 + r_0 n_1 d_1 + t_0 n_2 d_2 \end{bmatrix}. \tag{23}$$

Based on this c, we consider an element of $\mathcal{H}(p)$, that is, a matrix below in the set of the right-hand side of (17) with the equations of (14), (15), and (16):

$$\begin{bmatrix} \omega_{11} & \omega_{12} \\ \omega_{21} & \omega_{11} \end{bmatrix}. \tag{24}$$

Now we let

$$\begin{aligned} q_{11} &= (d_1n_1 + d_1^2n_1(x_1 + r_0n_1) + d_1^2n_2(x_2 + t_0n_2))(r + r_0) + 2d_1n_2s \\ &\quad + (d_2n_2 + d_2^2n_1(x_1 + r_0n_1) + d_2^2n_2(x_2 + t_0n_2))(t + t_0), \qquad (25) \\ q_{12} &= (n_1^2 + d_1n_1^2(x_1 + r_0n_1) + d_1n_1n_2(x_2 + t_0n_2))(r + r_0) + n_1n_2s \\ &\quad + (n_2^2 + d_2n_1n_2(x_1 + r_0n_1) + d_2n_2^2(x_2 + t_0n_2))(t + t_0), \qquad (26) \\ q_{21} &= d_1^2(r + r_0) + d_1d_2s + d_2^2(t + t_0). \qquad (27) \end{aligned}$$

Then, a straightforward but tedious computation shows that the matrix of (24) becomes equal to the matrix in the right-hand side of (22). Hence we have $\overline{\mathcal{H}}(p) \supset \mathcal{H}(p)$.

($\overline{\mathcal{H}}(p) \subset \mathcal{H}(p)$). Suppose an element of $\overline{\mathcal{H}}(p)$ of (22). Then we let

$$\begin{aligned} r &= (y_1 - r_0d_1)(x_1 + r_0n_1)q_{11} + (y_1 - r_0d_1)^2q_{12} + (x_1 + r_0n_1)^2q_{21} - r_0, \qquad (28) \\ s &= ((y_1 - r_0d_1)(x_2 + t_0n_2) + (y_2 - t_0d_2)(x_1 + r_0n_1))q_{11} \\ &\quad + 2(y_1 - r_0d_1)(y_2 - t_0d_2)q_{12} + 2(x_1 + r_0n_1)(x_2 + t_0n_2)q_{21}, \qquad (29) \\ t &= (y_2 - t_0d_2)(x_2 + t_0n_2)q_{11} + (y_2 - t_0d_2)^2q_{12} + (x_2 + t_0n_2)^2q_{21} - t_0. \qquad (30) \end{aligned}$$

By a straightforward but tedious computation again, the matrix in the right-hand side of (22) becomes equal to $\Omega(Q)$ of (11) with

$$\begin{aligned} Q &= \begin{bmatrix} q_{11} & q_{12} \\ q_{21} & 0 \end{bmatrix}, \\ c &= \frac{y_1d_1 + y_2d_2 + r_0d_1^2 + t_0d_2^2}{x_1d_1 + x_2d_2 + r_0n_1d_1 + t_0n_2d_2}, \quad \text{and} \\ H_0 &= H(p, c). \end{aligned}$$

Therefore, we have $\overline{\mathcal{H}}(p) \subset \mathcal{H}(p)$. □

Proof of Theorem 3. We consider two cases: $c = 0$ and $c \neq 0$.

($c = 0$). In this case, p is in $\mathcal{A}$. Then, $h_{11} = 1$, $h_{12} = -p$, and $h_{21} = 0$. The fraction in (7) is expressed as $\frac{r}{1-rp}$ provided $1 - rp$ is nonzero. This is just the Youla–Kučera parametrization of the plant p in $\mathcal{A}$ by noting that the coprime factorization of $p \in \mathcal{A}$ is $p = n/d$ with $n = p$, $d = 1$, and the Bézout identity $0 \cdot n + 1 \cdot d = 1$.

($c \neq 0$). As in the proof of Theorem 1, letting $n_1 = -h_{12}$, $d_1 = h_{11}$, $n_2 = 1 - h_{11}$, $d_2 = h_{21}$, $x_1 = y_2 = 1$, $y_1 = x_2 = 0$, we obtain (4) and (5). Applying Theorem 2 to them, we have (7). □

6. Example

6.1. Coprime Factorization

Suppose that a plant admits a coprime factorization, say $p = n/d$ and $yn + xd = 1$ with $n, d, y, x \in \mathcal{A}$. Letting $n_1 = n_2 = n$, $d_1 = d_2 = d$, $y_1 = y$, $x_1 = x$, $y_2 = x_2 = 0$, one can apply Theorems 1 and 2 to the plant in order to obtain stabilizing controllers. The expression in the right-hand side of (6) is expressed as

$$\frac{y + (r+s+t)d}{x - (r+s+t)n}.$$

By replacing $(r+s+t)$ by new parameter u of $\mathcal{A}$, we have

$$\frac{y + ud}{x - un},$$

which is equivalent to the Youla–Kučera parametrization.

