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Accelerating Update of Variable Precision Multigranulation Approximations While Adding Granular Structures

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Abstract: In multigranulation environments, variable precision multigranulation rough set (VP-MGRS) is a useful framework that has a tolerance for errors. Approximations are basic concepts for knowledge acquisition and attribute reductions. Accelerating update of approximations can enhance the efficiency of acquiring decision rules by utilizing previously saved information. In this study, we focus on exploiting update mechanisms of approximations in VPMGRS with the addition of granular structures. By analyzing the basic changing trends of approximations in VP-MGRS, we develop accelerating update mechanisms for acquiring approximations. In addition, an incremental algorithm to update variable precision multigranulation approximations is proposed when adding multiple granular structures. Finally, extensive comparisons elaborate the efficiency of the incremental algorithm.

Keywords: incremental mechanisms; data mining; granular structure; approximations

check for updates

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1. Introduction

As a classical framework for representing and handling uncertain or vague data, the model of rough set theory (RST) has been typically applied in diverse areas from theoretic and application viewpoints, such as rule induction, feature selection, three-way decisions, and so on [1–5].

In real applications, various kinds of data vary over time. Due to the dynamic changing of data, the naive method for knowledge discovery often cost too much running time. As a consequence, how to effectively accelerate the calculation of useful information becomes an essential issue. To alleviate this problem, incremental knowledge discovery under various rough set models when the information system varies have gained continuous interest [6–10]. Usually, the objects [11,12], attributes [13,14], and attribute values [15] may change dynamically. Moreover, Li et al. established a dynamical framework to obtain useful knowledge based on missing data [16]. Yang et al. developed an efficient framework to acquire fuzzy probability three-way decisions rules [17]. Chen et al. established an incremental model for handling noise data problems while objects dynamically alter [18]. Furthermore, Niu et al. presented the mechanisms for computing granular reduct and rules with the newly available data [19]. Huang et al. established an effective strategy applied for computing reduct under time-evolving data [20]. What is more, Hao et al. proposed an effective scale selection method when adding an object [21]. Li et al. introduced a unique approach to calculate dominated classes when a set of objects vary over time [22]. Moreover, Luo et al. designed a matrix method under evolving three-way decision framework [23]. To process neighborhood data, Zhang et al. put forward fast approaches for obtaining neighborhood-based approximations [24]. Luo et al. established an incremental set-valued rough framework [25]. In addition, Chen et al. solved decision rule updating problems when attribute values evolve [26]. To summarize, these extensive studies have attracted continuously growing interests.

Although RST has significant benefits in enhancing the performance of knowledge reasoning, several well-established soft computing frameworks have been exploited for intelligent decision making [27–29]. As an effective extension of RST, multigranulation rough set (MGRS) was proposed [30]. In MGRS, the approximations are established by a group of information granules derived from a set of equivalence relations and there are two different models, called the optimistic and pessimistic MGRS, respectively. Since then, multigranulation rough set offered useful insights into data analysis and its theory progresses rapidly [31–33]. For example, Qian et al. developed a unique pessimistic MGRS-based decision model based on seeking common ground [34]. Zhan et al. developed two novel covering based multigranulation fuzzy sets [35]. Zhang et al. developed a multi-attribute group decision mechanism in light of the multi-granularity three-way fuzzy context [36]. Li et al. investigated a unique method to construct three way cognitive concept in presence of multi-granularity [37]. Dou et al. discussed two kind strategies for constructing the framework of VPMGRS [38]. In dynamic environments, incremental update of multigranulation knowledge has attracted widespread concerns [39–43]. For instance, Li et al. discussed local multigranulation rough frameworks to handle incremental evolving ordered systems [44]. Zhang et al. presented updating mechanisms to acquire multigranulation knowledge in interval-valued approximate space [45]. The approximations of a concept which can be further used for rule acquisition and attribute reduction provide an essential help in intelligence analysis [46,47]. Nevertheless, in variable precision multigranulation rough sets, how to effectively dynamic updating approximations has rarely been concerned. Motivated from the above observations, we concentrate on exploiting an incremental approach to accelerate updating of approximations in VPMGRS with addition of granular structures.

The remainder of this study is structured as below. In Section 2, essential knowledge of RST, and VPMGRS are given. In Section 3, the accelerate update mechanisms and algorithm for computing approximations based on VPMGRS are proposed when adding multiple granular structures. Extensive comparisons are performed to validate the performance in Section 4. In Section 5, we give several remarks on the paper.

2. Preliminaries

This section briefly gives some notions in terms of RST and its extensions [33,34,38,48–50].

2.1. Rough Set and Variable Precision Rough Set

Definition 1 ([49,50])**.** *Let $S = (U, A, V, f)$ be an information system, where $U = \{x_1, x_2, \ldots, x_n\}$ is a group of objects, namely the universe. $A = \{a_1, a_2, \ldots, a_m\}$ is a set of attributes. $V = \bigcup_{a \in A} V_a$ is a set of attribute values, where V_a denotes the domain of attribute a. f refers to a decision function and $f(x, a) \in V_a$, for $\forall x \in U$.*

Notably, an information system $S = (U, A, V, f)$ is abbreviated as $S = (U, A)$ in this study.

Definition 2 ([49,50])**.** *Given $S = (U, A)$ and $B \subseteq A$, an equivalence relation on U can be expressed by:*

$$R_B = \{(x, y) | (x, y) \in U \times U : f(x, a) = f(y, a), \forall a \in B\} \tag{1}$$

Obviously, a pair (U, R_B) is treated as an approximate space. The equivalence relation R_B generates a partition of the universe, represented by U/R_B, namely, $U/R_B = \{E_1, E_2, \ldots, E_m\}$. The equivalence class in terms of x is denoted by $[x]_R = \{y \in U : (x, y) \in R_B\}$.

Definition 3 ([49,50])**.** *Given $S = (U, A)$ and $X \subseteq U$. $[x]_R$ refers to the equivalence class in terms of R_B. The lower and upper approximations of X are formalized by:*

$$\underline{R}(X) = \{x \in U | [x]_R \subseteq X\} \tag{2}$$

$$\overline{R}(X) = \{x \in U | [x]_R \cap X \neq \varnothing\} \tag{3}$$

Definition 4 ([48])**.** *Given $S = (U, A)$ and $X, Y \subseteq U$. Then, the relative degree of misclassification $c(X, Y)$ is described by*

$$c(X,Y) = \begin{cases} 1 - \frac{|X| \cap |Y|}{|X|}, & |X| > 0; \\ 0, & |X| = 0. \end{cases} \tag{4}$$

where $|\cdot|$ denotes the cardinality.

Definition 5 ([48])**.** *Given $S = (U, A)$ with $X \subseteq U$, and $0 \leq \beta < 0.5$. Then, β-lower approximation, β-upper approximation are defined by:*

$$\underline{R}_\beta(X) = \cup\{E \in U/R_B | c(E, X) \leq \beta\} \tag{5}$$

$$\overline{R}_\beta(X) = \cup\{E \in U/R_B | c(E, X) < 1 - \beta\} \tag{6}$$

Generally, β-positive region $pos_\beta(X)$, β-boundary region $bnd_\beta(X)$ and β-negative region $neg_\beta(X)$ are expressed by: $pos_\beta(X) = \underline{R}_\beta(X)$, $bnd_\beta(X) = \overline{R}_\beta(X) - \underline{R}_\beta(X)$, and $neg_\beta(X) = U - \overline{R}_\beta(X)$.

2.2. Variable Precision Multigranulation Rough Sets

Qian and Liang investigated multigranulation rough sets [33,34]. In MGRS, two effective frameworks were investigated, i.e., optimistic MGRS and pessimistic MGRS.

