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Improved Stability Criteria on Linear Systems with Distributed Interval Time-Varying Delays and Nonlinear Perturbations

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Abstract: The problem of delay-range-dependent stability analysis for linear systems with distributed time-varying delays and nonlinear perturbations is studied without using the model transformation and delay-decomposition approach. The less conservative stability criteria are obtained for the systems by constructing a new augmented Lyapunov–Krasovskii functional and various inequalities, which are presented in terms of linear matrix inequalities (LMIs). Four numerical examples are demonstrated for the results given to illustrate the effectiveness and improvement over other methods.

Keywords: distributed interval time-varying delay; linear system; Lyapunov–Krasovskii functional; linear matrix inequality; integral inequality

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1. Introduction

The system of linear delay differential equations appears naturally in many branches of science and engineering. The delay is very often encountered in various technical systems, such as robotics, electric systems, hydraulic networks, chemical processes, long transmission lines, communication networks, etc. [1]. The stability problem of the investigation of a linear system with delay has been exploited over many years [1–28]. The stability criteria for time-delay systems are generally divided into two classes: the delay-independent one and the delay-dependent one. Delay-independent stability criteria tend to be more conservative, especially for a small size delay; such criteria do not give any information on the size of the delay. On the other hand, delay-dependent stability criteria are concerned with the size of the delay and usually provide a maximal delay size. The stability criteria of dynamical systems with time-varying delays and nonlinear perturbations have received the attention of many theoreticians and engineers in this field over the last few decades [4,9,14,18,25,28]. For delay-dependent stability criteria, the main concern is to enlarge the feasible region of the criteria to guarantee the asymptotic stability of time-delay systems in a given time-delay interval [2,3,6,7,22,23]. A descriptor model transformation and a corresponding Lyapunov–Krasovskii functional were introduced for the stability analysis of systems with delays in [19,21]. Moreover, delay-range-dependent stability criteria for dynamical systems with interval time-varying delays have been attracting the attention of several researchers [4,9,14,20,25,28]. Delay-range-dependent stability criteria make use of information on the lower and upper bounds of delay. In the past few years, there have been various approaches to reduce the conservatism of delay-dependent conditions by using a new Lyapunov–Krasovskii functional [18,26], improved inequalities [7,11–13,16–18,27], the free-weighting matrices technique [3,18], the delay-decomposition approach [2], and model transformation [9,28]. However, the results do not take into account the presence of nonlinear perturbations and distributed time-varying delay in the system.

In the existing literature, the linear system with an interval time-varying state delay and nonlinear perturbations has been considered in [25] in the form:

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bx(t-h(t)) + f(t, x(t)) + g(t, x(t-h(t))), && (1) \\ x(t) &= \phi(t), \qquad t \in [-h_2, 0] && (2) \end{aligned}$$

where $h(t)$ is a interval time-varying delay,

$$0 \le h_1 \le h(t) \le h_2, \quad \dot{h}(t) \le u, \tag{3}$$

$x(t) \in \mathbb{R}^n$, A, $B \in \mathbb{R}^{n\times n}$, h_1, h_2 and u are positive real constants representing the lower and upper bounds of the delay, and $\phi(t)$ is a given continuously differentiable function on $t \in [-h_2, 0]$. The uncertainties $f(t, x(t))$ and $g(t, x(t-h(t)))$ represent the nonlinear parameter perturbations with respect to the current state $x(t)$ and the delayed state $x(t-h(t))$, respectively, and are bounded in magnitude in the form,

$$\begin{aligned} f^T(t, x(t)) f(t, x(t)) &\le \eta^2 x^T(t) x(t), && (4) \\ g^T(t, x(t-h(t))) g(t, x(t-h(t))) &\le \rho^2 x^T(t-h(t)) x(t-h(t)), && (5) \end{aligned}$$

where η and ρ are known real positive constants.

More recently, the authors studied the problem of stability analysis for linear systems with distributed time-varying delays in [22] in the form:

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bx(t-h(t)) + C\int_{t-r(t)}^{t} x(s)ds, && (6) \\ x(t) &= \phi(t), \qquad t \in [-h_2, 0], && (7) \end{aligned}$$

where $h(t)$ is a time-varying delay,

$$0 \le h(t) \le h_2, \quad -u \le \dot{h}(t) \le u < 1, \tag{8}$$

$x(t) \in \mathbb{R}^n$, A, B, $C \in \mathbb{R}^{n\times n}$, h_2 and u are positive real constants representing the d upper bounds of the delay, and $\phi(t)$ is a given continuously differentiable function on $t \in [-h_2, 0]$.

Motivated by the above statement, we consider the system with distributed interval time-varying delays and nonlinear perturbations in the form:

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bx(t-h(t)) + f(t, x(t)) + g(t, x(t-h(t))) \\ &\quad + C\int_{t-r(t)}^{t} x(s)ds, && (9) \\ x(t) &= \phi(t), \qquad t \in [-\max\{h_2, r_2\}, 0], && (10) \end{aligned}$$

where $h(t)$ and $r(t)$ are interval time-varying delays,

$$\begin{aligned} &0 \le h_1 \le h(t) \le h_2, \quad -u \le \dot{h}(t) \le u < 1, && (11) \\ &0 \le r_1 \le r(t) \le r_2, && (12) \end{aligned}$$

$x(t) \in \mathbb{R}^n$, A, B, $C \in \mathbb{R}^{n\times n}$, h_1, h_2, r_1, r_2 and u are positive real constants, and $\phi(t)$ is a given continuously differentiable function on $t \in [-\max\{h_2, r_2\}, 0]$. The uncertainties $f(t, x(t))$ and $g(t, x(t-h(t)))$ represent the nonlinear parameter perturbations with respect to the current state $x(t)$ and the delayed state $x(t-h(t))$, respectively, and are bounded in magnitude in the form (4) and (5).

In this paper, we consider the problem of stability criteria for a linear system with distributed interval time-varying delays and nonlinear perturbations (9). Based on some new inequalities, some new augmented Lyapunov–Krasovskii functionals, an improved Peng–Park integral inequality, a novel triple integral inequality, and the utilization of the zero equation, less conservative stability criteria are obtained in terms of linear matrix inequalities (LMIs) without using model transformation and the delay-decomposition approach. Numerical examples illustrate the results.

2. Problem Formulation and Preliminaries

Throughout this paper, $\mathbb{R}$ and $\mathbb{R}^n$ represent the set of real numbers and the n-dimensional Euclidean spaces, respectively. $M > (\geq)0$ means that the symmetric matrix M is positive (semi-positive) definite. $M < (\leq)0$ denotes that the symmetric matrix M is negative (semi-negative) definite. M^T and M^{-1} denote the transpose and the inverse of M, respectively. The symbol $*$ represents the symmetric block in a symmetric matrix. I is the identity matrix with appropriate dimensions. S^n denotes the set of symmetric matrices. S^n_+ denotes the set of symmetric positive definite matrices. $\mathcal{C}([a_1, a_2], \mathbb{R}^n)$ denotes the set of continuous functions mapping the interval $[a_1, a_2]$ to $\mathbb{R}^n$. For any square matrix M, we define $Sym(M) = M + M^T$.

