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A Worst Case Performance Analysis of Approximated Zero Forcing Vectoring for DSL Systems

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Abstract: In this paper we introduce a Gaussian approximation for the achievable downstream bit rate per user in modern broadband and ultra-broadband digital subscriber loop-based access systems. The considered formulation allows one to account for the main characteristics of the interference scenario (e.g., number and positions of interferers along the cable), the far-end crosstalk (FEXT) fluctuations, bit loading limitation per sub-carrier, and approximated zero forcing in vectoring pre-coding. Formulas are obtained assuming log-normal statistics for the signal-to-interference plus noise ratio per sub-carrier. The validity of the considered approximation has been assessed by computer calculation. A very good agreement between the exact and the approximated bit rates is obtained. The bit rate approximation is then used to analyze the performance of a very high-speed digital subscriber line type 2 (VDSL2, Profile 35b) with vectoring and used to assess G.fast performance degradation when approximated zero forcing in vectoring pre-coding is applied. It is observed for G.fast performance that degradation due to residual FEXT after vectoring pre-coding can be relevant. A significant performance improvement can be achieved at the expense of increased computational complexity of the vectoring pre-coding.

Keywords: subscriber loop; crosstalk; VDSL; G.fast; vectoring

1. Introduction

Modern digital subscriber loop (DSL) technologies, such as the very high-speed digital subscriber line type 2 (VDSL2, ITU-T G.993.2) in the fiber-to-the-cabinet (FttC) architecture and G.fast (ITU-T G9701) in the fiber-to-the-distribution point (FttDp) architecture, are adopted by telecommunication operators to provide ultra-broadband services to their subscribers. For preliminary performance analysis of FttC/FttDp access networks in terms of the achievable bit rate per user, simulation tools are adopted. However, due to the large number of variables, parameters, and constraints, simulation-based approaches for preliminary planning can be time consuming due to the very large number of possibilities to be explored. To avoid this inconvenience, analytical bit rate models or semi-analytical approaches can be helpful to reduce calculation time and to speed up preliminary planning. The main goal of this paper is to provide an analytical framework allowing one to rapidly assess bit rate performance for the considered ultra-broadband DSL technologies. The proposed formulation allows one to easily account for the access network geometry, for far-end crosstalk (FEXT) random dispersion, and for vectoring pre-coding, by incorporating models and techniques widely accepted in the current literature and in practice. We consider the co-located users scenario which allows one to assess achievable performance in the worst case interference situation for downstream (DS) transmissions. This scenario is simple to reproduce in the laboratory and it is commonly considered by telecommunication operators to be the base of their commercial offers.

Concerning FEXT characterization, several models (mostly based on experimental data) have been proposed by International Telecommunication Union (ITU) [1] and other organizations/institutions [2].

In current practice, the overall FEXT model is obtained by multiplying the 1% worst-case model including FEXT dispersion [3] by the selectivity term at sub-carrier frequency f_k [4,5]. FEXT is the main limiting factor of performance for VDSL as well as for G.fast. Vectoring techniques, originally proposed in Reference [6], are commonly adopted to mitigate or cancel FEXT and to improve performance on DS transmissions by means of pre-coding and by interference cancellation at the digital subscriber line access multiplexer (DSLAM) on upstream (US) signals.

Concerning DS transmissions, several pre-coding algorithms have been proposed over the years. Zero forcing (ZF) algorithms based on channel matrix inversion are presented in References [7,8] and their performance limits are investigated in References [9,10]. In Reference [11], the authors propose an alternative pre-coding technique based on the concept of least favourable noise. However, this algorithm is not ideally suited for practical implementation because of its computational complexity and large feedback overhead. A low complexity implementation of the generalized decision feedback (GDFE) technique of Reference [11] is presented in Reference [12]. All the proposed techniques require the calculation of the channel matrix inverse as well as the evaluation of QR decomposition (QRD) and singular value decomposition (SVD). Due to high computational complexity of these operations, their implementation in modern VDSL2 and G.fast systems could require high costs and power hungry vectoring processors in the DSLAM to implement the pre-coding algorithms.

The approximated zero forcing pre-coding technique (AZF) in Reference [7] is based on successive approximations of the channel matrix inverse. The AZF approach has been shown (Reference [13], Appendix III) as a viable, low cost, and effective practical implementation for vectoring pre-coding. The AZF doesn't allow complete FEXT cancelation from the received observables i.e., a residual FEXT term still limits performance. However, the AZF algorithm does not require the calculation of the channel matrix inverse, thus saving significant computational power especially when a large number of copper lines (e.g., up to 200) are processed. The AZF performance can be improved by successively refining the pre-coding matrix at the expense of an increased computational complexity. The effectiveness of the AZF technique has been analyzed in References [7,10] in terms of upper and lower bounds of the signal-to-interference plus noise per sub-carrier ratio and in terms of the achievable bit rate per user. However, in these analyses no random fluctuations of FEXT statistics have been considered and results are restricted to simple scenarios (e.g., from 4–28 VDSL lines in the 0–17 MHz band [10]).

Results on AZF performance are also presented in the current literature. In Reference [14], performance results are obtained using a simulation approach without providing any theoretical results on the statistics of the residual FEXT. Authors in Reference [15] present a spectrum optimization algorithm for both zero-forcing and minimum mean squared error (MMSE) pre-coding methods. Performance of both techniques are compared for G.fast showing MMSE outperforms AZF. Finally, in Reference [9], the authors provide a lower bound on the achievable DS bit rate which is based on a calculation of the achievable bit rate for the worst-case user. This differs from our paper in that no discussion or stochastic characterization of the residual FEXT is provided.

The stochastic framework presented in this paper permits performance analysis of the AZF pre-coding techniques in the co-located users case. In detail, the main contributions of this paper are summarized in the following points:

- The stochastic bit rate formulation in Reference [16] is extended to include vectoring. It is shown that a log-normal approximation of the signal-to-interference plus noise ratio (SINR) per sub-carrier is still valid in the case of vectoring, allowing one to derive the corresponding Gaussian characterization of the DS bit rate. For typical implementation of AZF pre-coding, the mean and the standard deviation of the bit rate are expressed in an analytical closed form.

The Gaussian bit rate approximation can also be used to obtain the user bit rate even in the case of a bit-loading limitation per sub-carrier. The validity of the Gaussian bit rate approximation including vectoring is assessed by computer calculation. Very good agreement with the bit rate values obtained from the exact calculation are achieved for the considered access network topology and FEXT statistics.

- We provide the analytical closed form expressions for the statistics, e.g., the mean and the quadratic mean of the residual FEXT for the AZF implementation suggested in Reference [13]. These expressions explicitly account for the position of users in the access network and for statistics of the random variables modeling FEXT fluctuations.
- We provide analytical closed form expressions for the mean and the quadratic mean of the residual FEXT in the case of co-located interferers for first and second order AZF pre-coding. These results are used to assess the effectiveness of second order pre-coding for improving bit rate performance in VDSL2 and G.fast. In particular, it is shown that second order pre-coding allows one to obtain a bit rate loss (evaluated with respect to the ideal case) of few percent for VDSL2 and of about 10% for G.fast.

In Reference [17], the authors have introduced a simplified stochastic framework for assessing the VDSL2 DS bit rate in the presence of vectoring. However, their results do not refer to any specific implementation of the DS pre-coding vectoring algorithm and vectoring effects are accounted for by a constant multiplicative factor (less than one) in front of the FEXT power [17]. Instead, in our work the residual FEXT term obtained after vectoring is characterized in terms of a log-normal approximation and its corresponding moments.