Definition 6 ([33])**.** *Given $S = (U, A)$ with $a_1, a_2, \ldots, a_m \in A$, $X \subseteq U$. The lower and upper approximations of X in optimistic MGRS are formalized by $\underline{\sum_{i=1}^m a_i^O}(X)$ and $\overline{\sum_{i=1}^m a_i^O}(X)$, where,*

$$\underline{\sum_{i=1}^{m} a_i^O}(X) = \{x \in U | [x]_{a_1} \subseteq X \vee [x]_{a_2} \subseteq X \vee \ldots \vee [x]_{a_m} \subseteq X\} \tag{7}$$

$$\overline{\sum_{i=1}^{m} a_i^O}(X) = \sim \underline{\sum_{i=1}^{m} a_i^O}(\sim X) \tag{8}$$

where $\sim X$ denotes the complement of X.

Theorem 1. *Given $S = (U, A)$, and $a_1, a_2, \ldots, a_m \in A$, $X \subseteq U$. Then, the following result holds:*

$$\overline{\sum_{i=1}^{m} a_i^O}(X) = \{x \in U | [x]_{a_1} \cap X \neq \varnothing \wedge [x]_{a_2} \cap X \neq \varnothing \wedge \ldots \wedge [x]_{a_m} \cap X \neq \varnothing\} \tag{9}$$

Definition 7 ([34])**.** *Given $S = (U, A)$, and $a_1, a_2, \ldots, a_m \in A$, $X \subseteq U$. The lower and upper approximations of X in pessimistic MGRS are formalized by $\underline{\sum_{i=1}^m a_i^P}(X)$ and $\overline{\sum_{i=1}^m a_i^P}(X)$, where,*

$$\underline{\sum_{i=1}^{m} a_i^P}(X) = \{x \in U | [x]_{a_1} \subseteq X \wedge [x]_{a_2} \subseteq X \wedge \ldots \wedge [x]_{a_m} \subseteq X\} \tag{10}$$

$$\overline{\sum_{i=1}^{m} a_i^P}(X) = \sim \underline{\sum_{i=1}^{m} a_i^P}(\sim X) \tag{11}$$

Theorem 2. *Given $S = (U, A)$ and $a_1, a_2, \ldots, a_m \in A$, $X \subseteq U$, the following result holds:*

$$\overline{\sum_{i=1}^{m} a_i^P}(X) = \{x \in U | [x]_{a_1} \cap X \neq \varnothing \vee [x]_{a_2} \cap X \neq \varnothing \vee \ldots \vee [x]_{a_m} \cap X \neq \varnothing\} \quad (12)$$

In what follows, the optimistic VPMGRS is established by using multiple granular structures.

Definition 8 ([38]). *Given $S = (U, A)$, and $a_1, a_2, \ldots, a_m \in A$, $X \subseteq U$, $0 \leq \beta < 0.5$. The lower and upper approximations of X in optimistic VPMGRS are defined as $\underline{\sum_{i=1}^{m} a_{i\beta}^O}(X)$ and $\overline{\sum_{i=1}^{m} a_{i\beta}^O}(X)$,*

$$\underline{\sum_{i=1}^{m} a_{i\beta}^O}(X) = \{x \in U | c([x]_{a_1}, X) \leq \beta \vee c([x]_{a_2}, X) \leq \beta \vee \ldots \vee c([x]_{a_m}, X) \leq \beta\} \quad (13)$$

$$\overline{\sum_{i=1}^{m} a_{i\beta}^O}(X) = \sim \underline{\sum_{i=1}^{m} a_{i\beta}^O}(\sim X) \quad (14)$$

Theorem 3. *Given $S = (U, A)$, and $a_1, a_2, \ldots, a_m \in A$, $X \subseteq U$. Assume $0 \leq \beta < 0.5$, then we have*

$$\overline{\sum_{i=1}^{m} a_{i\beta}^O}(X) = \{x \in U | c([x]_{a_1}, X) < 1 - \beta \wedge c([x]_{a_2}, X) < 1 - \beta \wedge \ldots \wedge c([x]_{a_m}, X) < 1 - \beta\} \quad (15)$$

Proof. It is immediate from Definitions 4 and 8. □

Similarly, in the following, we introduce pessimistic VPMGRS.

Definition 9 ([38]). *Given $S = (U, A)$ and $a_1, a_2, \ldots, a_m \in A$, $X \subseteq U$. Assume $0 \leq \beta < 0.5$, the lower and upper approximations of X in pessimistic VPMGRS are formalized by $\underline{\sum_{i=1}^{m} a_{i\beta}^P}(X)$ and $\overline{\sum_{i=1}^{m} a_{i\beta}^P}(X)$, where,*

$$\underline{\sum_{i=1}^{m} a_{i\beta}^P}(X) = \{x \in U | c([x]_{a_1}, X) \leq \beta \wedge c([x]_{a_2}, X) \leq \beta \wedge \ldots \wedge c([x]_{a_m}, X) \leq \beta\} \quad (16)$$

$$\overline{\sum_{i=1}^{m} a_{i\beta}^P}(X) = \sim \underline{\sum_{i=1}^{m} a_{i\beta}^P}(\sim X) \quad (17)$$

where $\sim X$ is the complement of X.

Theorem 4. *Given $S = (U, A)$ and $a_1, a_2, \ldots, a_m \in A$, $X \subseteq U$. Then, we have*

$$\overline{\sum_{i=1}^{m} a_{i\beta}^P}(X) = \{x \in U | c([x]_{a_1}, X) < 1 - \beta \vee c([x]_{a_2}, X) < 1 - \beta \vee \ldots \vee c([x]_{a_m}, X) < 1 - \beta\} \quad (18)$$

Proof. It can be easily proved by Definitions 4 and 9. □

Next, an illustrated example is shown to describe the process of computing approximations in VPMGRS.

Example 1. *Given an information system in terms of students' comprehensive qualities evaluation which is outlined in Table 1, where $U = \{x_1, \ldots, x_8\}$ is the universe of eight objects which are viewed as eight students, and $A = \{a_1, a_2, a_3, a_4\}$ is a multigranulation space consisting of four granular structures. These granular structures denotes four courses of students. The value of each granular structure indicates the grade level that each student achieves. Furthermore, d is a decision attribute. Assume $X = \{x_2, x_3, x_5, x_6\}$ is a target concept, then, by Definition 4, we can obtain the partitions with reference to each granular structure below:*

$U/a_1 = \{\{x_1, x_4, x_5\}, \{x_2, x_3, x_6\}, \{x_7, x_8\}\}$, $c([x_1]_{a_1}, X) = c([x_4]_{a_1}, X) = c([x_5]_{a_1}, X) = 2/3$, $c([x_2]_{a_1}, X) = c([x_3]_{a_1}, X) = c([x_6]_{a_1}, X) = 0$, $c([x_7]_{a_1}, X) = c([x_8]_{a_1}, X) = 1$.

$U/a_2 = \{\{x_1, x_3, x_4, x_5\}, \{x_2\}, \{x_6, x_7, x_8\}\}$, $c([x_1]_{a_2}, X) = c([x_3]_{a_2}, X) = c([x_4]_{a_2}, X) = c([x_5]_{a_2}, X) = 1/2$, $c([x_2]_{a_2}, X) = 0$, $c([x_6]_{a_2}, X) = c([x_7]_{a_2}, X) = c([x_8]_{a_2}, X) = 2/3$.

$U/a_3 = \{\{x_1, x_7\}, \{x_2, x_3\}, \{x_4, x_5, x_6, x_8\}\}$, $c([x_1]_{a_3}, X) = c([x_7]_{a_3}, X) = 1$, $c([x_2]_{a_3}, X) = c([x_3]_{a_3}, X) = 0$, $c([x_4]_{a_3}, X) = c([x_5]_{a_3}, X) = c([x_6]_{a_3}, X) = c([x_8]_{a_3}, X) = 1/2$.

$U/a_4 = \{\{x_1, x_4, x_5, x_6\}, \{x_2, x_3\}, \{x_7, x_8\}\}$, $c([x_1]_{a_4}, X) = c([x_4]_{a_4}, X) = c([x_5]_{a_4}, X) = c([x_6]_{a_4}, X) = 1/2$, $c([x_2]_{a_4}, X) = c([x_3]_{a_4}, X) = 0$, $c([x_7]_{a_4}, X) = c([x_8]_{a_4}, X) = 1$.