Lemma 1 (Schur complement [1]). *Given constant symmetric matrices X, Y, Z with appropriate dimensions satisfying $X = X^T, Y = Y^T > 0$, then $X + Z^T Y^{-1} Z < 0$ if and only if:*

$$\begin{bmatrix} X & Z^T \\ Z & -Y \end{bmatrix} < 0. \tag{13}$$

Lemma 2 ([2]). *For any constant matrix $Q \in \mathbb{R}^{n\times n}$, $Q = Q^T > 0$, positive real constant k_2, and a vector-valued function $\dot{x} : [-k_2, 0] \to \mathbb{R}^n$ such that the following integrals are well-defined, then:*

$$\begin{aligned} & -\int_{-k_2}^{0}\int_{t+s}^{t} \dot{x}^T(u)Q\dot{x}(u)duds \\ \leq & \begin{pmatrix} x(t) \\ \frac{1}{k_2}\int_{t-k_2}^{t} x(s)ds \end{pmatrix}^T \begin{pmatrix} -2Q & 2Q \\ * & -2Q \end{pmatrix} \begin{pmatrix} x(t) \\ \frac{1}{k_2}\int_{t-k_2}^{t} x(s)ds \end{pmatrix}. \end{aligned} \tag{14}$$

Lemma 3 (Sun et al. [20]). *For any constant matrix $Q \in \mathbb{R}^{n\times n}$, $Q = Q^T > 0$, positive real constants k_1, k_2, and a vector-valued function $\dot{x} : [-k_2, 0] \to \mathbb{R}^n$ such that the following integrals are well-defined, then:*

$$\begin{aligned} & -k_2\int_{t-k_2}^{t} \dot{x}^T(s)Q\dot{x}(s)ds \\ \leq & -\Big(\int_{t-k_2}^{t} \dot{x}(s)ds\Big)^T Q\Big(\int_{t-k_2}^{t} \dot{x}(s)ds\Big), \end{aligned} \tag{15}$$

$$\begin{aligned} & -\frac{(k_2^2 - k_1^2)}{2}\int_{-k_2}^{-k_1}\int_{t+s}^{t} x^T(u)Qx(u)duds \\ \leq & -\Big(\int_{-k_2}^{-k_1}\int_{t+s}^{t} x(u)duds\Big)^T Q\Big(\int_{-k_2}^{-k_1}\int_{t+s}^{t} x(u)duds\Big). \end{aligned} \tag{16}$$

Lemma 4 ([18]). *For any constant matrices $Q_1, Q_2, Q_3 \in \mathbb{R}^{n\times n}$, $Q_1 \geq 0$, $Q_3 > 0$, $\begin{bmatrix} Q_1 & Q_2 \\ * & Q_3 \end{bmatrix} \geq 0$, $k(t)$ are time-varying delays with $0 \leq k_1 \leq k(t) \leq k_2$, $k_1, k_2 \in \mathbb{R}$, vector-valued functions x, and $\dot{x} : [-k_2, -k_1] \to \mathbb{R}^n$ such that the following integrals are well-defined, then:*

$$
\begin{aligned}
&-(k_2-k_1)\int_{t-k_2}^{t-k_1} \begin{bmatrix} x(s) \\ \dot{x}(s) \end{bmatrix}^T \begin{bmatrix} Q_1 & Q_2 \\ * & Q_3 \end{bmatrix} \begin{bmatrix} x(s) \\ \dot{x}(s) \end{bmatrix} ds \\
\leq & \begin{bmatrix} x(t-k_1) \\ x(t-k(t)) \\ x(t-k_2) \\ \int_{t-k(t)}^{t-k_1} x(s)ds \\ \int_{t-k_2}^{t-k(t)} x(s)ds \end{bmatrix}^T \begin{bmatrix} -Q_3 & Q_3 & 0 & -Q_2^T & 0 \\ * & -Q_3-Q_3^T & Q_3 & Q_2^T & -Q_2^T \\ * & * & -Q_3 & 0 & Q_2^T \\ * & * & * & -Q_1 & 0 \\ * & * & * & * & -Q_1 \end{bmatrix} \\
& \times \begin{bmatrix} x(t-k_1) \\ x(t-k(t)) \\ x(t-k_2) \\ \int_{t-k(t)}^{t-k_1} x(s)ds \\ \int_{t-k_2}^{t-k(t)} x(s)ds \end{bmatrix}.
\end{aligned}
\tag{17}
$$

Lemma 5 (Peng–Park's integral inequality [12,13]). *For any constant matrices $Q, S \in \mathbb{R}^{n\times n}$, $Q \geq 0$, $\begin{bmatrix} Q & S \\ * & Q \end{bmatrix} \geq 0$, nonnegative real constants k_2, and a vector-valued function $\dot{x} : [-k_2, 0] \to \mathbb{R}^n$ such that the concerned integrations are well-defined, then:*

$$
-k_2 \int_{t-k_2}^{t} \dot{x}^T(s) Q \dot{x}(s) ds \leq \omega^T(t) \ominus \omega(t),
\tag{18}
$$

where $\omega(t) = \begin{bmatrix} x^T(t) & x^T(t-k(t)) & x^T(t-k_2) \end{bmatrix}^T$ *and* $\ominus = \begin{bmatrix} -Q & Q-S & S \\ * & -2Q+S+S^T & Q-S \\ * & * & -Q \end{bmatrix}$.

Lemma 6 ([27]). *For any constant matrix $Q \in \mathbb{R}^{n\times n}$, $Q = Q^T > 0$, real constants a, b, and a vector-valued function $\dot{x} : [a, b] \to \mathbb{R}^n$ such that the following integrals are well-defined, then:*

$$
\int_a^b \int_u^b \dot{x}^T(s) Q \dot{x}(s) ds du \geq 2\Omega_1^T Q \Omega_1 + 4\Omega_2^T Q \Omega_2 + 6\Omega_3^T Q \Omega_3,
\tag{19}
$$

where:

$$
\begin{aligned}
\Omega_1 &= x(b) - \frac{1}{b-a}\int_a^b x(s)ds, \\
\Omega_2 &= x(b) + \frac{2}{b-a}\int_a^b x(s)ds - \frac{6}{(b-a)^2}\int_a^b \int_u^b x(s)dsdu, \\
\Omega_3 &= x(b) - \frac{3}{b-a}\int_a^b x(s)ds + \frac{24}{(b-a)^2}\int_a^b \int_u^b x(s)dsdu \\
& \quad - \frac{60}{(b-a)^3}\int_a^b \int_u^b \int_s^b x(r)drdsdu.
\end{aligned}
$$

Lemma 7 ([16]). *For any matrices $\Theta \in S_+^n$, $M_1, M_2 \in \mathbb{R}^{m\times n}$, $\Upsilon \in \mathbb{R}^{2n\times m}$, $\forall \alpha \in (0,1)$, the inequality:*

$$
\begin{aligned}
-\Upsilon^T \begin{bmatrix} \frac{1}{\alpha}\Theta & 0 \\ 0 & \frac{1}{1-\alpha}\Theta \end{bmatrix} \Upsilon \leq & -\Upsilon^T \Sigma(\alpha) \Upsilon - sym\left(\Upsilon^T \begin{bmatrix} (1-\alpha)M_1^T \\ \alpha M_2^T \end{bmatrix}\right) \\
& + \alpha M_1 \Theta^{-1} M_1^T + (1-\alpha) M_2 \Theta^{-1} M_2^T,
\end{aligned}
\tag{20}
$$

holds, where:

$$\Sigma(\alpha) = \begin{bmatrix} (2-\alpha)\Theta & 0 \\ 0 & (1+\alpha)\Theta \end{bmatrix}.$$

Lemma 8 ([7]). *For a matrix $Q \in S_+^n$, and any continuously differentiable function function $x : [a,b] \to \mathbb{R}^n$, the equality:*

$$\begin{aligned} \int_a^b \dot{x}^T(s)Q\dot{x}(s)ds \quad \geq \quad & \frac{1}{b-a}\Omega_4^T Q\Omega_4 + \frac{3}{b-a}\Omega_5^T Q\Omega_5 + \frac{5}{b-a}\Omega_6^T Q\Omega_6 \\ & + \frac{7}{b-a}\Omega_7^T Q\Omega_7, \end{aligned} \tag{21}$$

holds, where:

$$\begin{aligned} \Omega_4 &= x(b) - x(a), \\ \Omega_5 &= x(b) + x(a) - \frac{2}{b-a}\int_a^b x(s)ds, \\ \Omega_6 &= x(b) - x(a) + \frac{6}{b-a}\int_a^b x(s)ds - \frac{12}{(b-a)^2}\int_a^b \int_u^b x(s)dsdu, \\ \Omega_7 &= x(b) + x(a) - \frac{12}{b-a}\int_a^b x(s)ds + \frac{60}{(b-a)^2}\int_a^b \int_u^b x(s)dsdu \\ & \quad - \frac{120}{(b-a)^3}\int_a^b \int_u^b \int_v^b x(s)dsdvdu. \end{aligned}$$