The paper is organized as follows. In Section 2 we describe the typical access network architecture for VDSL2/G.fast and we define the characteristics of the considered scenario. The AZF pre-coding is detailed in Section 3. The general characterization of FEXT and of residual FEXT is presented in Section 4. In Section 5 we introduce log-normal approximations for FEXT and residual FEXT in the first and second order AZF and we provide closed form expressions for their mean and quadratic mean, which are valid in the co-located users case. Starting from the log-normal (residual) FEXT approximation, the Gaussian bit rate approximation is derived in Section 6. The validity of the considered Gaussian bit rate model in the case of vectoring is assessed by computer calculation in Section 7. The proposed formulation is then applied to assess bit rate performance of VDSL2 and G.fast in Section 8. Finally, conclusions are drawn.

2. Access Scenario

The typical geometry of an FttC (or FttDp) access network is depicted in Figure 1.

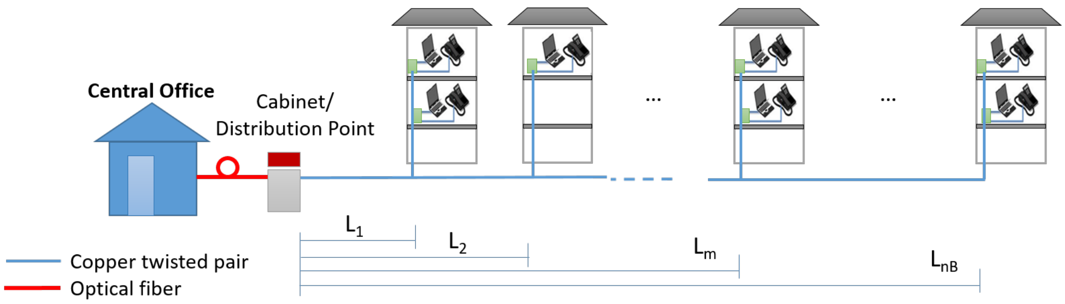

Figure 1. Considered FttC/FttDp generic interference scenario, where $\{L_i\}$ are the distances of the buildings from the Cabinet.

Interferers in the buildings are connected to the cabinet or to the distribution point (Dp) by pairs using the same copper cable. The copper pairs assigned to users can lie in the same or in other binders in the same cable. The typical number of copper pairs in each binder varies from 25 up to 50; the cable from cabinet or from the Dp typically contains four or eight binders and the overall number of pairs in the cable typically varies from 100 to 200, and up to 400. The m-th building is at a distance L_m from the cabinet or Dp. In general, users in the same building can be randomly located over the floors. In the following we assume users are co-located i.e., they are at the same distance from the cabinet/Dp. This refers to the worst case crosstalk condition since the coupling lengths among users take their maximum values equal to the distance from the cabinet. This scenario is very important for two reasons. It is simple to emulate for laboratory tests and bit rate performance obtained in this case provides a lower bound on the achievable bit rate in realistic conditions where users are distributed among the buildings. Bit rate performance in the co-located case are commonly used by telecommunication operators to define their commercial offers.

3. Vectoring Pre-Coding

The vector of observables $\mathbf{r}_k$ at the output of the N receivers at sub-carrier frequency f_k is:

$$\mathbf{r}_k = \mathbf{H}_k \mathbf{b}_k + \mathbf{n}_k, \tag{1}$$

where $\mathbf{b}_k$ is the $N \times 1$ vector of symbols transmitted by the N users on the sub-carrier frequency f_k and $\mathbf{n}_k$ is the $N \times 1$ vector accounting for background noise at each receiver. Symbols $b_{i,k}$, $i = 1, \ldots, N$ are assumed to be zero mean and identically distributed with the same power P_k. The $N \times N$ matrix $\mathbf{H}_k$ is the channel matrix at sub-carrier frequency f_k. The main diagonal terms of $\mathbf{H}_k$ account for direct propagation, i.e., $h_{ii}^{(k)} = H_{D,k}(d_i)$, where d_i is the distance of the reference user from the Cabinet.

The off-diagonal terms account for FEXT, i.e., $h_{ij}^{(k)} = H_{F,k}(d_i, l_{ij})$ for $i \neq j$ with:

$$H_{F,k}(d_i, l_{ij}) = \sqrt{\chi_F l_{ij}} f_k 10^{X_{ij}/20} 10^{\iota \phi_{ij}} g_{ij}(f_k) H_{D,k}(d_i), \tag{2}$$

where $g_{ij}(f_k)$ is the frequency selectivity term of the FEXT transfer function accounting for random fluctuations with respect to the average FEXT level, i.e., $\sqrt{\chi_F l_{ij}} f_k 10^{X_{ij}/20}$ at frequency f_k. We assume $g_{ij}(f_k)$ are identically distributed and statistically independent (i.i.d.) over i and j and for each k; they have zero mean and unit variance. Additionally, ϕ_{ij} is a random phase term independent of k and uniformly distributed in $[0, 2\pi)$; l_{ij} is the coupling length between the i-th and j-th active users; and χ_F is the FEXT coupling coefficient. For a given reference distance d_i, the coupling lengths l_{ij} can be easily obtained from the access network geometry such as that depicted in Figure 1. In the co-located case, we assume $l_{ij} = d_i$ for each $j = 1, 2, \ldots, N$ with $j \neq i$.

X_{ij} (in dB) are random variables accounting for FEXT fluctuation with respect to the 1% FEXT condition [3]. Here, X_{ij} are assumed to be Gaussian (in dB), with mean μ_{dB} and standard deviation σ_{dB}, and are assumed to be independent of the distance from the cabinet/Dp and of the sub-carrier frequency. Furthermore, we consider the $\{X_{ij}\}$ do not vary with frequency and are i.i.d. A discussion on the validity of the log-normal assumption even when X_{ij} are considered to be Beta distributed [2] has been presented in Reference [18]. In the following we consider random variables $e^{y_{ij}}$ in place of $10^{X_{ij}/20}$ i.e., :

$$y_{ij} = X_{ij} \ln(10)/20 = \mu + \sigma \nu_{ij}, \tag{3}$$

with $\mu = -\mu_{dB} \ln(10)/20$, $\sigma = \sigma_{dB} \ln(10)/20$, and ν_{ij} is a zero mean Gaussian random variable with unit variance. From now on, the index i will refer to a (generic) reference user.

The channel matrix $\mathbf{H}_k$ in Equation (1) can be conveniently re-written as:

$$\mathbf{H}_k = \mathbf{D}_k \left(\mathbf{I} + \mathbf{C}_k \right), \tag{4}$$

where $\mathbf{I}$ is the identity matrix, $\mathbf{D}_k$ is a diagonal matrix containing the direct propagation terms; the matrix $\mathbf{C}_k$ has zeros along the main diagonal and contains the FEXT terms normalized by row for the corresponding direct propagation term i.e., $c_{ij}^{(k)} = H_{F,k}^{(i,j)}(d_i, l_{ij})/H_{D,k}(d_i)$ with $l_{ij} = d_i$ for all $j \neq i$. The pre-coding matrix to be applied to the DS symbols approximates the inverse of $\mathbf{I} + \mathbf{C}_k$. For the considered frequency intervals, we can assume $|c_{ij}^{(k)}| \ll 1$ (i.e., diagonal dominance assumption) in $\mathbf{C}_k$. Thus, a good approximation of the inverse of $\mathbf{I} + \mathbf{C}_k$ is:

$$\mathbf{L}_k^{(p)} = \mathbf{I} + \sum_{m=1}^{p} (-1)^m \mathbf{C}_k^m, \tag{5}$$

and $p \geq 1$ is referred as the order of the pre-coding matrix. Let $\mathbf{L}_k^{(p)}\mathbf{b}_k$ be the vector of pre-coded symbols, assuming an exact estimate of $\mathbf{C}_k$, the vector of the received observables on the k-th sub-carrier is:

$$\mathbf{r}_k^{(p)} = \mathbf{D}_k \left(\mathbf{I} - \mathbf{C}_k^{p+1} \right) \mathbf{b}_k + \mathbf{n}_k. \tag{6}$$

When $p = 0$ no vectoring pre-coding is applied and $\mathbf{L}_k^{(0)} = \mathbf{I}$.