Assume $\beta = 0.3$, based on Definitions 8 and 9 and Theorems 3 and 4, we obtain the approximations of VPMGRS as follows: $\underline{\sum_{i=1}^{m} a_{i\beta}^{O}}(X) = \{x_2, x_3, x_6\}$, $\overline{\sum_{i=1}^{m} a_{i\beta}^{O}}(X) = \{x_2, x_3, x_4, x_5, x_6\}$, $\underline{\sum_{i=1}^{m} a_{i\beta}^{P}}(X) = \{x_2\}$, *and* $\overline{\sum_{i=1}^{m} a_{i\beta}^{P}}(X) = \{x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8\}$.

Table 1. A description of information system.

U	a_1	a_2	a_3	a_4	d
x_1	1	1	1	1	1
x_2	2	3	2	2	2
x_3	2	1	2	2	2
x_4	1	1	3	1	1
x_5	1	1	3	1	2
x_6	2	2	3	1	2
x_7	3	2	1	3	1
x_8	3	2	3	3	1

From Example 1, when the parameter β keeps constant, the lower approximation of optimistic VPMGRS is larger than that of pessimistic VPMGRS. Meanwhile, the upper approximation of optimistic VPMGRS is smaller than that of pessimistic VPMGRS. These decision strategies can help us to make better decision analysis according to different requirements. Based on above discussion, we present a traditional algorithm for calculating approximations in VPMGRS by using above definitions.

We outline the procedures of non-incrementally calculating variable precision multigranulation approximations in Algorithm 1 (Algorithm NACA). The complexity of obtaining equivalence classes and relative degree of misclassification in Lines 4–8 takes $O(mn^2)$, where m and n respectively denote the number of granular structures and objects. The complexity of computing approximations of optimistic VPMGRS based on Definition 8 in Lines 9–26 takes $O(2mn)$. The complexity of computing approximations of pessimistic VPMGRS based on Definition 9 in Lines 27–44 takes $O(2mn)$. The complexity of outputting the approximations of VPMGRS in Line 45 takes $O(n)$. Accordingly, the whole complexity of Algorithm NACA takes $O(mn^2)$.

Algorithm 1: A non-incremental algorithm for computing approximations in VPMGRS (NACA).

Input: An information system $S = (U, A)$, the granular structures $A = \{a_1, a_2, \ldots, a_m\}$, a target concept $X \subseteq U$, β.

Output: The approximations of X in VPMGRS: $\underline{\sum_{i=1}^{m} a_{i\beta}^{O}}(X)$, $\overline{\sum_{i=1}^{m} a_{i\beta}^{O}}(X)$, $\underline{\sum_{i=1}^{m} a_{i\beta}^{P}}(X)$, $\overline{\sum_{i=1}^{m} a_{i\beta}^{P}}(X)$.

begin
  $\underline{\sum_{i=1}^{m} a_{i\beta}^{O}}(X) \leftarrow \varnothing$, $\overline{\sum_{i=1}^{m} a_{i\beta}^{O}}(X) \leftarrow \varnothing$, `// Initialization`;
  $\underline{\sum_{i=1}^{m} a_{i\beta}^{P}}(X) \leftarrow \varnothing$, $\overline{\sum_{i=1}^{m} a_{i\beta}^{P}}(X) \leftarrow \varnothing$.
  for $i = 1$ **to** m **do**
    for each $x \in U$ **do**
      Compute $[x]_{a_i}$ and $c([x]_{a_i}, X)$;
    end
  end
  for each $x \in U$ **do**
    for $i = 1$ **to** m **do**
      if $c([x]_{a_i}, X) \leq \beta$ **then**
        $\underline{\sum_{i=1}^{m} a_{i\beta}^{O}}(X) = \underline{\sum_{i=1}^{m} a_{i\beta}^{O}}(X) \cup \{x\}$, **break**;
      end
    end
  end
  for each $x \in U$ **do**
    for $i = 1$ **to** m **do**
      flag=1;
      if $c([x]_{a_i}, X) \geq 1 - \beta$ **then**
        flag=0, **break**;
      end
    end
    if *flag==1* **then**
      $\overline{\sum_{i=1}^{m} a_{i\beta}^{O}}(X) = \overline{\sum_{i=1}^{m} a_{i\beta}^{O}}(X) \cup \{x\}$;
    end
  end
  for each $x \in U$ **do**
    for $i = 1$ **to** m **do**
      flag=1;
      if $c([x]_{a_i}, X) > \beta$ **then**
        flag=0, **break**;
      end
    end
    if *flag==1* **then**
      $\underline{\sum_{i=1}^{m} a_{i\beta}^{P}}(X) = \underline{\sum_{i=1}^{m} a_{i\beta}^{P}}(X) \cup \{x\}$;
    end
  end
  for each $x \in U$ **do**
    for $i = 1$ **to** m **do**
      if $c([x]_{a_i}, X) < 1 - \beta$ **then**
        $\overline{\sum_{i=1}^{m} a_{i\beta}^{P}}(X) = \overline{\sum_{i=1}^{m} a_{i\beta}^{P}}(X) \cup \{x\}$, **break**;
      end
    end
  end
  return $\underline{\sum_{i=1}^{m} a_{i\beta}^{O}}(X)$, $\overline{\sum_{i=1}^{m} a_{i\beta}^{O}}(X)$, $\underline{\sum_{i=1}^{m} a_{i\beta}^{P}}(X)$, $\overline{\sum_{i=1}^{m} a_{i\beta}^{P}}(X)$.
end

3. Accelerating Update of Approximations of VPMGRS

In order to reduce computational cost, this section focuses on exploiting dynamical strategies to obtain approximations in the case of addition of a single granular structure. Under the addition of a new granular structure, the update of variable precision multigranulation approximations is by using the traditional approach to recompute the new data sets, which requires to scan all the objects and multigranulation structures. Nevertheless, it is very time-consuming to obtain the valuable knowledge. To deal with this problem, a novel incremental approach is investigated to update variable precision multigranulation rough approximations by using the previous computational results which avoids to recalculate all the objects and granular structures. Notably, the change of multiple granular structures is treated as the iterative change of a single granular structure. Assume $a_1, a_2, \ldots, a_m \in A$ are m granular structures in a multigranulation space. After adding a granular structure

$m+1$, the approximations should be updated in an incremental manner. To facilitate the description, we denote the updated lower and upper approximations of X in optimistic VPMGRS as $\underline{\sum_{i=1}^{m+1} a_{i\beta}^{\wedge O}}(X)$, and $\overline{\sum_{i=1}^{m+1} a_{i\beta}^{\wedge O}}(X)$, respectively. Similarly, after adding a granular structure $m+1$, we denote the updated lower and upper approximations of X in pessimistic VPMGRS as $\underline{\sum_{i=1}^{m+1} a_{i\beta}^{\wedge P}}(X)$, and $\overline{\sum_{i=1}^{m+1} a_{i\beta}^{\wedge P}}(X)$, respectively.

Proposition 1. *Given $S=(U,A)$ and $a_1,a_2,\ldots,a_m \in A$, $0 \le \beta < 0.5$, $X \subseteq U$. After adding a single granular structure a_{m+1}, for the optimistic variable precision multigranulation approximations of X, we have,*
(1) $\underline{\sum_{i=1}^{m+1} a_{i\beta}^{\wedge O}}(X) \supseteq \underline{\sum_{i=1}^{m} a_{i\beta}^{O}}(X)$;
(2) $\overline{\sum_{i=1}^{m+1} a_{i\beta}^{\wedge O}}(X) \subseteq \overline{\sum_{i=1}^{m} a_{i\beta}^{O}}(X)$.

Proof.
(1) If $\forall x \in \underline{\sum_{i=1}^{m} a_{i\beta}^{O}}(X)$, based on Definition 8, then $c([x]_{a_i},X) \le \beta$ holds, $\exists i \in \{1,2,\ldots,m\}$. With reference to addition of a_{m+1}, it implies $c([x]_{a_i},X) \le \beta$ for a given $i \in \{1,2,\ldots,m+1\}$. Therefore, $x \in \underline{\sum_{i=1}^{m+1} a_{i\beta}^{\wedge O}}(X)$. Hence, $\underline{\sum_{i=1}^{m+1} a_{i\beta}^{\wedge O}}(X) \supseteq \underline{\sum_{i=1}^{m} a_{i\beta}^{O}}(X)$.