Lemma 9 ([11]). *Suppose $\Omega, \Omega_{ij}(i,j = 1,2)$ are the constant matrices of appropriate dimensions, $\alpha \in [0,1]$, $\beta \in [-u,u], 0 \leq u < 1$, then:*

$$\Omega + \alpha\Omega_{11} + (1-\alpha)\Omega_{12} + \beta\Omega_{21} + (1-\beta)\Omega_{22} < 0, \tag{22}$$

holds if and only if the following inequalities hold,

$$\Omega + \Omega_{11} - u\Omega_{21} + (1+u)\Omega_{22} < 0, \tag{23}$$
$$\Omega + \Omega_{12} - u\Omega_{21} + (1+u)\Omega_{22} < 0, \tag{24}$$
$$\Omega + \Omega_{11} + u\Omega_{21} + (1-u)\Omega_{22} < 0, \tag{25}$$
$$\Omega + \Omega_{12} + u\Omega_{21} + (1-u)\Omega_{22} < 0. \tag{26}$$

3. Main Results

We define a new parameter:

$$X_1 = \begin{bmatrix} P_2 & P_3 \\ * & P_4 \end{bmatrix}, \quad X_2 = \begin{bmatrix} P_5 & P_6 \\ * & P_7 \end{bmatrix}, \quad X_3 = \begin{bmatrix} P_8 & P_9 \\ * & P_{10} \end{bmatrix}, \quad X_4 = \begin{bmatrix} P_{11} & P_{12} \\ * & P_{13} \end{bmatrix},$$
$$X_5 = \begin{bmatrix} Q_1 & Q_2 \\ * & Q_3 \end{bmatrix}, \quad X_6 = \begin{bmatrix} Q_4 & Q_5 \\ * & Q_6 \end{bmatrix}, \quad X_7 = \begin{bmatrix} R_1 & R_2 \\ * & R_3 \end{bmatrix}, \quad X_8 = \begin{bmatrix} R_4 & R_5 \\ * & R_6 \end{bmatrix},$$
$$X_9 = \begin{bmatrix} R_7 & R_8 \\ * & R_9 \end{bmatrix}, \quad X_{10} = \begin{bmatrix} R_{10} & R_{11} \\ * & R_{12} \end{bmatrix}, \quad X_{11} = \begin{bmatrix} -Z_2 & Z_2 - S & S \\ * & -2Z_2 + S + S^T & Z_2 - S \\ * & * & -Z_2 \end{bmatrix},$$
$$X_{12} = \begin{bmatrix} -R_3 & R_3 & 0 & -R_2^T & 0 \\ * & -R_3 - R_3^T & R_3 & R_2^T & -R_2^T \\ * & * & -R_3 & 0 & R_2^T \\ * & * & * & -R_1 & 0 \\ * & * & * & * & -R_1 \end{bmatrix},$$