The $\mathbf{C}_k^{p+1}$ term in Equation (6) is responsible for residual FEXT in the received observables. In the following Sections we derive a stochastic characterization of residual FEXT power for $p = 1$, while for $p = 2$ we derive the first two moments of the residual FEXT power in the co-located users case i.e., all receivers are placed at the same distance d from the cabinet. For given FEXT fluctuations and allocated transmitted power per sub-carrier, co-location of interferers corresponds to the worst case interference scenario for the reference user at distance d from the cabinet, since all coupling lengths assume their maximum values. This assumption allows one to obtain a lower bound on the achievable performance in the case of AZF vectoring pre-coding.

4. FEXT Characterization and SINR Calculation

Let $c_{ij}^{(k,p)}(d_i)$ be the (i,j) element of $\mathbf{C}_k^{p+1}$. The residual FEXT term in the i-th received observable $r_{i,k}$ is:

$$x_{F,i,k}^{(p)}(d_i) = -H_{D,k}(d_i) \sum_{j=1}^{N} c_{ij}^{(k,p)} b_{j,k} (1 - \delta_{ij}), \tag{7}$$

and δ_{ij} is the Kronecker symbol. The residual FEXT power is obtained from Equation (7) by first averaging the square modulus of Equation (7) with respect to transmitted symbols $\{b_{i,k}\}$ for $i = 1, \ldots, N$, and then averaging with respect to the FEXT frequency selective $g_{ij}(f_k)$ and phase terms ϕ_{ij} in Equation (2), thus obtaining:

$$I_{F,i,k}^{(p)}(d_i) = |H_{D,k}(d_i)|^2 \sum_{j=1}^{N} E\left\{ \left| c_{ij}^{(k,p)} \right|^2 \right\} P_k \cdot (1 - \delta_{ij}). \tag{8}$$

The expectation $E\{\cdot\}$ in Equation (8) is evaluated with respect to the selective FEXT and phase terms.

As shown in the following, performance can be expressed in terms of the $SINR_{k,p}^{(i)}(d_i)$ which is the SINR on the k-th sub-carrier at frequency f_k for the (generic) i-th reference user at distance d_i from the cabinet/Dp. When considering AZF vectoring pre-coding of order p, the $SINR_{k,p}^{(i)}(d_i)$ can be written as:

$$SINR_{k,p}^{(i)}(d_i) = \frac{|H_{D,k}(d_i)|^2 E\left\{ \left| 1 - c_{ii}^{(k,p)} \right|^2 \right\} P_k}{\eta_k + I_{F,i,k}^{(p)}(d_i)}, \tag{9}$$

where η_k is the background noise power; and P_k is the power transmitted on the k-th sub-carrier. In the co-located scenario, we assume $P_k = P$ are the same for DS users transmitting from the same cabinet. This avoids the harmful FEXT of high power users on low power ones in the non-vectoring case. In the case of vectoring, the power of transmitted symbols can be increased/decreased with respect to P in accordance with the resulting pre-coding matrix. Under the diagonal dominance assumption, which is valid for the considered frequency range (i.e., up to 100 MHz), we can assume $|c_{ii}^{(k,p)}| \ll 1$ and hence it can be neglected in Equation (9) for VDSL2.

For a given distance of users from the cabinet/Dp and a given FEXT situation described by the random variables y_{ij} in Equation (3), it is not difficult to observe that the residual FEXT in Equation (8) is given by the sum of (correlated) log-normal random variables for any p. Then, following the approach of Reference [16], the $SINR_{k,p}^{(i)}(d_i)$ in Equation (9), for the non-vectoring case (i.e., $p = 0$) as well as for vectoring pre-coding of order p, can be approximated as:

$$SINR_{k,p}^{(i)}(d_i) \approx \frac{D_k(d_i)}{1 + v_D D_k(d_i) \chi_F^{p+1} f_k^{2(p+1)} N_{p,i} d_i^{p+1} e^{\tilde{y}_{p,i}}}, \tag{10}$$

where $D_k(d) = |H_{D,k}(d)|^2 P/\eta_k$ is the signal-to-background noise ratio in the no FEXT case. From Equation (10) we have implicitly assumed that the FEXT in the non-vectoring case (i.e., $p = 0$) or the residual FEXT power after pre-coding in Equation (8) can be approximated as:

$$I_{F,i,k}^{(p)}(d_i) \approx v_D \chi_F^{p+1} f_k^{2(p+1)} d^{p+1} N_{p,i} e^{\tilde{y}_{p,i}} |H_{D,k}(d_i)|^2 P, \tag{11}$$

where $N_{p,i}$ is a coefficient, $e^{\tilde{y}_{p,i}}$ is the log-normal sum of random variables approximating the FEXT/residual FEXT term, and $\tilde{y}_{p,i}$ is a Gaussian variable with mean $\tilde{\mu}_{p,i}$ and standard deviation $\tilde{\sigma}_{p,i}$. Finally, v_D is the degradation due to practical implementation of the vectoring pre-coding algorithm and it may account for losses due to an imperfect channel estimate.

In the non-vectoring case i.e., $p = 0$, the results of Reference [18] are re-obtained. To render the paper self-contained, some of the results of this reference have been repeated in the Appendix.

5. First and Second Order Statistics of FEXT and of Residual FEXT

In this Section we provide the closed form expression of the first and second order moments of the FEXT in the non-vectoring case, $p = 0$, and the residual FEXT in the co-located case for $p = 1, 2$. Results are reported in the following sub-sections.

5.1. Non-Vectoring Case, $p = 0$

Considering DS transmission, in the non-vectoring case and assuming co-located terminals, following the approach in Reference [16] the FEXT power (after averaging out selective FEXT) is:

$$I_{F,i,k}^{(0)}(d_i) = v_D \chi_F f_k^2 d_i |H_{D,k}(d_i)|^2 P \sum_{j=1}^{N} e^{2y_{ij}}, \tag{12}$$

It can be observed that $I_{F,i,k}^{(0)}(d_i)$ is the sum of log-normal random variables which can be approximated as log-normal itself [18]. The mean and the quadratic mean of Equation (12) with respect to the log-normal terms are:

$$E\{I_{F,i,k}^{(0)}(d_i)\} = v_D \chi_F f_k^2 d_i |H_{D,k}(d_i)|^2 P e^{2\mu + 2\sigma^2} (N-1), \tag{13}$$

$$E\{I_{F,i,k}^{(0)}(d_i)^2\} = v_D^2 \chi_F^2 f_k^4 d_i^2 |H_{D,k}(d_i)|^4 P^2 e^{4\mu + 4\sigma^2} (N-1) \left(e^{4\sigma^2} + (N-2) \right). \tag{14}$$