(2) With addition of a_{m+1}, for $\forall x \in \overline{\sum_{i=1}^{m+1} a_{i\beta}^{O}}(X)$, according to Theorem 3, it implies $c([x]_{a_i},X) < 1-\beta$ for every $i \in \{1,2,\ldots,m+1\}$. Obviously, $c([x]_{a_i},X) < 1-\beta$ for every $i \in \{1,2,\ldots,m\}$. As a result, $x \in \overline{\sum_{i=1}^{m} a_{i\beta}^{\wedge O}}(X)$. Hence, $\overline{\sum_{i=1}^{m+1} a_{i\beta}^{\wedge O}}(X) \subseteq \overline{\sum_{i=1}^{m} a_{i\beta}^{O}}(X)$. □

According to Proposition 1, the lower approximation of optimistic VPMGRS has a trend of increase whereas the upper approximation of optimistic VPMGRS has a trend of decrease.

Proposition 2. *Given $S=(U,A)$, and $a_1,a_2,\ldots,a_m \in A$, $0 \le \beta < 0.5$, $X \subseteq U$. After adding a single granular structure a_{m+1}, for the pessimistic variable precision multigranulation approximations, we have,*
(1) $\underline{\sum_{i=1}^{m+1} a_{i\beta}^{\wedge P}}(X) \subseteq \underline{\sum_{i=1}^{m} a_{i\beta}^{P}}(X)$;
(2) $\overline{\sum_{i=1}^{m+1} a_{i\beta}^{\wedge P}}(X) \supseteq \overline{\sum_{i=1}^{m} a_{i\beta}^{P}}(X)$.

Proof.
(1) If $\forall x \in \underline{\sum_{i=1}^{m+1} a_{i\beta}^{\wedge P}}(X)$, then, $c([x]_{a_i},X) \le \beta$ holds for any $i \in \{1,2,\ldots,m+1\}$ according to Definition 9. As a result, $c([x]_{a_i},X) \le \beta$ holds for any $i \in \{1,2,\ldots,m\}$. Therefore, $x \in \underline{\sum_{i=1}^{m} a_{i\beta}^{P}}(X)$. This implies that $\underline{\sum_{i=1}^{m+1} a_{i\beta}^{\wedge P}}(X) \subseteq \underline{\sum_{i=1}^{m} a_{i\beta}^{P}}(X)$.

(2) If $\forall x \in \overline{\sum_{i=1}^{m} a_{i\beta}^{\wedge P}}(X)$, based on Theorem 4, we have $c([x]_{a_i},X) < 1-\beta$ holds for a given $i \in \{1,2,\ldots,m\}$. Therefore, $c([x]_{a_i},X) < 1-\beta$ holds for a given $i \in \{1,2,\ldots,m+1\}$. Hence, $x \in \overline{\sum_{i=1}^{m+1} a_{i\beta}^{\wedge P}}(X)$. As a result, $\overline{\sum_{i=1}^{m+1} a_{i\beta}^{\wedge P}}(X) \supseteq \overline{\sum_{i=1}^{m} a_{i\beta}^{P}}(X)$. □

In light of Proposition 2, we can conclude that the pessimistic variable precision multigranulation upper approximation has a trend of increase while the pessimistic variable precision multigranulation lower approximation has a trend of decrease.

In what follows, updating mechanisms for acquiring approximations of optimistic and pessimistic VPMGRS are demonstrated.

Theorem 5. *Given $S = (U, A)$, and $a_1, a_2, \ldots, a_m \in A$, $0 \le \beta < 0.5$, $X \subseteq U$, while adding a_{m+1}, if $x \notin \underline{\sum_{i=1}^{m} a_{i\beta}^{O}}(X)$, the following result holds,*

(1) if $c([x]_{a_{m+1}}, X) \le \beta$, then $\underline{\sum_{i=1}^{m+1} a_{i\beta}^{\wedge O}}(X) = \underline{\sum_{i=1}^{m} a_{i\beta}^{O}}(X) \cup \{x\}$;

(2) otherwise $\underline{\sum_{i=1}^{m+1} a_{i\beta}^{\wedge O}}(X) = \underline{\sum_{i=1}^{m} a_{i\beta}^{O}}(X)$.

Proof.

(1) Due to $x \notin \underline{\sum_{i=1}^{m} a_{i\beta}^{O}}(X)$, according to Definition 8, it implies $c([x]_{a_i}, X) > \beta$ for any $i \in \{1, 2, \ldots, m\}$, while adding a_{m+1}, if $c([x]_{a_{m+1}}, X) \le \beta$, then $\exists i \in \{1, 2, \ldots, m+1\}$, $c([x]_{a_i}, X) \le \beta$ holds. So $x \in \underline{\sum_{i=1}^{m+1} a_{i\beta}^{\wedge O}}(X)$. Hence, $\underline{\sum_{i=1}^{m+1} a_{i\beta}^{\wedge O}}(X) = \underline{\sum_{i=1}^{m} a_{i\beta}^{O}}(X) \cup \{x\}$.

(2) If $x \notin \underline{\sum_{i=1}^{m} a_{i\beta}^{O}}(X)$, then we obtain $c([x]_{a_i}, X) > \beta$ for any $i \in \{1, 2, \ldots, m\}$ by Definition 8, while adding a_{m+1}, if $c([x]_{a_{m+1}}, X) > \beta$, then we obtain $c([x]_{a_i}, X) > \beta$ for any $i \in \{1, 2, \ldots, m+1\}$. Consequently, $x \notin \underline{\sum_{i=1}^{m+1} a_{i\beta}^{\wedge O}}(X)$. Thus, $\underline{\sum_{i=1}^{m+1} a_{i\beta}^{\wedge O}}(X) = \underline{\sum_{i=1}^{m} a_{i\beta}^{O}}(X)$. □

In light of Theorem 5, we can update lower approximation of optimistic VPMGRS with addition of a granular structure.

Example 2 (Continuation of Example 1)**.** *Table 2 indicates the candidate five courses that will be added into the original multigranulation space. After adding a granular structure, the approximations applied for decision analysis may change. To verify incremental mechanism for updating approximations of VPMGRS, we suppose that the candidate granular structure a_5 shown in Table 2 will be added into Table 1. Then, we can compute $U/a_5 = \{\{x_1, x_3, x_4\}, \{x_2, x_5\}, \{x_6, x_7, x_8\}\}$. Based on Definition 4, we obtain, $c([x_1]_{a_5}, X) = c([x_3]_{a_5}, X) = c([x_4]_{a_5}, X) = 2/3$, $c([x_2]_{a_5}, X) = c([x_5]_{a_5}, X) = 0$, $c([x_6]_{a_5}, X) = c([x_7]_{a_5}, X) = c([x_8]_{a_5}, X) = 2/3$. Because $x_5 \notin \underline{\sum_{i=1}^{m} a_{i\beta}^{O}}(X)$, and $c([x_5]_{a_5}, X) = 0 < \beta = 0.3$, according to Theorem 5, we obtain $\underline{\sum_{i=1}^{m+1} a_{i\beta}^{\wedge O}}(X) = \underline{\sum_{i=1}^{m} a_{i\beta}^{O}}(X) \cup \{x_5\} = \{x_2, x_3, x_5, x_6\}$.*

Table 2. A description of added candidate granular structures.