$$X_{13} = \begin{bmatrix} -R_6 & R_6 & 0 & -R_5^T & 0 \\ * & -R_6 - R_6^T & R_6 & R_5^T & -R_5^T \\ * & * & -R_6 & 0 & R_5^T \\ * & * & * & -R_4 & 0 \\ * & * & * & * & -R_4 \end{bmatrix},$$
$$X_{14} = \begin{bmatrix} -R_9 & R_9 & 0 & -R_8^T & 0 \\ * & -R_9 - R_9^T & R_9 & R_8^T & -R_8^T \\ * & * & -R_9 & 0 & R_8^T \\ * & * & * & -R_7 & 0 \\ * & * & * & * & -R_7 \end{bmatrix},$$
$$X_{15} = \begin{bmatrix} -R_{12} & R_{12} & 0 & -R_{11}^T & 0 \\ * & -R_{12} - R_{12}^T & R_{12} & R_{11}^T & -R_{11}^T \\ * & * & -R_{12} & 0 & R_{11}^T \\ * & * & * & -R_{10} & 0 \\ * & * & * & * & -R_{10} \end{bmatrix},$$
$$X_{16} = \begin{bmatrix} -2W_2 & 2W_2 \\ * & -2W_2 \end{bmatrix}, \quad X_{17} = \begin{bmatrix} -2W_4 & 2W_4 \\ * & -2W_4 \end{bmatrix},$$
$$X_{18} = \begin{bmatrix} -12W_1 & 12W_1 & -120W_1 & 360W_1 \\ 12W_1 & -72W_1 & 480W_1 & -1080W_1 \\ -120W_1 & 480W_1 & -3600W_1 & 8640W_1 \\ 360W_1 & -1080W_1 & 8640W_1 & -21{,}600W_1 \end{bmatrix},$$
$$X_{19} = \begin{bmatrix} -12W_3 & 12W_3 & -120W_3 & 360W_3 \\ 12W_3 & -72W_3 & 480W_3 & -1080W_3 \\ -120W_3 & 480W_3 & -3600W_3 & 8640W_3 \\ 360W_3 & -1080W_3 & 8640W_3 & -21{,}600W_3 \end{bmatrix},$$
$$\begin{aligned} \phi(\alpha,\beta) = {} & Sym(\Pi_1^T P_1 \Pi_2) + Sym(\Pi_3^T X_1 \Pi_4) + Sym(\Pi_5^T X_2 \Pi_6) + Sym(\Pi_7^T X_3 \Pi_8) \\ & + Sym(\Pi_9^T X_4 \Pi_{10}) + \Pi_{11}^T X_5 \Pi_{11} - (1-\beta)\Pi_{12}^T X_5 \Pi_{12} + \Pi_{13}^T X_6 \Pi_{14} \\ & + h_2^2 \Pi_{15}^T Z_1 \Pi_{15} + h_2^2 \Pi_{15}^T Z_2 \Pi_{15} + r_2^2 \Pi_1^T Z_3 \Pi_1 + \Pi_{16}^T X_{11} \Pi_{16} + \Pi_{17}^T Z_3 \Pi_{17} \\ & + h_2^2 \Pi_{11}^T X_7 \Pi_{11} + (h_2 - h_1)^2 \Pi_{11}^T X_8 \Pi_{11} + r_2^2 \Pi_{11}^T X_9 \Pi_{11} + \Pi_{18}^T X_{12} \Pi_{18} \\ & + (r_2 - r_1)^2 \Pi_{11}^T X_{10} \Pi_{11} + \Pi_{19}^T X_{13} \Pi_{19} + \Pi_{20}^T X_{14} \Pi_{20} + \Pi_{21}^T X_{15} \Pi_{21} \\ & + \frac{h_2^2}{2} \Pi_{15}^T W_1 \Pi_{15} + \frac{h_1^4}{2} \Pi_{15}^T W_2 \Pi_{15} + \frac{r_2^2}{2} \Pi_{15}^T W_3 \Pi_{15} + \frac{r_1^4}{2} \Pi_{15}^T W_4 \Pi_{15} \\ & + h_1^2 \Pi_{22}^T X_{16} \Pi_{22} + r_1^2 \Pi_{23}^T X_{17} \Pi_{23} + \Pi_{24}^T X_{18} \Pi_{24} + \Pi_{25}^T X_{19} \Pi_{25} + \Pi_{15}^T L_1 \Pi_{26} \\ & + \epsilon_1 \eta^2 \Pi_1^T I \Pi_1 - \epsilon_2 \Pi_{27}^T I \Pi_{27} + \epsilon_2 \rho^2 \Pi_{28}^T I \Pi_{28} - \epsilon_2 \Pi_{29}^T I \Pi_{29}, \end{aligned}$$
$$\Pi_1 = \varepsilon_1, \quad \Pi_2 = A\varepsilon_1 + B\varepsilon_3 + C\varepsilon_{12} + \varepsilon_{22} + \varepsilon_{23}, \quad \Pi_3 = \begin{bmatrix} \varepsilon_1^T & h_1\varepsilon_{28}^T \end{bmatrix}^T,$$
$$\Pi_4 = \begin{bmatrix} \varepsilon_2^T & \varepsilon_1^T - \varepsilon_{26}^T \end{bmatrix}^T, \quad \Pi_5 = \begin{bmatrix} \varepsilon_1^T & h_2\varepsilon_8^T \end{bmatrix}^T, \quad \Pi_6 = \begin{bmatrix} \varepsilon_2^T & \varepsilon_1^T - \varepsilon_6^T \end{bmatrix}^T,$$
$$\Pi_7 = \begin{bmatrix} \varepsilon_1^T & r_1\varepsilon_{29}^T \end{bmatrix}^T, \quad \Pi_8 = \begin{bmatrix} \varepsilon_2^T & \varepsilon_1^T - \varepsilon_{27}^T \end{bmatrix}^T, \quad \Pi_9 = \begin{bmatrix} \varepsilon_1^T & r_2\varepsilon_9^T \end{bmatrix}^T,$$
$$\Pi_{10} = \begin{bmatrix} \varepsilon_2^T & \varepsilon_1^T - \varepsilon_7^T \end{bmatrix}^T, \quad \Pi_{11} = \begin{bmatrix} \varepsilon_1^T & \varepsilon_2^T \end{bmatrix}^T, \quad \Pi_{12} = \begin{bmatrix} \varepsilon_3^T & \varepsilon_4^T \end{bmatrix}^T, \quad \Pi_{13} = \begin{bmatrix} \varepsilon_1^T & \varepsilon_3^T \end{bmatrix}^T,$$
$$\Pi_{14} = \begin{bmatrix} \varepsilon_2^T & (1-\beta)\varepsilon_4^T \end{bmatrix}^T, \quad \Pi_{15} = \varepsilon_2, \quad \Pi_{16} = \begin{bmatrix} \varepsilon_1^T & \varepsilon_3^T & \varepsilon_6^T \end{bmatrix}^T, \quad \Pi_{17} = \varepsilon_{12},$$
$$\Pi_{18} = \begin{bmatrix} \varepsilon_1^T & \varepsilon_3^T & \varepsilon_6^T & h_2\alpha\varepsilon_{10}^T & h_2(1-\alpha)\varepsilon_{11}^T \end{bmatrix}^T, \quad \Pi_{20} = \begin{bmatrix} \varepsilon_1^T & \varepsilon_5^T & \varepsilon_7^T & \varepsilon_{12}^T & \varepsilon_{13}^T \end{bmatrix}^T,$$
$$\Pi_{19} = \begin{bmatrix} \varepsilon_{26}^T & \varepsilon_3^T & \varepsilon_6^T & \varepsilon_{24}^T & h_2(1-\alpha)\varepsilon_{11}^T \end{bmatrix}^T, \quad \Pi_{21} = \begin{bmatrix} \varepsilon_{27}^T & \varepsilon_5^T & \varepsilon_7^T & \varepsilon_{25}^T & \varepsilon_{13}^T \end{bmatrix}^T,$$
$$\Pi_{22} = \begin{bmatrix} \varepsilon_1^T & \varepsilon_{28}^T \end{bmatrix}^T, \quad \Pi_{23} = \begin{bmatrix} \varepsilon_1^T & \varepsilon_{29}^T \end{bmatrix}^T, \quad \Pi_{24} = \begin{bmatrix} \varepsilon_1^T & \varepsilon_8^T & \varepsilon_{15}^T & \varepsilon_{19}^T \end{bmatrix}^T,$$
$$\Pi_{25} = \begin{bmatrix} \varepsilon_1^T & \varepsilon_9^T & \varepsilon_{17}^T & \varepsilon_{21}^T \end{bmatrix}^T, \quad \Pi_{26} = -\varepsilon_2 + A\varepsilon_1 + B\varepsilon_3 + C\varepsilon_{12} + \varepsilon_{22} + \varepsilon_{23},$$
$$\Pi_{27} = \varepsilon_{22}, \quad \Pi_{28} = \varepsilon_3, \quad \Pi_{29} = \varepsilon_{23}, \quad \chi_1 = \varepsilon_1 - \varepsilon_3, \quad \chi_2 = \varepsilon_1 + \varepsilon_3 - 2\varepsilon_{10},$$
$$\chi_3 = \varepsilon_1 - \varepsilon_3 + 6\varepsilon_{10} - 12\varepsilon_{14}, \quad \chi_4 = \varepsilon_1 + \varepsilon_3 - 12\varepsilon_{10} + 60\varepsilon_{14} - 120\varepsilon_{18},$$
$$\chi_5 = \varepsilon_3 - \varepsilon_6, \quad \chi_6 = \varepsilon_3 + \varepsilon_6 - 2\varepsilon_{11}, \quad \chi_7 = \varepsilon_3 - \varepsilon_6 + 6\varepsilon_{11} - 12\varepsilon_{16},$$
$$\chi_8 = \varepsilon_3 + \varepsilon_6 - 12\varepsilon_{11} + 60\varepsilon_{16} - 120\varepsilon_{20}, \Upsilon = \begin{bmatrix} \chi_1^T & \chi_2^T & \chi_3^T & \chi_4^T & \chi_5^T & \chi_6^T & \chi_7^T & \chi_8^T \end{bmatrix}^T,$$
$$\Theta = diag(Z_1,\ 3Z_1,\ 5Z_1,\ 7Z_1), \quad \Sigma(\alpha) = \begin{bmatrix} (2-\alpha)\Theta & 0 \\ 0 & (1+\alpha)\Theta \end{bmatrix},$$

$$\Phi(\alpha,\beta) = \begin{bmatrix} \phi(\alpha,\beta) - \Upsilon^T\Sigma(\alpha)\Upsilon - Sym\left(\Upsilon^T \begin{bmatrix} (1-\alpha)M_1^T \\ \alpha M_2^T \end{bmatrix}\right) & \alpha M_1 + (1-\alpha)M_2 \\ * & -\Theta \end{bmatrix},$$

and $\varepsilon_i \in \mathbb{R}^{n\times 29n}$ is defined as $\varepsilon_i = \begin{bmatrix} 0_{n\times(i-1)n} & I_n & 0_{n\times(29-i)n} \end{bmatrix}$ for $i = 1, 2, \cdots, 29$.

Theorem 1. *For given positive constants h_1, h_2, r_1, r_2, u, if there exist symmetric positive definite matrices P_1, Z_1, Z_2, Z_3, R_1, R_3, R_4, R_6, R_7, R_9, R_{10}, R_{12}, W_1, W_2, W_3, W_4, X_l, $l = 1, 2, \ldots, 10$, any appropriate dimensional matrices $M_1, M_2, S, P_i, Q_j, R_k$, $i = 2, 3, \ldots, 13$, $j = 1, 2, \ldots, 6$, $k = 2, 5, 8, 11$, and positive constants η, ρ such that the following LMIs hold:*

$$\Phi(\alpha,\beta) < 0, \tag{27}$$

$$\begin{bmatrix} Z_2 & S \\ * & Z_2 \end{bmatrix} > 0, \tag{28}$$

holds for $\alpha = \{0,1\}$, $\dot{r}(t) = \beta = \{-u, u\}$, i.e.,:

$$\Phi(0,-u) < 0, \tag{29}$$

$$\Phi(0,u) < 0, \tag{30}$$

$$\Phi(1,-u) < 0, \tag{31}$$

$$\Phi(1,u) < 0, \tag{32}$$

then the system (9) is asymptotically stable.