This can be used to evaluate $N_{0,i}$ and the statistics (mean and standard deviation) of the random variable $\tilde{y}_{0,i}$ of Equation (11) in accordance with the Wilkinson method [19] as follows. The coefficient $N_{0,i}$ can be obtained by assuming equality between the mean in Equation (11) (with $p = 0$) and Equation (13), thus obtaining:

$$N_{0,i} E\left\{e^{\tilde{y}_{0,i}}\right\} = (N-1) e^{2\mu + 2\sigma^2}. \tag{15}$$

Here, $E\{e^{\tilde{y}_{0,i}}\} = e^{\tilde{\mu}_{0,i}+\tilde{\sigma}^2_{0,i}/2}$ is the mean of the log-normal random variable $e^{\tilde{y}_{0,i}}$; and $\tilde{\mu}_{0,i}$, and $\tilde{\sigma}_{0,i}$ are the mean and the standard deviation of $\tilde{y}_{0,i}$, respectively, which should be evaluated. From Equation (15) we assume $E\{e^{\tilde{y}_{0,i}}\} = e^{2\mu+2\sigma^2}$ so that:

$$N_{0,i} = N - 1. \tag{16}$$

The second equation required to calculate $\tilde{\mu}_{0,i}$, $\tilde{\sigma}_{0,i}$ can be obtained by equating the quadratic mean of Equation (11) (with $p = 0$) and the quadratic mean in Equation (14) i.e.,:

$$N^2_{0,i} E\left\{e^{2\tilde{y}_{0,i}}\right\} = e^{4\mu+4\sigma^2}(N-1)\left(e^{4\sigma^2} + (N-2)\right), \tag{17}$$

and hence:

$$e^{2\tilde{\mu}_{0,i}+2\tilde{\sigma}^2_{0,i}} = \frac{e^{4\mu+4\sigma^2}}{N-1}\left(e^{4\sigma^2} + (N-2)\right). \tag{18}$$

The two Equations (15) and (18) can be used to evaluate the moments of $\tilde{y}_{0,i}$ as:

$$\tilde{\mu}_{0,i} = 2\ln A - \frac{1}{2}\ln B \quad , \quad \tilde{\sigma}_{0,i} = \ln B - 2\ln A, \tag{19}$$

where, A and B are:

$$A = e^{2\mu+2\sigma^2} \quad , \quad B = \frac{e^{4\mu+4\sigma^2}}{N-1}\left(e^{4\sigma^2} + (N-2)\right). \tag{20}$$

5.2. Vectoring Pre-Coding: $p = 1$

The mean and the quadratic mean of the residual FEXT in the $p = 1$ case can be obtained from Equation (8) and hence we have:

$$E\{I^{(2)}_{F,i,k}(d_i)\} = v_D f^4_k \chi^2_F d^2_i |H_{D,k}(d_i)|^2 P \cdot e^{4(\eta+\sigma^2)}(N-1)(N-2), \tag{21}$$

and:

$$E\{I^{(2)}_{F,i,k}(d_i)^2\} = v^2_D f^8_k \chi^4_F d^4_i |H_{D,k}(d_i)|^4 P^2 \cdot e^{8(\eta+\sigma^2)}(N-1)(N-2)\cdot\left(e^{8\sigma^2} + (N-3)e^{4\sigma^2} + (N-2)^2\right). \tag{22}$$

Equations (21) and (22) have been obtained using a symbolic mathematical tool. We have resorted to manual calculations of Equations (21) and (22) for small values of N so to further verify their correctness.

In order to completely characterize the statistics of the approximated expression of the residual FEXT in Equation (11) with $p = 1$, we need to derive $N_{1,i}$ and $\tilde{\mu}_{1,i}$, $\tilde{\sigma}_{1,i}$ which are the moments of the random variable $\tilde{y}_{1,i}$ in Equation (11). To this aim we follow the same calculation approach presented in Section 5.1 for the non-vectoring case, thus obtaining:

$$N_1 = (N-1)(N-2), \tag{23}$$

and $\tilde{\mu}_{1,i}$, $\tilde{\sigma}_{1,i}$ are obtained using Equation (19) with A and B as:

$$A = e^{4\mu+4\sigma^2} \quad , \quad B = \frac{e^{8(\eta+\sigma^2)}}{(N-1)(N-2)}\left(e^{8\sigma^2} + (N-3)e^{4\sigma^2} + (N-2)^2\right). \tag{24}$$

5.3. Vectoring Pre-Coding: $p = 2$

In the case of vectoring pre-coding with $p = 2$, the mean and the quadratic mean of the residual FEXT are:

$$E\{I^{(3)}_{F,i,k}(d_i)\} = v_D f^6_k \chi^3_F d^3_i |H_{D,k}(d_i)|^2 P e^{6(\eta+\sigma^2)}(N-1)\cdot\left(6e^{4\sigma^2} + (N-2)(N+1)\right), \tag{25}$$

$$E\{I_{F,i,k}^{(3)}(d_i)^2\} = v_D^2 f_k^{12}\chi_F^6 d_i^6 |H_{D,k}(d_i)|^4 P^2 e^{12(\eta+\sigma^2)}(N-1)\cdot \\ \cdot\left(a_1 e^{28\sigma^2} + a_2 e^{16\sigma^2} + a_3 e^{12\sigma^2} + a_4 e^{8\sigma^2} + a_5 e^{4\sigma^2} + a_6\right), \tag{26}$$

where

$$\begin{aligned} a_1 &= 36 \\ a_2 &= 24(N-2) \\ a_3 &= (13(N-3)+56)(N-2) \\ a_4 &= (N-2)(44+(N-3)(N+4)) \\ a_5 &= (N-2)(40+(N-3)(139+(N-4)(N+25))) \\ a_6 &= (N-2)(N-3)(42+(N-4)(63+(N-5)(N+10))). \end{aligned} \tag{27}$$

As in Section 5.1, to completely characterize the statistics of the approximated expression of the residual FEXT in Equation (11) with $p = 2$, we need to derive $N_{2,i}$, $\tilde{\mu}_{2,i}$, and $\tilde{\sigma}_{2,i}$, where the latter two are the moments of the random variable $\tilde{y}_{2,i}$ appearing in Equation (11). Again following the same calculation approach we obtain:

$$N_{2,i} = (N-1)\cdot\left(6e^{4\sigma^2} + (N-2)(N+1)\right), \tag{28}$$

and $\tilde{\mu}_{2,i}$, and $\tilde{\sigma}_{2,i}$ are obtained using Equation (19) with A and B as:

$$A = e^{6\mu+6\sigma^2} \quad , \quad B = \frac{e^{12(\eta+\sigma^2)}(N-1)}{N_{2,i}^2}\cdot\left(a_1 e^{28\sigma^2} + a_2 e^{16\sigma^2} + a_3 e^{12\sigma^2} + a_4 e^{8\sigma^2} + a_5 e^{4\sigma^2} + a_6\right). \tag{29}$$

The a_i are the coefficients given in Equation (27).

6. Bit Rate per User and Its Approximation

The log-normal approximation of FEXT and residual FEXT in Equation (11) can be used to derive a Gaussian approximation of the achievable bit rate per user as shown in this Section.