U	a_5	a_6	a_7	a_8
x_1	1	2	2	1
x_2	2	1	2	1
x_3	1	1	3	1
x_4	1	2	1	2
x_5	2	1	3	3
x_6	3	1	3	2
x_7	3	3	1	3
x_8	3	3	1	3

Theorem 6. *Given $S = (U, A)$, and $a_1, a_2, \ldots, a_m \in A$, $0 \le \beta < 0.5$, $X \subseteq U$, while adding a_{m+1} into the information system, if $x \in \overline{\sum_{i=1}^{m} a_{i\beta}^{O}}(X)$, the following result holds,*

(1) if $c([x]_{a_{m+1}}, X) \ge 1 - \beta$, then $\overline{\sum_{i=1}^{m+1} a_{i\beta}^{\wedge O}}(X) = \overline{\sum_{i=1}^{m} a_{i\beta}^{O}}(X) - \{x\}$;

(2) otherwise, $\overline{\sum_{i=1}^{m+1} a_{i\beta}^{\wedge O}}(X) = \overline{\sum_{i=1}^{m} a_{i\beta}^{O}}(X)$.

Proof.

(1) If $x \in \overline{\sum_{i=1}^{m} a_{i\beta}^{O}}(X)$, according to Theorem 3, we obtain $c([x]_{a_i}, X) < 1 - \beta$ for any $i \in \{1, 2, \ldots, m\}$, while adding a_{m+1}, if $c([x]_{a_{m+1}}, X) \ge 1 - \beta$, it follows that $c([x]_{a_i}, X) \ge 1 - \beta$ for a given $i \in \{1, 2, \ldots, m+1\}$. So $x \notin \overline{\sum_{i=1}^{m+1} a_{i\beta}^{\wedge O}}(X)$. Hence, $\overline{\sum_{i=1}^{m+1} a_{i\beta}^{\wedge O}}(X) = \overline{\sum_{i=1}^{m} a_{i\beta}^{O}}(X) - \{x\}$.

(2) If $x \in \overline{\sum_{i=1}^{m} a_{i\beta}^{O}}(X)$, according to Theorem 3, then we obtain $c([x]_{a_i}, X) < 1-\beta$ for any $i \in \{1,2,\ldots,m\}$, while adding a_{m+1}, if $c([x]_{a_{m+1}}, X) < 1-\beta$ holds, then $c([x]_{a_i}, X) < 1-\beta$ for any $i \in \{1,2,\ldots,m+1\}$. Thus, $x \in \overline{\sum_{i=1}^{m+1} a_{i\beta}^{\wedge O}}(X)$. As a result, $\overline{\sum_{i=1}^{m+1} a_{i\beta}^{\wedge O}}(X) = \overline{\sum_{i=1}^{m} a_{i\beta}^{O}}(X)$. □

Based on Theorem 6, we can effectively acquire upper approximation of optimistic VPMGRS with addition of a granular structure.

Example 3 (Continuation of Example 1)**.** *Assume the granular structure a_6 in Table 2 is added into Table 1, then we obtain $U/a_6 = \{\{x_1,x_4\},\{x_2,x_3,x_5,x_6\},\{x_7,x_8\}\}$. By Definition 4, we have, $c([x_1]_{a_6}, X) = c([x_4]_{a_6}, X) = 1$, $c([x_2]_{a_6}, X) = c([x_3]_{a_6}, X) = c([x_5]_{a_6}, X) = c([x_6]_{a_6}, X) = 0$ and $c([x_7]_{a_6}, X) = c([x_8]_{a_6}, X) = 1$. Because $x_4 \in \overline{\sum_{i=1}^{m} a_{i\beta}^{O}}(X)$ and $c([x_4]_{a_6}, X) = 1 \geq 1-\beta = 0.7$, according to Theorem 6, it follows that $\overline{\sum_{i=1}^{m+1} a_{i\beta}^{\wedge O}}(X) = \overline{\sum_{i=1}^{m} a_{i\beta}^{O}}(X) - \{x_4\} = \{x_2,x_3,x_5,x_6\}$.*

Theorem 7. *Given $S = (U,A)$ and $a_1,a_2,\ldots,a_m \in A$, $X \subseteq U$, $0 \leq \beta < 0.5$, while adding a_{m+1} into the information system, if $x \in \underline{\sum_{i=1}^{m} a_{i\beta}^{P}}(X)$, we have*

(1) if $c([x]_{a_{m+1}}, X) > \beta$, then $\underline{\sum_{i=1}^{m+1} a_{i\beta}^{\wedge P}}(X) = \underline{\sum_{i=1}^{m} a_{i\beta}^{P}}(X) - \{x\}$;

(2) otherwise $\underline{\sum_{i=1}^{m+1} a_{i\beta}^{\wedge P}}(X) = \underline{\sum_{i=1}^{m} a_{i\beta}^{P}}(X)$.

Proof.

(1) If $x \in \underline{\sum_{i=1}^{m} a_{i\beta}^{P}}(X)$, according to Definition 9, $c([x]_{a_i}, X) \leq \beta$ holds for any $i \in \{1,2,\ldots,m\}$, while adding a_{m+1}, if $c([x]_{a_{m+1}}, X) > \beta$, then $c([x]_{a_i}, X) > \beta$ holds for a given $i \in \{1,2,\ldots,m+1\}$. Therefore, $x \notin \underline{\sum_{i=1}^{m+1} a_{i\beta}^{\wedge P}}(X)$. As a result, $\underline{\sum_{i=1}^{m+1} a_{i\beta}^{\wedge P}}(X) = \underline{\sum_{i=1}^{m} a_{i\beta}^{P}}(X) - \{x\}$.

(2) If $x \in \underline{\sum_{i=1}^{m} a_{i\beta}^{P}}(X)$, according to Definition 9, then $c([x]_{a_i}, X) \leq \beta$ for any $i \in \{1,2,\ldots,m\}$, while adding a_{m+1}, if $c([x]_{a_{m+1}}, X) \leq \beta$, obviously, $c([x]_{a_i}, X) \leq \beta$ holds for any $i \in \{1,2,\ldots,m+1\}$. Therefore, $x \in \underline{\sum_{i=1}^{m+1} a_{i\beta}^{\wedge P}}(X)$. As a result, $\underline{\sum_{i=1}^{m+1} a_{i\beta}^{\wedge P}}(X) = \underline{\sum_{i=1}^{m} a_{i\beta}^{P}}(X)$ holds. □

On the basis of Theorem 7, we can update lower approximation of pessimistic VPMGRS with addition of a granular structure.

Example 4 (Continuation of Example 1)**.** *Assume the granular structure a_7 in Table 2 is added into Table 1, then we can compute $U/a_7 = \{\{x_1,x_2\},\{x_3,x_5,x_6\},\{x_4,x_7,x_8\}\}$. Based on Definition 4, we can achieve $c([x_1]_{a_7}, X) = c([x_2]_{a_7}, X) = 1/2$, $c([x_3]_{a_7}, X) = c([x_5]_{a_7}, X) = c([x_6]_{a_7}, X) = 0$, $c([x_4]_{a_7}, X) = c([x_7]_{a_7}, X) = c([x_8]_{a_7}, X) = 1$. Because $x_2 \in \underline{\sum_{i=1}^{m} a_{i\beta}^{P}}(X)$ and $c([x_2]_{a_7}, X) = 1/2 > \beta = 0.3$, according to Theorem 7, we have $\underline{\sum_{i=1}^{m+1} a_{i\beta}^{\wedge P}}(X) = \underline{\sum_{i=1}^{m} a_{i\beta}^{P}}(X) - \{x_2\} = \varnothing$.*

Theorem 8. *Given $S = (U,A)$, and $a_1,a_2,\ldots,a_m \in A$, $X \subseteq U$, $0 \leq \beta < 0.5$, while adding a_{m+1} into the information system, if $x \notin \overline{\sum_{i=1}^{m} a_{i\beta}^{P}}(X)$, we have*

(1) if $c([x]_{a_{m+1}}, X) < 1-\beta$, then $\overline{\sum_{i=1}^{m+1} a_{i\beta}^{\wedge P}}(X) = \overline{\sum_{i=1}^{m} a_{i\beta}^{P}}(X) \cup \{x\}$;

(2) otherwise $\overline{\sum_{i=1}^{m+1} a_{i\beta}^{\wedge P}}(X) = \overline{\sum_{i=1}^{m} a_{i\beta}^{P}}(X)$.

Proof.