Proof of Theorem 1. Consider the system (9) with the following Lyapunov–Krasovskii functional:

$$V(t) = \sum_{i=1}^{7} V_i(t),$$

where:

$$\begin{aligned}
V_1(t) &= x^T(t)P_1x(t), \\
V_2(t) &= \begin{bmatrix} x(t) \\ \int_{t-h_1}^t x(s)ds \end{bmatrix}^T \begin{bmatrix} P_2 & P_3 \\ * & P_4 \end{bmatrix} \begin{bmatrix} x(t) \\ \int_{t-h_1}^t x(s)ds \end{bmatrix} \\
&\quad + \begin{bmatrix} x(t) \\ \int_{t-h_2}^t x(s)ds \end{bmatrix}^T \begin{bmatrix} P_5 & P_6 \\ * & P_7 \end{bmatrix} \begin{bmatrix} x(t) \\ \int_{t-h_2}^t x(s)ds \end{bmatrix} \\
&\quad + \begin{bmatrix} x(t) \\ \int_{t-r_1}^t x(s)ds \end{bmatrix}^T \begin{bmatrix} P_8 & P_9 \\ * & P_{10} \end{bmatrix} \begin{bmatrix} x(t) \\ \int_{t-r_1}^t x(s)ds \end{bmatrix} \\
&\quad + \begin{bmatrix} x(t) \\ \int_{t-r_2}^t x(s)ds \end{bmatrix}^T \begin{bmatrix} P_{11} & P_{12} \\ * & P_{13} \end{bmatrix} \begin{bmatrix} x(t) \\ \int_{t-r_2}^t x(s)ds \end{bmatrix}, \\
V_3(t) &= \int_{t-h(t)}^t \begin{bmatrix} x(s) \\ \dot{x}(s) \end{bmatrix}^T \begin{bmatrix} Q_1 & Q_2 \\ * & Q_3 \end{bmatrix} \begin{bmatrix} x(s) \\ \dot{x}(s) \end{bmatrix} ds, \\
V_4(t) &= \begin{bmatrix} x(t) \\ x(t-h(t)) \end{bmatrix}^T \begin{bmatrix} Q_4 & Q_5 \\ * & Q_6 \end{bmatrix} \begin{bmatrix} x(t) \\ x(t-h(t)) \end{bmatrix}, \\
V_5(t) &= h_2 \int_{t-h_2}^t \int_u^t \dot{x}^T(s)(Z_1+Z_2)\dot{x}(s)\, dsdu \\
&\quad + r_2 \int_{t-r_2}^t \int_u^t x^T(s)Z_3x(s)\, dsdu,
\end{aligned}$$

$$
\begin{aligned}
V_6(t) &= h_2 \int_{-h_2}^{0} \int_{t+s}^{t} \begin{bmatrix} x(\theta) \\ \dot{x}(\theta) \end{bmatrix}^T \begin{bmatrix} R_1 & R_2 \\ * & R_3 \end{bmatrix} \begin{bmatrix} x(\theta) \\ \dot{x}(\theta) \end{bmatrix} d\theta ds \\
&\quad +(h_2 - h_1) \int_{-h_2}^{-h_1} \int_{t+s}^{t} \begin{bmatrix} x(\theta) \\ \dot{x}(\theta) \end{bmatrix}^T \begin{bmatrix} R_4 & R_5 \\ * & R_6 \end{bmatrix} \begin{bmatrix} x(\theta) \\ \dot{x}(\theta) \end{bmatrix} d\theta ds \\
&\quad +r_2 \int_{-r_2}^{0} \int_{t+s}^{t} \begin{bmatrix} x(\theta) \\ \dot{x}(\theta) \end{bmatrix}^T \begin{bmatrix} R_7 & R_8 \\ * & R_9 \end{bmatrix} \begin{bmatrix} x(\theta) \\ \dot{x}(\theta) \end{bmatrix} d\theta ds \\
&\quad +(r_2 - r_1) \int_{-r_2}^{-r_1} \int_{t+s}^{t} \begin{bmatrix} x(\theta) \\ \dot{x}(\theta) \end{bmatrix}^T \begin{bmatrix} R_{10} & R_{11} \\ * & R_{12} \end{bmatrix} \begin{bmatrix} x(\theta) \\ \dot{x}(\theta) \end{bmatrix} d\theta ds, \\
V_7(t) &= \int_{t-h_2}^{t} \int_{s}^{t} \int_{\theta}^{t} \dot{x}^T(u) W_1 \dot{x}(u) du d\theta ds \\
&\quad +h_1^2 \int_{-h_1}^{0} \int_{s}^{0} \int_{t+\theta}^{t} \dot{x}^T(u) W_2 \dot{x}(u) du d\theta ds \\
&\quad + \int_{t-r_2}^{t} \int_{s}^{t} \int_{\theta}^{t} \dot{x}^T(u) W_3 \dot{x}(u) du d\theta ds \\
&\quad +r_1^2 \int_{-r_1}^{0} \int_{s}^{0} \int_{t+\theta}^{t} \dot{x}^T(u) W_4 \dot{x}(u) du d\theta ds.
\end{aligned}
$$

The time derivatives of $V_i(t), i = 1,2,3,4,$ along the trajectories of system (9) are given by

$$
\begin{aligned}
\dot{V}_1(t) &= 2x^T(t) P_1 \dot{x}(t), \\
&= 2x^T(t) P_1 [Ax(t) + Bx(t-h(t)) + f(t,x(t)) + g(t,x(t-h(t))) \\
&\quad +C \int_{t-r(t)}^{t} x(s) ds],
\end{aligned} \tag{33}
$$

$$
\begin{aligned}
\dot{V}_2(t) &= 2 \begin{bmatrix} x(t) \\ \int_{t-h_1}^{t} x(s) ds \end{bmatrix}^T \begin{bmatrix} P_2 & P_3 \\ * & P_4 \end{bmatrix} \begin{bmatrix} \dot{x}(t) \\ x(t) - x(t-h_1) \end{bmatrix} \\
&\quad +2 \begin{bmatrix} x(t) \\ \int_{t-h_2}^{t} x(s) ds \end{bmatrix}^T \begin{bmatrix} P_5 & P_6 \\ * & P_7 \end{bmatrix} \begin{bmatrix} \dot{x}(t) \\ x(t) - x(t-h_2) \end{bmatrix} \\
&\quad +2 \begin{bmatrix} x(t) \\ \int_{t-r_1}^{t} x(s) ds \end{bmatrix}^T \begin{bmatrix} P_8 & P_9 \\ * & P_{10} \end{bmatrix} \begin{bmatrix} \dot{x}(t) \\ x(t) - x(t-r_1) \end{bmatrix} \\
&\quad +2 \begin{bmatrix} x(t) \\ \int_{t-r_2}^{t} x(s) ds \end{bmatrix}^T \begin{bmatrix} P_{11} & P_{12} \\ * & P_{13} \end{bmatrix} \begin{bmatrix} \dot{x}(t) \\ x(t) - x(t-r_2) \end{bmatrix},
\end{aligned} \tag{34}
$$

$$
\begin{aligned}
\dot{V}_3(t) &= \begin{bmatrix} x(t) \\ \dot{x}(t) \end{bmatrix}^T \begin{bmatrix} Q_1 & Q_2 \\ * & Q_3 \end{bmatrix} \begin{bmatrix} x(t) \\ \dot{x}(t) \end{bmatrix} \\
&\quad -(1-\dot{h}(t)) \begin{bmatrix} x(t-h(t)) \\ \dot{x}(t-h(t)) \end{bmatrix}^T \begin{bmatrix} Q_1 & Q_2 \\ * & Q_3 \end{bmatrix} \begin{bmatrix} x(t-h(t)) \\ \dot{x}(t-h(t)) \end{bmatrix} \\
&= \begin{bmatrix} x(t) \\ \dot{x}(t) \end{bmatrix}^T \begin{bmatrix} Q_1 & Q_2 \\ * & Q_3 \end{bmatrix} \begin{bmatrix} x(t) \\ \dot{x}(t) \end{bmatrix} \\
&\quad -(1-\beta) \begin{bmatrix} x(t-h(t)) \\ \dot{x}(t-h(t)) \end{bmatrix}^T \begin{bmatrix} Q_1 & Q_2 \\ * & Q_3 \end{bmatrix} \begin{bmatrix} x(t-h(t)) \\ \dot{x}(t-h(t)) \end{bmatrix},
\end{aligned} \tag{35}
$$

$$
\begin{aligned}
\dot{V}_4(t) &= 2 \begin{bmatrix} x(t) \\ x(t-h(t)) \end{bmatrix}^T \begin{bmatrix} Q_4 & Q_5 \\ * & Q_6 \end{bmatrix} \begin{bmatrix} \dot{x}(t) \\ (1-\dot{h}(t))\dot{x}(t-h(t)) \end{bmatrix} \\
&= 2 \begin{bmatrix} x(t) \\ x(t-h(t)) \end{bmatrix}^T \begin{bmatrix} Q_4 & Q_5 \\ * & Q_6 \end{bmatrix} \begin{bmatrix} \dot{x}(t) \\ (1-\beta)\dot{x}(t-h(t)) \end{bmatrix}.
\end{aligned} \tag{36}
$$