The performance index used to assess DSL performance is the achievable bit rate for the (generic) i-th user at distance d_i from the cabinet,:

$$R_{b,p}^{(i)}(d_i) = R_s \sum_{k\in I_c} B[\rho_{k,p}^{(i)}(d_i)], \tag{30}$$

where p is the order of the AZF vectoring pre-coding, I_c is the set of sub-carrier indices assigned to DS transmission, R_s is the symbol rate, and $B[\cdot]$ is the number of bits on the k-th sub-carrier assigned in accordance with the following bit-loading limitation criterion:

$$B[x] = \begin{cases} b_{max} & \text{if } x \geq b_{max} \\ x & \text{if } b_{min} \leq x < b_{max} \\ 0 & \text{otherwise} \end{cases} \tag{31}$$

where b_{min} and b_{max} are the minimum and maximum number of bits, respectively, that can be allocated per sub-carrier. Additionally, $\rho_{k,p}^{(i)}(d)$ in Equation (30) is:

$$\rho_{k,p}^{(i)}(d) = \log_2\left(1 + \frac{SINR_{k,p}^{(i)}(d_i)}{\Gamma}\right), \tag{32}$$

and Γ is the performance gap [6]. The log-normal approximation in Equation (11) allows one to express the argument of the logarithm in Equation (32) as:

$$1 + SINR_{k,p}^{(i)}(d_i)/\Gamma \approx e^{z_{k,i,p}(d)},\ \forall k \in I_c \tag{33}$$

where $z_{k,i,p}(d_i) = \mu_{z_{k,i,p}}(d_i) + \sigma_{z_{k,i,p}}(d_i)(-\nu)$ is Gaussian with mean $\mu_{z_{k,i,p}}(d_i)$ and standard deviation $\sigma_{z_{k,i,p}}(d_i)$.

From Equation (30) and using the result of Equation (33) in the case of no bit-loading limitation per sub-carrier (e.g., theoretically $b_{max} \to \infty$), the bit rate per user can be modelled with the following Gaussian random variable [16]:

$$R_{b,p}^{(i)}(d_i, \nu) = \overline{R}_{b,p}^{(i)}(d_i) + \sigma_{R_{b,p}}^{(i)}(d_i)(-\nu), \tag{34}$$

whose mean $\overline{R}_{b,p}^{(i)}(d_i)$ and standard deviation $\sigma_{R_{b,p}}^{(i)}(d_i)$ are:

$$\overline{R}_{b,p}^{(i)}(d_i) = R_s \log_2(e) \sum_{k \in S_c} \mu_{z_{k,i,p}}(d_i) \quad , \quad \sigma_{R_{b,p}}^{(i)}(d_i) = R_s \log_2(e) \sum_{k \in S_c} \sigma_{z_{k,i,p}}(d_i). \tag{35}$$

In the bit rate formulas, we have shown the dependence of the bit rate on p, d_i, and ν. In the bit-loading limitation case the bit rate is given in Equation (30) with:

$$\rho_{k,p}^{(i)}(d_i) = \left(\mu_{z_{k,i,p}}(d_i) + \sigma_{z_{k,i,p}}(d_i)(-\nu)\right) \log_2(e) \tag{36}$$

The mean and the standard deviation of the Gaussian random variables $z_{k,i,p}(d_i)$ can be related to the corresponding moments $\tilde{\mu}_{p,i}$ and $\tilde{\sigma}_{p,i}$ of the random variable $\tilde{y}_{p,i}$ in Equation (11). In Appendix A we derive relations between the moments of $z_{k,i,p}(d_i)$ those of $\tilde{y}_{p,i}$. The Wilkinson approach case [19] has been considered for the derivation.

7. Validity of the Bit Rate Approximation

The effectiveness of the bit rate approximation in Equation (34) for VDSL2 in the non-vectoring case has been discussed in Reference [16] for variable μ_{dB} and/or σ_{dB}. These results are not repeated here. In this Section we assess the validity of the bit rate expression in Equation (34) in the AZF vectoring case. The assessment is carried out in terms of the cumulative distribution function (CDF) of the bit rate under variable FEXT conditions i.e., variable number of active terminals, N, and variable distance d from the cabinet. VDSL2 and G.fast technologies have been considered. The maximum VDSL2 frequency is set to $f_M = 35.32$ MHz and the overall transmission power is set to 14.5 dBm. The VDSL2 gap Γ is 12 dB. For G.fast the maximum frequency is set to $f_M = 106.1$ MHz, the overall transmission power is 4 dBm and the gap is $\Gamma = 10.75$ dB. In both cases, VDSL2 and G.fast, a flat transmitter power spectrum is assumed (i.e., no (optimal) bit loading algorithm has been considered). Furthermore, the considered FEXT coupling coefficient is $\chi_F \cong 3.6 \times 10^{-20}$ [20]. For validation purposes, the exact value of the constant χ_F is not important.

In Figure 2 we plot the CDF of the bit rate for AZF vectoring with $p = 1$. Results have been obtained for $N = 20$ and $v_D = 3$ dB. Results in Figure 2 show very good agreement between the exact (solid) and approximated (dots) curves at every distance d and for the considered FEXT statistics. Exact results have been obtained by applying Equations (30) and (32) with the $SINR_{k,p}^{(i)}(d_i)$ from Equation (9) with $p = 1$. Differences between approximated bit rates are on the order of some Mbit/s and are not clearly distinguishable in the Figures. The adoption of vectoring allows for significant improvement of performance. Ideally, performances with vectoring should be independent of the positions of the interferers. However, due to imperfect vectoring cancellation, performance degradations due to residual FEXT still occur. Furthermore, for distances greater than $d = 500$ m the FEXT contribution is no-longer dominant with respect to the background (BN) noise.

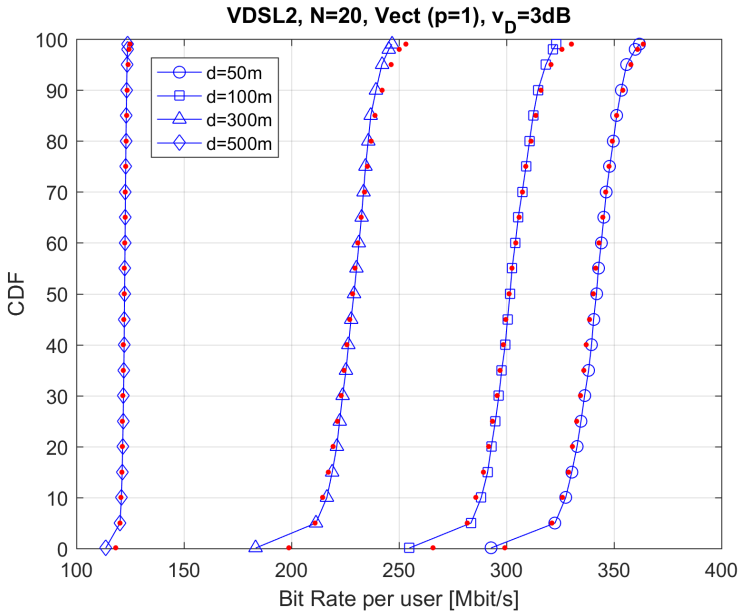

Figure 2. Cumulative distribution functions (CDFs) for the bit rate per user in the vectoring case with $p = 1$ for $N = 20$; exact (solid) and Gaussian approximation (dots).

The validity of the proposed bit rate Gaussian model is further confirmed by looking at the results in Figure 3, which reports the CDFs of the exact and approximate bit rate for AZF vectoring with $p = 1$ for variable N and for a given distance of the reference user $d = 200$ m from the cabinet.