(1) $x \notin \overline{\sum_{i=1}^{m} a_{i\beta}^{P}}(X)$, in light of Theorem 4, we have $c([x]_{a_i}, X) \geq 1-\beta$ for every $i \in \{1, 2, \ldots, m\}$, while adding a_{m+1}, if $c([x]_{a_{m+1}}, X) < 1-\beta$, then $c([x]_{a_i}, X) < 1-\beta$ holds for a given $i \in \{1, 2, \ldots, m+1\}$. So, $x \in \overline{\sum_{i=1}^{m+1} a_{i\beta}^{\wedge P}}(X)$. Hence, $\overline{\sum_{i=1}^{m+1} a_{i\beta}^{\wedge P}}(X) = \overline{\sum_{i=1}^{m} a_{i\beta}^{P}}(X) \cup \{x\}$.

(2) $x \notin \overline{\sum_{i=1}^{m} a_{i\beta}^{P}}(X)$, according to Theorem 4, it implied that $c([x]_{a_i}, X) \geq 1-\beta$ holds for every $i \in \{1, 2, \ldots, m\}$, while adding a_{m+1}, if $c([x]_{a_{m+1}}, X) \geq 1-\beta$ holds, then $c([x]_{a_i}, X) > 1-\beta$ for every $i \in \{1, 2, \ldots, m+1\}$. As a result, $x \notin \overline{\sum_{i=1}^{m+1} a_{i\beta}^{\wedge P}}(X)$. Therefore, we have $\overline{\sum_{i=1}^{m+1} a_{i\beta}^{\wedge P}}(X) = \overline{\sum_{i=1}^{m} a_{i\beta}^{P}}(X)$. □

Based on Theorem 8, we can dynamically update upper approximation of pessimistic VPMGRS with addition of a granular structure.

Example 5 (Continuation of Example 1)**.** *Assume a granular structure a_8 in Table 2 is added into Table 1, then we obtain $U/a_8 = \{\{x_1, x_2, x_3\}, \{x_4, x_6\}, \{x_5, x_7, x_8\}\}$. Based on Definition 4, we obtain $c([x_1]_{a_8}, X) = c([x_2]_{a_8}, X) = c([x_3]_{a_8}, X) = 1/3$, $c([x_4]_{a_8}, X) = c([x_6]_{a_8}, X) = 1/2$, $c([x_5]_{a_8}, X) = c([x_7]_{a_8}, X) = c([x_8]_{a_8}, X) = 2/3$. For $x_2 \in U$, $c([x_2]_{a_7}, X) = 1/3 > \beta = 0.3$, by Theorem 7, we obtain $\underline{\sum_{i=1}^{m+1} a_{i\beta}^{\wedge P}}(X) = \underline{\sum_{i=1}^{m} a_{i\beta}^{P}}(X) - \{x_2\} = \varnothing$. Due to $\forall x_i \in \overline{\sum_{i=1}^{m} a_{i\beta}^{P}}(X)$, there is no need to update $\overline{\sum_{i=1}^{m+1} a_{i\beta}^{\wedge P}}(X)$. According to Theorem 8, we obtain $\overline{\sum_{i=1}^{m+1} a_{i\beta}^{\wedge P}}(X) = \overline{\sum_{i=1}^{m} a_{i\beta}^{P}}(X) = \{x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8\}$.*

In light of proposed theorems, we present an incremental algorithm for efficiently obtaining approximations of VPMGRS when adding multiple granular structures.

We summarize the procedures of updating variable precision multigranulation approximations in Algorithm 2 (Algorithm IAUA). The complexity of initializing the approximations in Lines 2–3 takes $O(n)$, where n is the number of objects. The complexity of calculating equivalence classes and relative degree of misclassification in Lines 4–8 takes $O(m'n^2)$, where m is the number of granular structures. The complexity of updating lower approximation of optimistic VPMGRS by Theorem 5 in Lines 9–15 takes $O(m'n)$. The complexity of updating upper approximation of optimistic VPMGRS based on Theorem 6 in Lines 16–22 takes $O(m'n)$. The complexity of updating lower approximation of pessimistic VPMGRS based on Theorem 7 in Lines 23–29 takes $O(m'n)$. The complexity of updating upper approximation of pessimistic VPMGRS based on Theorem 8 in Lines 30–36 takes $O(m'n)$. The complexity of outputting updated approximations of VPMGRS in Line 37 takes $O(n)$. Thus, the whole complexity of algorithm IAUA takes $O(m'n^2)$, which is less than that of algorithm NACA when adding granular structures.

Algorithm 2: An incremental algorithm for updating approximations of VPM-GRS while adding multiple granular structures (IAUA).

Input: (1) $S = (U, A)$, the original granular structures $A = \{a_1, a_2, \ldots, a_m\}$, $X \subseteq U$. (2) Equivalence classes: $[x]_{a_i}, i = 1, 2, \ldots, m$ for each $x \in U$. (3) The added granular structures: $a_{m+1}, a_{m+2}, \ldots, a_{m+m'}$. (4) The parameter β. (5) $\underline{\sum_{i=1}^{m} a_{i\beta}^{O}}(X)$, $\overline{\sum_{i=1}^{m} a_{i\beta}^{O}}(X)$, $\underline{\sum_{i=1}^{m} a_{i\beta}^{P}}(X)$, $\overline{\sum_{i=1}^{m} a_{i\beta}^{P}}(X)$.

Output: The updated approximations of X in VPMGRS: $\underline{\sum_{i=1}^{m+m'} a_{i\beta}^{\wedge O}}(X)$, $\overline{\sum_{i=1}^{m+m'} a_{i\beta}^{\wedge O}}(X)$, $\underline{\sum_{i=1}^{m+m'} a_{i\beta}^{\wedge P}}(X)$, $\overline{\sum_{i=1}^{m+m'} a_{i\beta}^{\wedge P}}(X)$.

begin
  $\underline{\sum_{i=1}^{m+m'} a_{i\beta}^{\wedge O}}(X) \leftarrow \underline{\sum_{i=1}^{m} a_{i\beta}^{O}}(X)$, $\overline{\sum_{i=1}^{m+m'} a_{i\beta}^{\wedge O}}(X) \leftarrow \overline{\sum_{i=1}^{m} a_{i\beta}^{O}}(X)$,
  $\underline{\sum_{i=1}^{m+m'} a_{i\beta}^{\wedge P}}(X) \leftarrow \underline{\sum_{i=1}^{m} a_{i\beta}^{P}}(X)$, $\overline{\sum_{i=1}^{m+m'} a_{i\beta}^{\wedge P}}(X) \leftarrow \overline{\sum_{i=1}^{m} a_{i\beta}^{P}}(X)$.
  for each $x \in U$ **do** `// Computer the equivalence classes and the relative degree with respect to the added granular structures;`
    for $i = m+1$ **to** $m+m'$ **do**
      Compute $[x]_{a_i}$ and $c([x]_{a_i}, X)$;
    end
  end
  for each $x \notin \underline{\sum_{i=1}^{m} a_{i\beta}^{O}}(X)$ **do** `// According to Theorem 5;`
    for $i = m+1$ **to** $m+m'$ **do**
      if $c([x]_{a_i}, X) \leq \beta$ **then**
        $\underline{\sum_{i=1}^{m+m'} a_{i\beta}^{\wedge O}}(X) = \underline{\sum_{i=1}^{m+m'} a_{i\beta}^{\wedge O}}(X) \cup \{x\}$, **break**;
      end
    end
  end
  for each $x \in \overline{\sum_{i=1}^{m} a_{i\beta}^{O}}(X)$ **do** `// According to Theorem 6;`
    for $i = m+1$ **to** $m+m'$ **do**
      if $c([x]_{a_i}, X) \geq 1-\beta$ **then**
        $\overline{\sum_{i=1}^{m+m'} a_{i\beta}^{\wedge O}}(X) = \overline{\sum_{i=1}^{m+m'} a_{i\beta}^{\wedge O}}(X) - \{x\}$, **break**;
      end
    end
  end
  for each $x \in \underline{\sum_{i=1}^{m} a_{i\beta}^{P}}(X)$ **do** `// According to Theorem 7;`
    for $i = m+1$ **to** $m+m'$ **do**
      if $c([x]_{a_i}, X) > \beta$ **then**
        $\underline{\sum_{i=1}^{m+m'} a_{i\beta}^{\wedge P}}(X) = \underline{\sum_{i=1}^{m+m'} a_{i\beta}^{\wedge P}}(X) - \{x\}$, **break**;
      end
    end
  end
  for each $x \notin \overline{\sum_{i=1}^{m} a_{i\beta}^{P}}(X)$ **do** `// According to Theorem 8;`
    for $i = m+1$ **to** $m+m'$ **do**
      if $c([x]_{a_i}, X) < 1-\beta$ **then**
        $\overline{\sum_{i=1}^{m+m'} a_{i\beta}^{\wedge P}}(X) = \overline{\sum_{i=1}^{m+m'} a_{i\beta}^{\wedge P}}(X) \cup \{x\}$, **break**;
      end
    end
  end
  return $\underline{\sum_{i=1}^{m+m'} a_{i\beta}^{\wedge O}}(X)$, $\overline{\sum_{i=1}^{m+m'} a_{i\beta}^{\wedge O}}(X)$, $\underline{\sum_{i=1}^{m+m'} a_{i\beta}^{\wedge P}}(X)$, $\overline{\sum_{i=1}^{m+m'} a_{i\beta}^{\wedge P}}(X)$.
end