The differential of $V_5(t)$ can be estimated as follows by Lemma 5

$$
\begin{aligned}
\dot{V}_5(t) &= h_2^2\dot{x}^T(t)(Z_1+Z_2)\dot{x}(t) - h_2\int_{t-h_2}^{t}\dot{x}^T(s)Z_1\dot{x}(s)\,ds \\
&\quad -h_2\int_{t-h_2}^{t}\dot{x}^T(s)Z_2\dot{x}(s)\,ds + r_2^2x^T(t)Z_3x(t) - r_2\int_{t-r_2}^{t}x^T(s)Z_3x(s)\,ds \\
&\le h_2^2\dot{x}^T(t)(Z_1+Z_2)\dot{x}(t) - h_2\int_{t-h_2}^{t}\dot{x}^T(s)Z_1\dot{x}(s)\,ds \\
&\quad +\begin{bmatrix} x(t) \\ x(t-h(t)) \\ x(t-h_2) \end{bmatrix}^T \begin{bmatrix} -Z_2 & Z_2-S & S \\ * & -2Z_2+S+S^T & Z_2-S \\ * & * & -Z_2 \end{bmatrix} \begin{bmatrix} x(t) \\ x(t-h(t)) \\ x(t-h_2) \end{bmatrix} \\
&\quad +r_2^2x^T(t)Z_3x(t) - \left(\int_{t-r(t)}^{t}x^T(s)ds\right)Z_3\left(\int_{t-r(t)}^{t}x(s)\,ds\right).
\end{aligned} \tag{37}
$$

The differential of $V_6(t)$ is computed by Lemma 4

$$
\begin{aligned}
\dot{V}_6(t) &= h_2^2\begin{bmatrix} x(t) \\ \dot{x}(t) \end{bmatrix}^T \begin{bmatrix} R_1 & R_2 \\ * & R_3 \end{bmatrix} \begin{bmatrix} x(t) \\ \dot{x}(t) \end{bmatrix} + (h_2-h_1)^2\begin{bmatrix} x(t) \\ \dot{x}(t) \end{bmatrix}^T \begin{bmatrix} R_4 & R_5 \\ * & R_6 \end{bmatrix} \begin{bmatrix} x(t) \\ \dot{x}(t) \end{bmatrix} \\
&\quad +r_2^2\begin{bmatrix} x(t) \\ \dot{x}(t) \end{bmatrix}^T \begin{bmatrix} R_7 & R_8 \\ * & R_9 \end{bmatrix} \begin{bmatrix} x(t) \\ \dot{x}(t) \end{bmatrix} + (r_2-r_1)^2\begin{bmatrix} x(t) \\ \dot{x}(t) \end{bmatrix}^T \begin{bmatrix} R_{10} & R_{11} \\ * & R_{12} \end{bmatrix} \begin{bmatrix} x(t) \\ \dot{x}(t) \end{bmatrix} \\
&\quad -h_2\int_{-h_2}^{0}\begin{bmatrix} x(t+s) \\ \dot{x}(t+s) \end{bmatrix}^T \begin{bmatrix} R_1 & R_2 \\ * & R_3 \end{bmatrix} \begin{bmatrix} x(t+s) \\ \dot{x}(t+s) \end{bmatrix} ds \\
&\quad -(h_2-h_1)\int_{-h_2}^{-h_1}\begin{bmatrix} x(t+s) \\ \dot{x}(t+s) \end{bmatrix}^T \begin{bmatrix} R_4 & R_5 \\ * & R_6 \end{bmatrix} \begin{bmatrix} x(t+s) \\ \dot{x}(t+s) \end{bmatrix} ds \\
&\quad -r_2\int_{-r_2}^{0}\begin{bmatrix} x(t+s) \\ \dot{x}(t+s) \end{bmatrix}^T \begin{bmatrix} R_7 & R_8 \\ * & R_9 \end{bmatrix} \begin{bmatrix} x(t+s) \\ \dot{x}(t+s) \end{bmatrix} ds \\
&\quad -(r_2-r_1)\int_{-r_2}^{-r_1}\begin{bmatrix} x(t+s) \\ \dot{x}(t+s) \end{bmatrix}^T \begin{bmatrix} R_{10} & R_{11} \\ * & R_{12} \end{bmatrix} \begin{bmatrix} x(t+s) \\ \dot{x}(t+s) \end{bmatrix} ds \\
&\le h_2^2\begin{bmatrix} x(t) \\ \dot{x}(t) \end{bmatrix}^T \begin{bmatrix} R_1 & R_2 \\ * & R_3 \end{bmatrix} \begin{bmatrix} x(t) \\ \dot{x}(t) \end{bmatrix} + (h_2-h_1)^2\begin{bmatrix} x(t) \\ \dot{x}(t) \end{bmatrix}^T \begin{bmatrix} R_4 & R_5 \\ * & R_6 \end{bmatrix} \begin{bmatrix} x(t) \\ \dot{x}(t) \end{bmatrix} \\
&\quad +r_2^2\begin{bmatrix} x(t) \\ \dot{x}(t) \end{bmatrix}^T \begin{bmatrix} R_7 & R_8 \\ * & R_9 \end{bmatrix} \begin{bmatrix} x(t) \\ \dot{x}(t) \end{bmatrix} + (r_2-r_1)^2\begin{bmatrix} x(t) \\ \dot{x}(t) \end{bmatrix}^T \begin{bmatrix} R_{10} & R_{11} \\ * & R_{12} \end{bmatrix} \begin{bmatrix} x(t) \\ \dot{x}(t) \end{bmatrix} \\
&\quad +\begin{bmatrix} x(t) \\ x(t-h(t)) \\ x(t-h_2) \\ \int_{t-h(t)}^{t}x(s)ds \\ \int_{t-h_2}^{t-h(t)}x(s)ds \end{bmatrix}^T \begin{bmatrix} -R_3 & R_3 & 0 & -R_2^T & 0 \\ * & -R_3-R_3^T & R_3 & R_2^T & -R_2^T \\ * & * & -R_3 & 0 & R_2^T \\ * & * & * & -R_1 & 0 \\ * & * & * & * & -R_1 \end{bmatrix} \begin{bmatrix} x(t) \\ x(t-h(t)) \\ x(t-h_2) \\ \int_{t-h(t)}^{t}x(s)ds \\ \int_{t-h_2}^{t-h(t)}x(s)ds \end{bmatrix} \\
&\quad +\begin{bmatrix} x(t-h_1) \\ x(t-h(t)) \\ x(t-h_2) \\ \int_{t-h(t)}^{t-h_1}x(s)ds \\ \int_{t-h_2}^{t-h(t)}x(s)ds \end{bmatrix}^T \begin{bmatrix} -R_6 & R_6 & 0 & -R_5^T & 0 \\ * & -R_6-R_6^T & R_6 & R_5^T & -R_5^T \\ * & * & -R_6 & 0 & R_5^T \\ * & * & * & -R_4 & 0 \\ * & * & * & * & -R_4 \end{bmatrix} \begin{bmatrix} x(t-h_1) \\ x(t-h(t)) \\ x(t-h_2) \\ \int_{t-h(t)}^{t-h_1}x(s)ds \\ \int_{t-h_2}^{t-h(t)}x(s)ds \end{bmatrix} \\
&\quad +\begin{bmatrix} x(t) \\ x(t-r(t)) \\ x(t-r_2) \\ \int_{t-r(t)}^{t}x(s)ds \\ \int_{t-r_2}^{t-r(t)}x(s)ds \end{bmatrix}^T \begin{bmatrix} -R_9 & R_9 & 0 & -R_8^T & 0 \\ * & -R_9-R_9^T & R_9 & R_8^T & -R_8^T \\ * & * & -R_9 & 0 & R_8^T \\ * & * & * & -R_7 & 0 \\ * & * & * & * & -R_7 \end{bmatrix} \begin{bmatrix} x(t) \\ x(t-r(t)) \\ x(t-r_2) \\ \int_{t-r(t)}^{t}x(s)ds \\ \int_{t-r_2}^{t-r(t)}x(s)ds \end{bmatrix} \\
&\quad +\begin{bmatrix} x(t-r_1) \\ x(t-r(t)) \\ x(t-r_2) \\ \int_{t-r(t)}^{t-r_1}x(s)ds \\ \int_{t-r_2}^{t-r(t)}x(s)ds \end{bmatrix}^T \begin{bmatrix} -R_{12} & R_{12} & 0 & -R_{11}^T & 0 \\ * & -R_{12}-R_{12}^T & R_{12} & R_{11}^T & -R_{11}^T \\ * & * & -R_{12} & 0 & R_{11}^T \\ * & * & * & -R_{10} & 0 \\ * & * & * & * & -R_{10} \end{bmatrix} \begin{bmatrix} x(t-r_1) \\ x(t-r(t)) \\ x(t-r_2) \\ \int_{t-r(t)}^{t-r_1}x(s)ds \\ \int_{t-r_2}^{t-r(t)}x(s)ds \end{bmatrix}.
\end{aligned} \tag{38}
$$