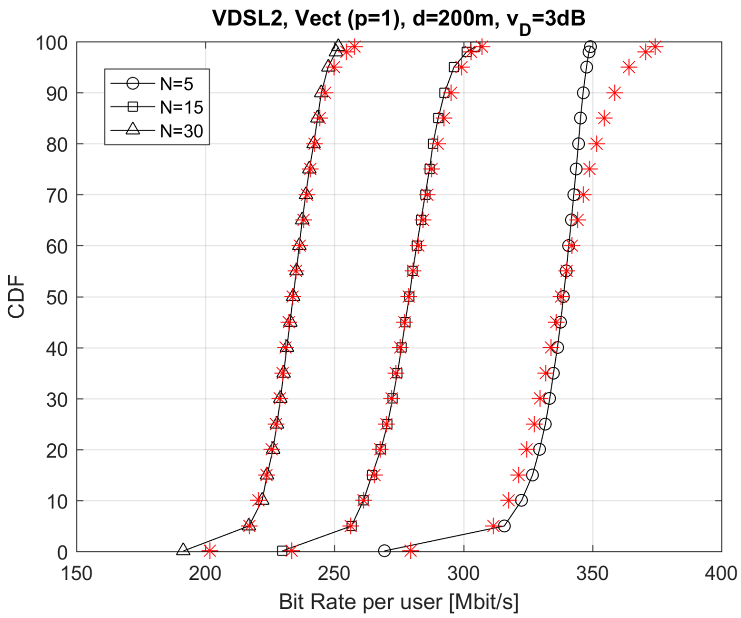

Figure 3. CDFs of the vit rate per user in the vectoring case ($p = 1$): $d = 200$ m, and variable N. Exact (solid) and Gaussian approximation (asterisks) results are shown.

Even in this case, exact and approximated CDFs are practically superimposed in all cases. Nevertheless, for very low FEXT conditions (e.g., $N = 5$), approximated results differ from exact ones for high percentiles in the vectoring case i.e., the model overestimates the achievable performance. Even then, the differences are on the order of several Mbit/s and then can be considered negligible. Residual FEXT increases with N thus degrading performance.

We have repeated the calculations of the exact and approximated bit rates in the G.fast case. Results of the CDFs of the bit rate are reported in Figures 4 and 5 for G.fast starting frequency of 2.2 MHz. Results are provided for the non-vectored ($p = 0$) and vectored ($p = 1$) cases.

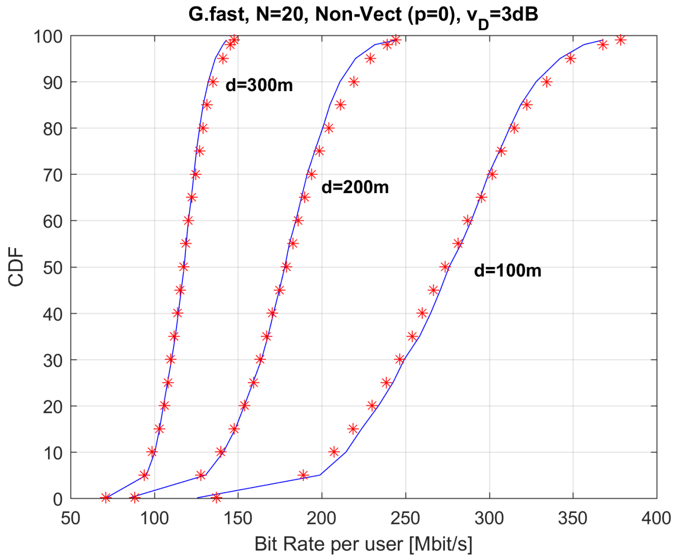

Figure 4. CDFs of the bit rate per user in the non-vectoring case for G.fast: $N = 20$. Exact (solid) and Gaussian approximation (asterisks) results are shown.

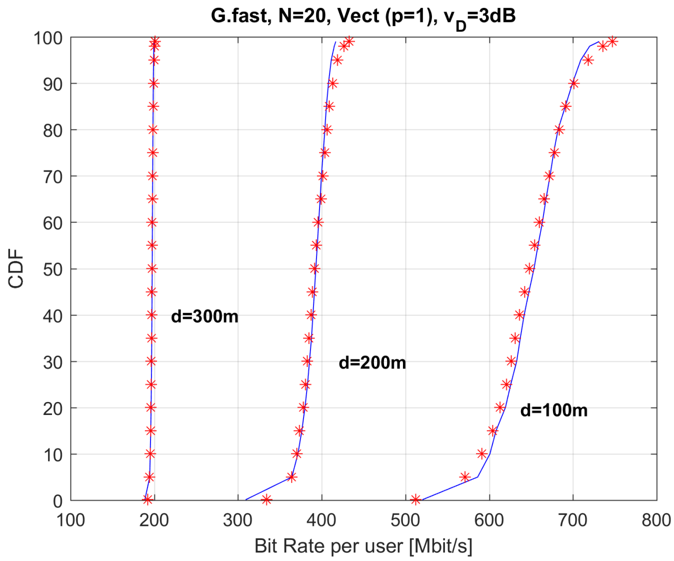

Figure 5. CDFs of the bit rate per user in the vectoring case ($p = 1$) for G.fast: $N = 20$. Exact (solid) and Gaussian approximation (asterisks) results are shown.

Considerations similar to those expressed in the VDSL2 case apply. As expected, vectoring is mandatory for G.fast to avoid the severe performance degradation shown in Figure 4.

8. Results

The approximated Gaussian bit rate formulation is now applied to assess the performance of VDSL2 and G.fast with AZF vectoring in the co-located scenario.

8.1. VDSL2 Performance Analysis

In Figure 6, we report the mean of the bit rate per user as a function of the distance from the cabinet for variable N with and without vectoring. In order to show the degradation due to residual FEXT with respect to the ideal case (i.e., background noise only), in the same figure we have plotted the maximum achievable bit rate curve obtained in the ideal case.

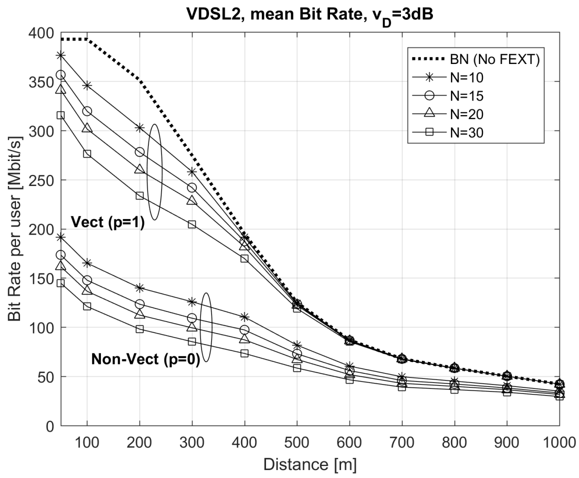

Figure 6. Bit rate per user vs. distance for VDSL2 with a variable number of users N. Non-vectoring ($p = 0$), and vectoring ($p = 1$) cases are displayed.

Due to the presence of residual FEXT, performance rapidly degrades with N. The residual FEXT becomes negligible with respect to background noise for distances greater than 500 m.

In Figure 7, we report the maximum distance vs. the number of users that can be supported for an assigned target bit rate. Results have been obtained considering the bit rate curves corresponding to the 5-th percentile. Results in Figure 7 can be related to the coverage provided by the VDSL2. In the same figure, the horizontal dotted lines indicate the maximum distance a single user can be served at the target bit rate in the ideal case. These curves show the coverage degradation due to residual FEXT. As an example, for target bit rate $R_{bT} = 250$ Mbit/s, in the ideal case users up to 330 m can be served at R_{bT}. However, in the presence of residual FEXT, only 12 users at a maximum distance of 250 m can be served at R_{bT}.