4. Experimental Analysis

This section aims to evaluate the performance of incremental algorithm IAUA by making use of extensive experiments. Experimental data sets are downloaded from UCI (http://archive.ics.uci.edu/ml/datasets.html, accessed on 15 August 2022). The description of all data sets is listed in Table 3. In addition, experiments are run on a PC with Win 10, Intel i5-1135G7 CPU @ 2.4 GHz, 16 GB RAM, and the algorithms are performed through Java.

Table 3. The description of data sets.

ID	Data Sets	# Objects	# Attributes	# Classes
1	Sonar	208	60	2
2	SPECTF	267	44	2
3	Ionosphere	351	34	2
4	Libras	360	90	15
5	Dermatology	366	33	6
6	Wdbc	569	30	2
7	Diabetic	1151	19	2
8	Segmentation	2310	19	7

In the subsequent sections, we try to evaluate the algorithm IAUA and the algorithm NACA from the following three aspects. The first one is to compare the efficiency between IAUA and NACA with different sizes of data sets. The second one is to verify the efficiency between IAUA and NACA under different updating ratios of adding granular structures. Furthermore, the last one is to evaluate the influence of the parameter β on the algorithms IAUA and NACA with respect to computational time. Moreover, for the first two experiments, when calculating variable precision multigranulation approximations, the parameter β is fixed as 0.3. Additionally, in our experiments, each attribute in the data sets is regarded as one single granular structure.

4.1. Comparison between IAUA and NACA with Different Sizes of Universe

In this subsection, we test algorithms IAUA and NACA under the circumstance of different sizes of universe. To show efficiency of IAUA and NACA more intuitively, the data sets are initially spit into ten equal parts. The first basic data is viewed as the first part, the second basic data is combined with first one and the second part, and so on, the tenth basic data is combined with all of ten parts. Additionally, in regard to the partition of granular structures, the first 50% are treated as original granular structures while the rest are treated as adding granular structures.

The computational times consumed by algorithms IAUA and NACA for acquiring approximations of VPMGRS are depicted in Figure 1, where x-axis reveals different size of universe, and y-axis indicates the time consumed for computing approximations. According to Figure 1, it is obvious that the NACA trend increases dramatically. This indicates that the algorithm NACA spends increasingly more computational time than that of IAUA. Computational time of the algorithm IAUA remains below that of the algorithm NACA, because the algorithm NACA performs the calculations of approximations in VPMGRS on the whole data sets without using the prior knowledge. On the contrary, the algorithm IAUA updates the approximations by making use of the prior valuable knowledge and thus reduces the computational time.

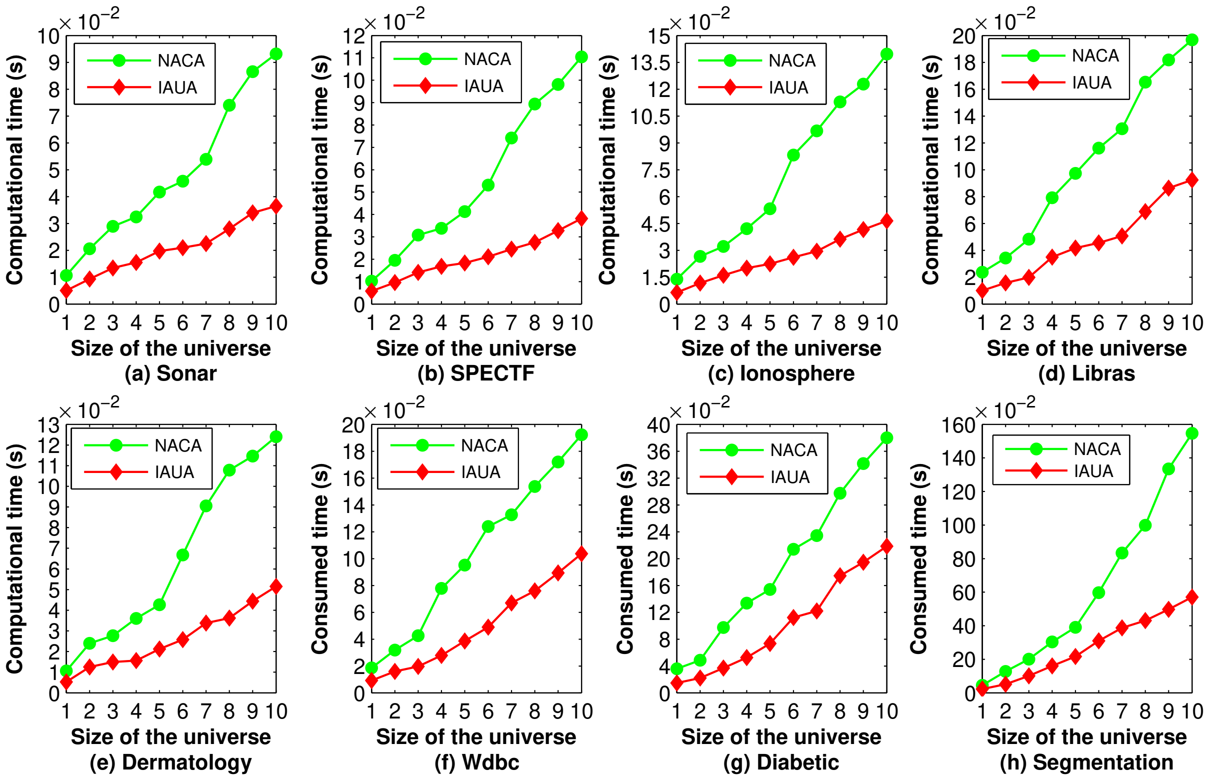

Figure 1. A comparison of NACA and IAUA with different sizes of universe.

4.2. Comparison between IAUA and NACA with Different Updating Ratios

In this subsection, we verify the incremental algorithm IAUA under different updating ratios of granular structures, and the number of objects is fixed. We select 30% of the whole granular structures as the original part and the rest part is regarded as the candidate part. Furthermore, updating radio is the ratio with reference to the number of adding granular structures and that of candidate granular structures. Then, we select 10% to 100% of granular structures with the increment step of 10% as adding granular structures.

Figure 2 depicts computational time regarding algorithms IAUA as well as NACA to update approximations, where x-axis pertains to different updating ratios, while y-axis denotes time consumed for updating approximations. As observed from Figure 2, the algorithm IAUA reduces the computational time of updating approximations from each data set in comparison with the algorithm NACA. The main reason is contributed to that NACA has no updating mechanisms for making fully use of the prior useful information. When adding granular structures, the algorithm NACA requires to be carried out from beginning to obtain approximations of VPMGRS. Therefore, the incremental algorithm IAUA is more efficient for dynamical maintenance of approximations of VPMGRS when adding granular structures.