6.2. Anantharam's Example

Let us consider an example given by Anantharam in [14]. He considered the case $\mathcal{A} = \mathbb{Z}[\sqrt{-5}] = \{u + v\sqrt{-5} \mid u, v \in \mathbb{Z}\}$, where $\mathbb{Z}$ denotes the set of integers. We also let $\mathcal{Z}$ be $\{0\}$. The ring $\mathcal{A}$ is isomorphic to $\mathbb{Z}[x]/(x^2+5)$ and is an integral domain but not a unique factorization domain [18] (pp. 134–135). In fact, $6 \in \mathcal{A}$ has two factorizations, $2 \cdot 3$ and $(1+\sqrt{-5}) \cdot (1-\sqrt{-5})$. He showed, in [14], that a plant $p = (1+\sqrt{-5})/2$ does not admit a coprime factorization but is stabilizable and $c = (1-\sqrt{-5})/(-2)$ is a stabilizing controller. Then, $H(p,c)$ is as follows:

$$H(p,c) = \begin{bmatrix} -2 & 1+\sqrt{-5} \\ 1-\sqrt{-5} & -2 \end{bmatrix}.$$

Based on Theorem 3, the set of stabilizing controllers of p, $\mathcal{S}(p)$ of (7), is given as

$$\mathcal{S}(p) = \left\{ \frac{(1-\sqrt{-5}) + 4r - 2(1-\sqrt{-5})s - 2(2+\sqrt{-5})t}{-2 - 2(1+\sqrt{-5})r + 6s - 3(1-\sqrt{-5})t} \;\middle|\; r,\ s,\ t \in \mathcal{A},\ (-2 - 2(1+\sqrt{-5})r + 6s - 3(1-\sqrt{-5})t) \text{ is nonzero} \right\}. \quad (31)$$

By replacing $r \to -u$, $s \to -u$, $t \to -u$ with new parameter u of $\mathcal{A}$, we have

$$\mathcal{S}(p) = \left\{ \frac{2u + (1-\sqrt{-5})}{-(1+\sqrt{-5})u - 2} \;\middle|\; u \in \mathcal{A},\ (-(1+\sqrt{-5})u - 2) \text{ is nonzero} \right\},$$

which is the same result as one shown in [19].

We can also obtain alternative parametrization from Theorem 2 based on Note 5. The plant $p = (1+\sqrt{-5})/2$ has alternative representation $3/(1-\sqrt{-5})$. From 3 and 2, which are the numerator of $3/(1-\sqrt{-5})$ and the denominator of $(1+\sqrt{-5})/2$, respectively, we can see $1 \cdot 3 + (-1) \cdot 2 = 1$, so that letting $n_1 = 3$, $n_2 = (1+\sqrt{-5})$, $d_1 = (1-\sqrt{-5})$, $d_2 = 2$, $y_1 = 1$, $x_1 = 0$, $y_2 = 0$, $x_2 = -1$, we have (4) and (5). Thus, (6) results

$$\left\{ \frac{(1-\sqrt{-5}) - 2(2+\sqrt{-5})r + 2(1-\sqrt{-5})s + 4t}{-2 - 3(1-\sqrt{-5})r - 6s - 2(1+\sqrt{-5})t} \;\middle|\; r,\ s,\ t \in \mathcal{A},\ (-2 - 3(1-\sqrt{-5})r - 6s - 2(1+\sqrt{-5})t) \text{ is nonzero} \right\}. \quad (32)$$

This set (32) is equal to (31) by appropriate changes of parameters r, s, t.

6.3. Discrete-Time Systems without Unit-Delay Element

Mori [16] considered the case $\mathcal{A} = \mathbb{R}[z^2, z^3]$, where $\mathbb{R}$ denotes the set of real numbers. We also let $\mathcal{Z}$ be $\{f \in \mathcal{A} \mid f \text{ has zero constant term.}\}$. This ring is an integral domain but not a unique factorization domain. In fact, $z^6 \in \mathcal{A}$ has two factorizations, $z^2 \cdot z^2 \cdot z^2$ and $z^3 \cdot z^3$.

Let us consider the plant $p = (1-z^2)/(1-z^3) \in \mathcal{P}$. Now we let

$$n_1 = 1 - z^2,\ d_1 = 1 - z^3,\ n_2 = 1 + z^3,\ d_2 = 1 + z^2 + z^4$$

of $\mathcal{A}$ with $p = n_1/d_1 = n_2/d_2$. Then we have $y_1 = (2+z^2)/3$, $x_1 = 0$, $y_2 = 0$, and $x_2 = 1/3$ with $n_1 y_1 + d_1 x_1 + n_2 y_2 + d_2 x_2 = 1$. Thus, the plant p is stabilizable by Theorem 1.

Based on Theorem 2, the set of stabilizing controllers of p, $\mathcal{S}(p)$ of (6), is given as

$$\mathcal{S}(p) = \{n_c/d_c \mid r,\ s,\ t \in \mathcal{A},\ d_c \text{ is nonzero}\},$$

where

$$\begin{aligned} n_c &= \tfrac{1}{3}(2+z^2)(1-z^3) + (1-z^3)^2 r + (1-z^3)(1+z^2+z^4)s + (1+z^2+z^4)^2 t, \\ d_c &= \tfrac{1}{3}(1+z^2+z^4) - (1-z^2)(1-z^3)r - (1-z^6)s - (1+z^3)(1+z^2+z^4)t. \end{aligned}$$

7. Conclusions and Future Work

This paper has presented a generalization of the Youla–Kučera parametrization to obtain all stabilizing controllers without coprime factorization. This is based on two rational expressions of a given plant. Alternative parametrization is also given by one stabilizing controller.

As future work, we will aim to investigate further generalization of the Youla–Kučera parametrization for multi-input multi-output plants with no coprime factorizations as well. Also, the possibility to extend Theorems 1–3 will be investigated.

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