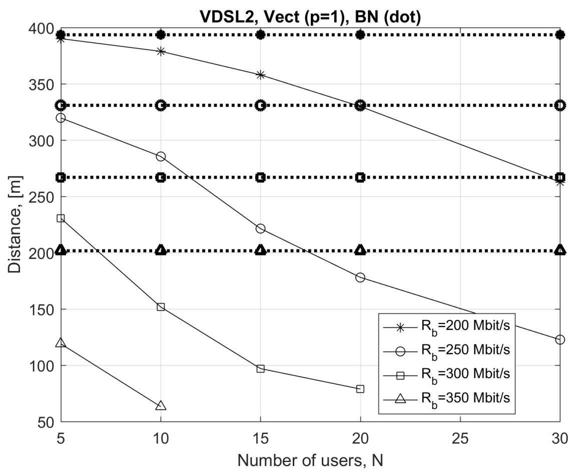

Figure 7. Maximum distance vs. number of users that can be served at a target bit rate R_b, for the vectoring case $p = 1$, 5-th percentile.

Detrimental effects of residual FEXT can be reduced by increasing the order of the pre-coding matrix. This is obtained in the AZF case by passing from $p = 1$ to $p = 2$, or to $p > 2$. In Figure 8, we plot the achievable bit rate performance in the considered interference scenario including vectoring with $p = 1$ and $p = 2$.

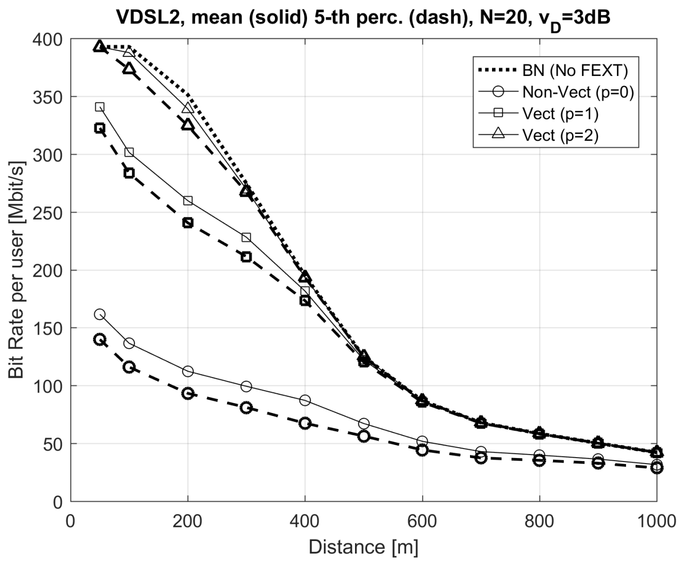

Figure 8. Mean (solid) and 5-th percentile (dashed) of the bit rate per user in the non-vectoring ($p = 0$) and vectoring cases with $p = 1$ and $p = 2$: $N = 20$.

As shown in Figure 8, the introduction of the quadratic term in the pre-coding matrix (e.g., $p = 2$) allows one to significantly improve the achievable bit rate which becomes closer to the ideal case both in terms of its mean and its 5-th percentile. This fact can be better appreciated looking at the results in Figure 9, where we plot the bit rate loss (percentage) as a function of the distance of the user from the cabinet. The bit rate loss is defined as the difference between the ideal bit rate and the bit rate obtained with vectoring, normalized with respect to the ideal bit rate at the same distance.

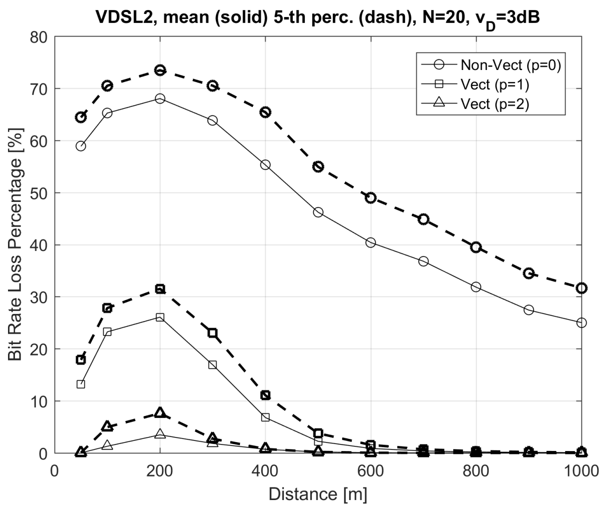

Figure 9. Bit rate loss percentage as a function of the distance. Non-vectoring ($p = 0$), and vectoring cases, with $p = 1$ and $p = 2$, are shown.

Passing from $p = 1$ to $p = 2$ performance improves up to 4 or 5 times. As an example, at distance $d = 200$ m, bit rate loss is reduced from 30% to about 8% (5-th percentile). This improvement is achieved at the expense of increased complexity and computational power required for the vectoring processors and to an increase in the corresponding costs and power consumption.

8.2. G.fast Performance Analysis

In this Section we report the DS performance of G.fast under variable interference scenarios. In Figure 10, we plot the user bit rate as a function of distance from the distribution point by varying G.fast starting transmission frequency f_S, which is selected in order to guarantee G.fast coexistence with other DSL technologies. Results have been presented for $N = 20$ and $v_D = 3$ dB with and without vectoring.

As expected, due to FEXT increase with frequency, vectoring is mandatory for G.fast. The increase of starting frequency, from 2.2 MHz to 17.664 MHz and 35.32 MHz, leads to severe performance reduction. This may raise serious concerns for telecommunications operators looking to adopt G.fast, especially when $f_S = 35.32$ MHz. Furthermore, the performance reduction due to residual FEXT in the $p = 1$ vectoring case is significant. This fact is shown in Figure 11, where we plot the mean and 5-th percentile of the bit rate as a function of the user's distance from the Dp.

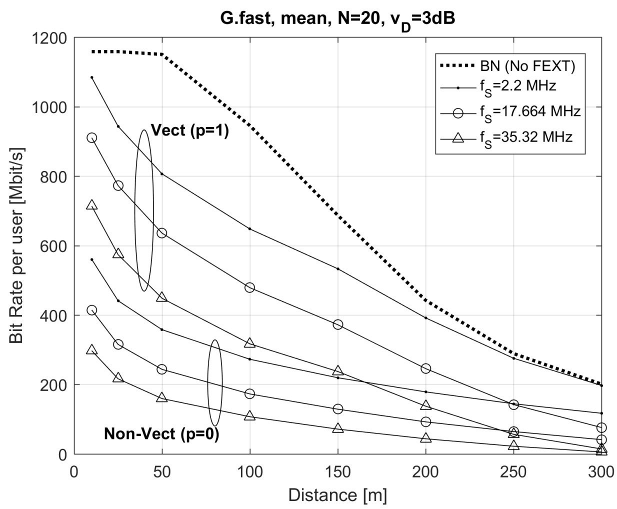

Figure 10. G.fast bit rate vs. distance for $f_S = 2.2$ MHz, $f_S = 17.664$ MHz, and $f_S = 35.32$ MHz.