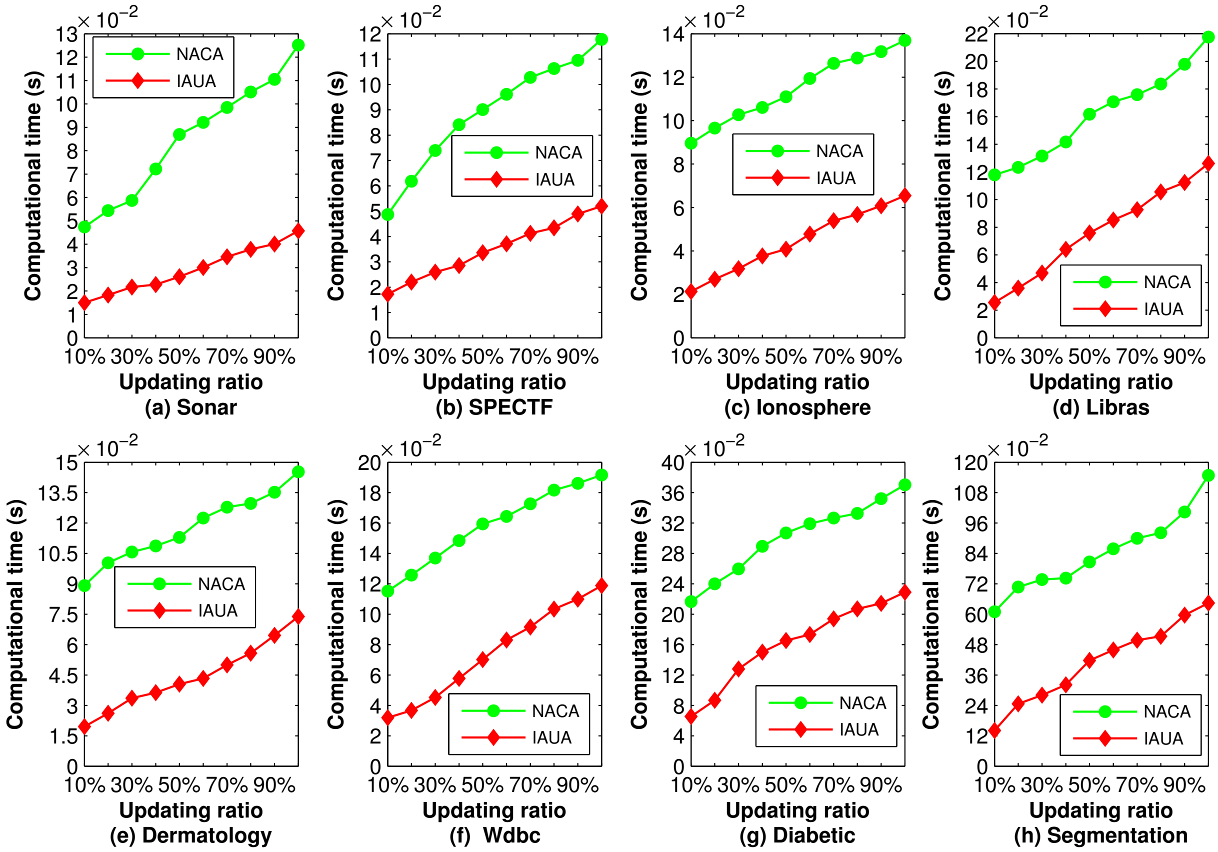

Figure 2. A comparison of NACA and IAUA with different updating ratios.

4.3. Comparison between IAUA and NACA with Changing Values of the Parameter β

In this subsection, we aim to elaborate on the influence of changing values of the parameter β on computational efficiency of the algorithms IAUA and NACA. To demonstrate the time consumed with changing values of β, the value of β is considered to change from 0.1 to 0.45 with incremental step of 0.05. For the algorithm NACA, we perform experiments on the whole universe of each data set from scratch. For the algorithm IAUA, the whole granular structures are divided into two parts, namely the original granular structures and adding granular structures. The original part consists of 40% of the whole granular structures while the adding part includes the rest of the granular structures. Furthermore, the size of test data set is fixed by the whole universe when the algorithm IAUA is carried out.

Figure 3 shows the times consumed by the algorithms IAUA and NACA on all data sets with changing values of β. In two sub-figures of Figure 3, the x-axis refers to changing values of β while the y-axis expresses computational time for all data sets. In light of Figure 3, it clearly indicates that the time consumed by IAUA is consistently lower than that of NACA. According to the algorithms IAUA and NACA, the computational times are stable in terms of different values of β. Therefore, the time consumed by the algorithms IAUA and NACA for updating approximations in VPMGRS fluctuates a little with the variation of β .

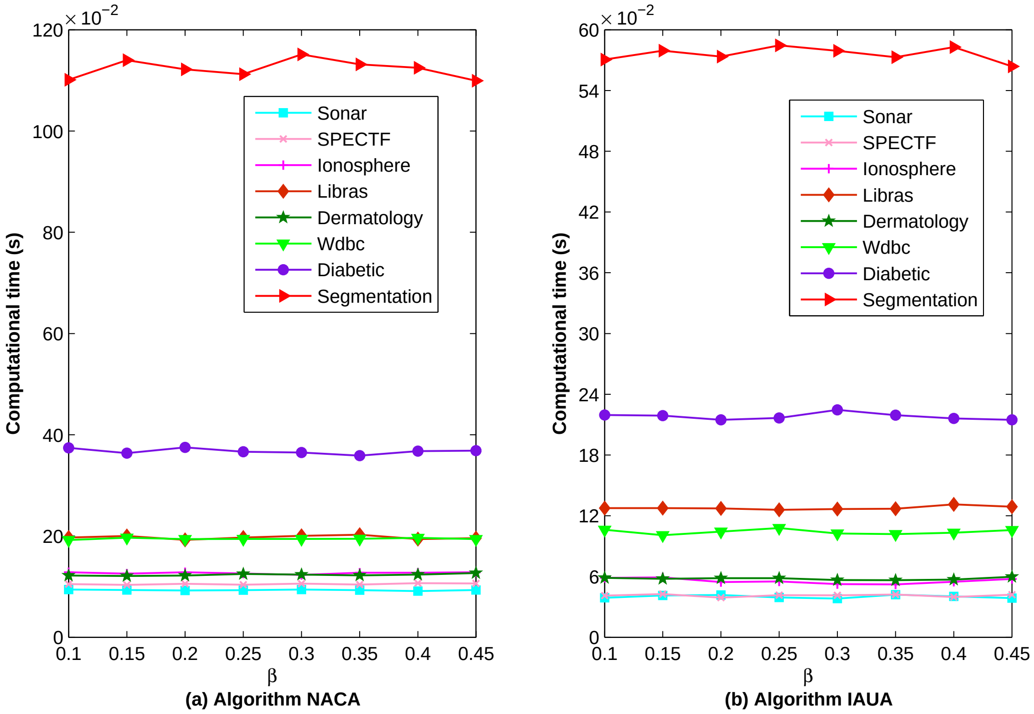

Figure 3. A comparison of NACA and IAUA with changing values of the parameter β.

5. Conclusions

In real applications, the granular structures in multigranulation environments evolve over time. Therefore, the incremental technique by making use of prior knowledge can efficiently maintain valuable knowledge in changing data context. Therefore, this study exploited an efficient algorithm for maintenance of approximations in VPMGRS after adding multiple granular structures. We developed dynamical mechanisms for obtaining approximations in the presence of adding granular structures. At the same time, the incremental algorithm was investigated with the purpose of enhancing the efficiency. Experimental results on public available data sets validated the feasibility for updating approximations in VPMGRS. In an information system, the values of attributes may change. Accordingly, accelerating strategies for maintaining valuable knowledge will be developed by taking the generalization of values of attributes into account so as to decrease the computational cost. Meanwhile, it is possible to use the proposed algorithm for addressing some real-world uncertainty reasoning problems.

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