An upper bound of $\dot{V}_7(t)$ can be obtained by using Lemmas 2 and 6

$$\begin{aligned}
\dot{V}_7(t) &= \frac{h_2^2}{2}\dot{x}^T(t)W_1\dot{x}(t) + \frac{h_1^4}{2}\dot{x}^T(t)W_2\dot{x}(t) + \frac{r_2^2}{2}\dot{x}^T(t)W_3\dot{x}(t) + \frac{r_1^4}{2}\dot{x}^T(t)W_4\dot{x}(t) \\
&\quad - \int_{t-h_2}^{t}\int_{\theta}^{t}\dot{x}^T(u)W_1\dot{x}(u)dud\theta - h_1^2\int_{-h_1}^{0}\int_{s}^{0}\dot{x}^T(t+\theta)W_2\dot{x}(t+\theta)d\theta ds \\
&\quad + \int_{t-r_2}^{t}\int_{\theta}^{t}\dot{x}^T(u)W_3\dot{x}(u)duds - r_1^2\int_{-r_1}^{0}\int_{s}^{0}\dot{x}^T(t+\theta)W_4\dot{x}(t+\theta)d\theta ds \\
&\leq \frac{h_2^2}{2}\dot{x}^T(t)W_1\dot{x}(t) + \frac{h_1^4}{2}\dot{x}^T(t)W_2\dot{x}(t) + \frac{r_2^2}{2}\dot{x}^T(t)W_3\dot{x}(t) + \frac{r_1^4}{2}\dot{x}^T(t)W_4\dot{x}(t) \\
&\quad + h_1^2\begin{bmatrix} x(t) \\ \frac{1}{h_1}\int_{t-h_1}^{t}x(s)ds \end{bmatrix}^T \begin{bmatrix} -2W_2 & 2W_2 \\ * & -2W_2 \end{bmatrix} \begin{bmatrix} x(t) \\ \frac{1}{h_1}\int_{t-h_1}^{t}x(s)ds \end{bmatrix} \\
&\quad + r_1^2\begin{bmatrix} x(t) \\ \frac{1}{r_1}\int_{t-r_1}^{t}x(s)ds \end{bmatrix}^T \begin{bmatrix} -2W_4 & 2W_4 \\ * & -2W_4 \end{bmatrix} \begin{bmatrix} x(t) \\ \frac{1}{r_1}\int_{t-r_1}^{t}x(s)ds \end{bmatrix} \\
&\quad + \begin{bmatrix} x(t) \\ \frac{1}{h_2}\int_{t-h_2}^{t}x(s)ds \\ \frac{1}{h_2^2}\int_{t-h_2}^{t}\int_{u}^{t}x(s)dsdu \\ \frac{1}{h_2^3}\int_{t-h_2}^{t}\int_{u}^{t}\int_{s}^{t}x(\theta)d\theta dsdu \end{bmatrix}^T \begin{bmatrix} -12W_1 & 12W_1 & -120W_1 & 360W_1 \\ 12W_1 & -72W_1 & 480W_1 & -1080W_1 \\ -120W_1 & 480W_1 & -3600W_1 & 8640W_1 \\ 360W_1 & -1080W_1 & 8640W_1 & -21{,}600W_1 \end{bmatrix} \\
&\quad \times \begin{bmatrix} x(t) \\ \frac{1}{h_2}\int_{t-h_2}^{t}x(s)ds \\ \frac{1}{h_2^2}\int_{t-h_2}^{t}\int_{u}^{t}x(s)dsdu \\ \frac{1}{h_2^3}\int_{t-h_2}^{t}\int_{u}^{t}\int_{s}^{t}x(\theta)d\theta dsdu \end{bmatrix} \\
&\quad + \begin{bmatrix} x(t) \\ \frac{1}{r_2}\int_{t-r_2}^{t}x(s)ds \\ \frac{1}{r_2^2}\int_{t-r_2}^{t}\int_{u}^{t}x(s)dsdu \\ \frac{1}{r_2^3}\int_{t-r_2}^{t}\int_{u}^{t}\int_{s}^{t}x(\theta)d\theta dsdu \end{bmatrix}^T \begin{bmatrix} -12W_3 & 12W_3 & -120W_3 & 360W_3 \\ 12W_3 & -72W_3 & 480W_3 & -1080W_3 \\ -120W_3 & 480W_3 & -3600W_3 & 8640W_3 \\ 360W_3 & -1080W_3 & 8640W_3 & -21{,}600W_3 \end{bmatrix} \\
&\quad \times \begin{bmatrix} x(t) \\ \frac{1}{r_2}\int_{t-r_2}^{t}x(s)ds \\ \frac{1}{r_2^2}\int_{t-r_2}^{t}\int_{u}^{t}x(s)dsdu \\ \frac{1}{r_2^3}\int_{t-r_2}^{t}\int_{u}^{t}\int_{s}^{t}x(\theta)d\theta dsdu \end{bmatrix}. \qquad (39)
\end{aligned}$$

From the utilization of the zero equation, the following equation is true for real matrix L_1 with appropriate dimensions:

$$2\dot{x}^T(t)L_1^T\left[-\dot{x}(t) + Ax(t) + Bx(t-h(t)) + f(t,x(t)) + g(t,x(t-h(t))) + C\int_{t-r(t)}^{t}x(s)ds\right] = 0. \qquad (40)$$

From (4) and (5), we obtain, for any positive real constants ϵ_1 and ϵ_2,

$$0 \leq \epsilon_1\eta^2x^T(t)x(t) - \epsilon_1 f^T(t,x(t))f(t,x(t)), \qquad (41)$$
$$0 \leq \epsilon_2\rho^2x^T(t-h(t))x(t-h(t)) - \epsilon_2 g^T(t,x(t-h(t)))g(t,x(t-h(t))). \qquad (42)$$

According to (33)–(42), we can obtain:

$$\dot{V}(t) = \zeta^T(t)[\phi(\alpha,\beta)]\zeta(t) - h_2\int_{t-h_2}^{t}\dot{x}^T(s)Z_1\dot{x}(s)\,ds \qquad (43)$$

where:

$\zeta(t) = \left[\pi_1^T(t) \quad \pi_2^T(t) \quad \pi_3^T(t) \quad \pi_4^T(t) \quad \pi_5^T(t) \quad \pi_6^T(t) \quad \pi_7^T(t) \quad \pi_8^T(t) \quad \pi_9^T(t)\right]^T,$
$\pi_1(t) = \left[x^T(t) \quad \dot{x}^T(t) \quad x^T(t-h(t)) \quad \dot{x}^T(t-h(t)) \quad x^T(t-r(t)) \quad x^T(t-h_2)\right]^T,$
$\pi_2(t) = \left[x^T(t-r_2) \quad \frac{1}{h_2}\int_{t-h_2}^{t}x^T(s)ds \quad \frac{1}{r_2}\int_{t-r_2}^{t}x^T(s)ds \quad \frac{1}{h(t)}\int_{t-h(t)}^{t}x^T(s)ds\right]^T,$