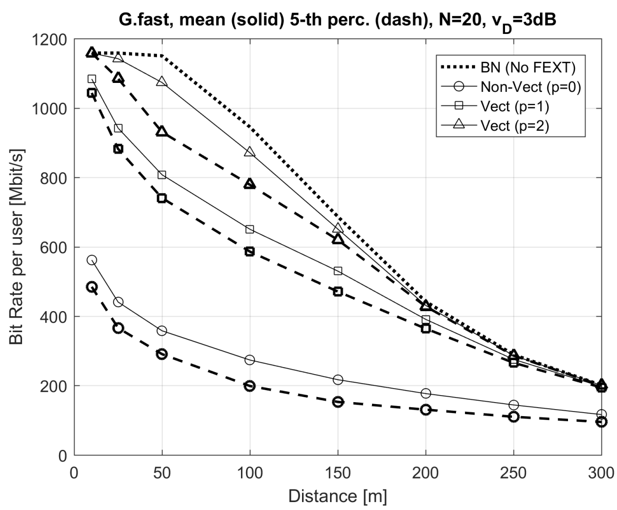

Figure 11. Mean and 5-th percentile in the non-vectoring $p = 0$ and vectoring cases, with $p = 1$ and $p = 2$, for G.fast with $N = 20$.

Even in this case, vectoring performance against residual FEXT can be improved by passing from $p = 1$ to $p = 2$. This fact is shown by the results in Figure 12 reporting the bit rate loss as a function of the distance from the Dp.

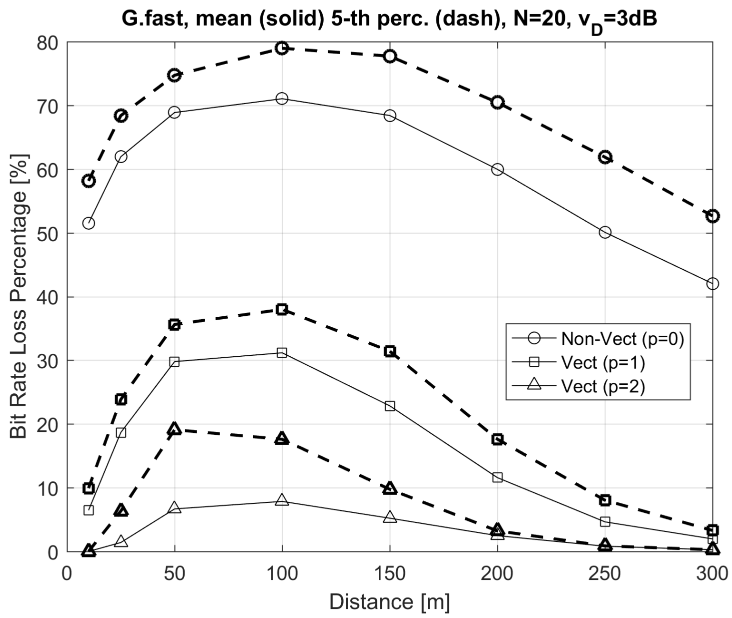

Figure 12. G.fast bit rate loss percentage as a function of distance. Non-vectoring ($p = 0$) and vectoring, with $p = 1$ and $p = 2$, results are shown.

Passing from $p = 1$ to $p = 2$ allows one to reduce the bit rate loss from 30% to less than 10% at distances between 50 m and 100 m. To further reduce bit rate loss the possibility of adopting $p = 3$ (not discussed here) should be considered at the expense of a significant increase in complexity and required computational power of the vectoring pre-coding processors.

9. Conclusions

Starting from a log-normal approximation of the SINR per sub-carrier, we have extended the approximated Gaussian representation for the user bit rate in Reference [16] to include the AZF vectoring algorithm. The effectiveness of the proposed formulations has been tested by computer calculation for variable FEXT conditions and co-located users scenario. Very good agreement between approximated and exact results have been obtained. The approximation has been used to evaluate the achievable performance for VDSL2 and G.fast showing the impact of residual FEXT on the achievable performance. It is shown that passing from $p = 1$ to $p = 2$ allows one to obtain performance closer to the ideal case in VDSL2, while for G.fast the $p = 2$ case should be considered mandatory and $p > 2$ could further improve performance.

Author Contributions: Authors contribute equally to this paper.

Conflicts of Interest: The authors declare no conflict of interest.

Appendix A. Calculation of Log-Normal Approximation

In this Appendix, we derive the moments of the random variable $z_{k,i,p}(d_i)$ in terms of the moments of $\tilde{y}_{p,i}$ in a closed analytical form by using the Wilkinson approach [19]. We observe Equation (33) can be conveniently written as:

$$e^{z_{k,i,p}(d_i)} \approx 1 + e^{w_{k,i,p}(d)}, \tag{A1}$$

and the random variable $w_{k,i,p}(d)$ is Gaussian with mean $\mu_{w_{k,i,p}}(d)$ and standard deviation $\sigma_{w_{k,i,p}}(d)$. The $u_{k,i,p} = \mu_{u_{k,i,p}} + \sigma_{u_{k,i,p}}\nu$ is the generic Gaussian random variable approximating the sum in the denominator of Equation (10).

The moments of the Gaussian random variables $\{u_{k,i,p}\}$, $\{w_{k,i,p}\}$ and $\{z_{k,i,p}\}$ can be obtained by applying any one of the methods in References [19,21]. Proper selection of the method should be based on the value of the standard deviation $\tilde{\sigma}_{p,i}$. As shown in the following, for $p = 2$ the Fenton–Wilkinson (FW) method can be successfully applied. For $p = 3$, the method in Reference [21] is more accurate.

When applying the FW method recursively, we obtain the following formulas. For notational simplicity, we have omitted the dependence on i, p, and d_i leaving only the dependence on the sub-carrier index k. Starting from the moments of $z_{k,i}$ we have:

$$\sigma_{z_k}^2 = \ln\left(1 + \frac{e^{2\mu_{w_k}+\sigma_{w_k}^2}\left(e^{\sigma_{w_k}^2}-1\right)}{\left(1+e^{\mu_{w_k}+\frac{1}{2}\sigma_{w_k}^2}\right)^2}\right), \tag{A2}$$

$$\mu_{z_k} = \ln\left(1+e^{\mu_{w_k}+\frac{1}{2}\sigma_{w_k}^2}\right) - \frac{1}{2}\sigma_{z_k}^2, \tag{A3}$$

where $\mu_{w_k} = -\mu_{u_k} - \ln\Gamma + \ln D_k(d)$. Additionally $\sigma_{w_k} = \sigma_{u_k}$ with:

$$\sigma_{u_k}^2 = \ln\left(1 + \frac{e^{2\mu_k+\sigma_k^2}\left(e^{\sigma_k^2}-1\right)}{\left(1+e^{\mu_k+\frac{1}{2}\sigma_k^2}\right)^2}\right), \tag{A4}$$

$$\mu_{u_k} = \ln\left(1+e^{\mu_k+\frac{1}{2}\sigma_k^2}\right) - \frac{1}{2}\sigma_{u_k}^2. \tag{A5}$$

Finally, $\mu_k = \ln\left(v_D D_k(d)\chi_F^{p+1} N_{p,i} d_i^{p+1} f_k^{2(p+1)}\right) + \tilde{\mu}_{p,i}$ and $\sigma_k = \tilde{\sigma}_{p,i}$ for $k \in I_c$. To overcome the limitations of Wilkinson's approach, other methods to evaluate the log-normal parameters can be considered [21]. However, in some cases it may be not possible to obtain closed form expressions for the parameters of the log-normal random variable which approximates the sum.

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