$\pi_3(t) = \left[\frac{1}{h_2-h(t)}\int_{t-h_2}^{t-h(t)} x^T(s)ds \quad \int_{t-r(t)}^{t} x^T(s)ds \quad \int_{t-r_2}^{t-r(t)} x^T(s)ds\right]^T,$
$\pi_4(t) = \left[\frac{1}{h^2(t)}\int_{t-h(t)}^{t}\int_u^t x^T(s)dsdu \quad \frac{1}{h_2^2}\int_{t-h_2}^{t}\int_u^t x^T(s)dsdu\right]^T,$
$\pi_5(t) = \left[\frac{1}{(h_2-h(t))^2}\int_{t-h_2}^{t-h(t)}\int_u^{t-h(t)} x^T(s)dsdu \quad \frac{1}{r_2^2}\int_{t-r_2}^{t}\int_u^t x^T(s)dsdu\right]^T,$
$\pi_6(t) = \left[\frac{1}{h^3(t)}\int_{t-h(t)}^{t}\int_u^t\int_v^t x^T(s)dsdvdu \quad \frac{1}{h_2^3}\int_{t-h_2}^{t}\int_u^t\int_v^t x^T(s)dsdvdu\right]^T,$
$\pi_7(t) = \left[\frac{1}{(h_2-h(t))^3}\int_{t-h_2}^{t-h(t)}\int_u^{t-h(t)}\int_v^{t-h(t)} x^T(s)dsdvdu \quad \frac{1}{r_2^3}\int_{t-r_2}^{t}\int_u^t\int_v^t x^T(s)dsdvdu\right]^T,$
$\pi_8(t) = \left[f^T(t,x(t)) \quad g^T(t,x(t-h(t))) \quad \int_{t-h(t)}^{t-h_1} x^T(s)ds \quad \int_{t-r(t)}^{t-r_1} x^T(s)ds\right]^T,$
$\pi_9(t) = \left[x^T(t-h_1) \quad x^T(t-r_1) \quad \frac{1}{h_1}\int_{t-h_1}^{t} x^T(s)ds \quad \frac{1}{r_1}\int_{t-r_1}^{t} x^T(s)ds\right]^T.$

Let $\alpha = \frac{h(t)}{h_2}$, then $1-\alpha = \frac{h_2-h(t)}{h_2}$, and applying Lemma 8, we have:

$$\begin{aligned}
& -h_2\int_{t-h_2}^{t}\dot{x}^T(s)Z_1\dot{x}(s)\,ds \\
= & -h_2\int_{t-h(t)}^{t}\dot{x}^T(s)Z_1\dot{x}(s)\,ds - h_2\int_{t-h_2}^{t-h(t)}\dot{x}^T(s)Z_1\dot{x}(s)\,ds \\
\leq & -\frac{h_2}{h(t)}\zeta^T(t)\left[\Pi_1^TZ_1\Pi_1 + 3\Pi_2^TZ_1\Pi_2 + 5\Pi_3^TZ_1\Pi_3 + 7\Pi_4^TZ_1\Pi_4\right]\zeta(t) \\
& -\frac{h_2}{h_2-h(t)}\zeta^T(t)\left[\Pi_5^TZ_1\Pi_5 + 3\Pi_6^TZ_1\Pi_6 + 5\Pi_7^TZ_1\Pi_7 + 7\Pi_8^TZ_1\Pi_8\right]\zeta(t) \\
= & -\frac{1}{\alpha}\zeta^T(t)\left[\Pi_1^TZ_1\Pi_1 + 3\Pi_2^TZ_1\Pi_2 + 5\Pi_3^TZ_1\Pi_3 + 7\Pi_4^TZ_1\Pi_4\right]\zeta(t) \\
& -\frac{1}{1-\alpha}\zeta^T(t)\left[\Pi_5^TZ_1\Pi_5 + 3\Pi_6^TZ_1\Pi_6 + 5\Pi_7^TZ_1\Pi_7 + 7\Pi_8^TZ_1\Pi_8\right]\zeta(t) \\
= & -\zeta^T(t)\Upsilon^T\begin{bmatrix}\frac{1}{\alpha}\Theta & 0\\ 0 & \frac{1}{1-\alpha}\Theta\end{bmatrix}\Upsilon\zeta(t).
\end{aligned} \tag{44}$$

For any matrices $M_1, M_2 \in \mathbb{R}^{29n\times 4n}$ and applying Lemma 7, we can obtain:

$$\begin{aligned}
& -\Upsilon^T\begin{bmatrix}\frac{1}{\alpha}\Theta & 0\\ 0 & \frac{1}{1-\alpha}\Theta\end{bmatrix}\Upsilon \\
\leq & -\Upsilon^T\Sigma(\alpha)\Upsilon - sym\left(\Upsilon^T\begin{bmatrix}(1-\alpha)M_1{}^T\\ \alpha M_2{}^T\end{bmatrix}\right) + \alpha M_1\Theta^{-1}M_1{}^T \\
& +(1-\alpha)M_2\Theta^{-1}M_2{}^T = \Delta(\alpha).
\end{aligned} \tag{45}$$

From (43)–(45), we get:

$$\dot{V}(t) \leq \zeta^T(t)(\phi(\alpha,\beta) + \Delta(\alpha))\zeta(t),$$

By Lemma 2 [10], if LMIs (27) and (28) are true for $\alpha = \{0,1\}, \beta = \{-u,u\}$, then $\phi(\alpha,\beta) + \Delta(\alpha) < 0$ holds for all $\alpha \in (0,1)$. $\beta \in [-u,u]$. By Lemma 9, LMIs (27) and (28) hold if and only if LMIs (29)–(32) hold. This completes the proof. □

We now introduce the following notations for later use

$$\hat{\Phi}(\alpha,\beta) = \begin{bmatrix}\hat{\phi}(\alpha,\beta) - \hat{\Upsilon}^T\Sigma(\alpha)\hat{\Upsilon} - Sym\left(\hat{\Upsilon}^T\begin{bmatrix}(1-\alpha)M_1^T\\ \alpha M_2^T\end{bmatrix}\right) & \alpha M_1 + (1-\alpha)M_2\\ * & -\hat{\Theta}\end{bmatrix},$$

$$\begin{aligned}
\hat{\phi}(\alpha,\beta) = & Sym(\hat{\Pi}_1^TP_1\hat{\Pi}_2) + Sym(\hat{\Pi}_5^T\hat{X}_2\hat{\Pi}_6) + Sym(\hat{\Pi}_9^T\hat{X}_4\hat{\Pi}_{10}) + \hat{\Pi}_{11}^T\hat{X}_5\hat{\Pi}_{11} \\
& -(1-\beta)\hat{\Pi}_{12}^T\hat{X}_5\hat{\Pi}_{12} + \hat{\Pi}_{13}^T\hat{X}_6\hat{\Pi}_{14} + h_2^2\hat{\Pi}_{15}^TZ_1\hat{\Pi}_{15} + h_2^2\hat{\Pi}_{15}^TZ_2\hat{\Pi}_{15} \\
& + r_2^2\hat{\Pi}_1^TZ_3\hat{\Pi}_1 + \hat{\Pi}_{16}^T\hat{X}_{11}\hat{\Pi}_{16} + \hat{\Pi}_{17}^TZ_3\hat{\Pi}_{17} + h_2^2\hat{\Pi}_{11}^T\hat{X}_7\hat{\Pi}_{11} \\
& + r_2^2\hat{\Pi}_{11}^T\hat{X}_9\hat{\Pi}_{11} + \hat{\Pi}_{18}^T\hat{X}_{12}\hat{\Pi}_{18} + \hat{\Pi}_{20}^T\hat{X}_{14}\hat{\Pi}_{20} + \frac{h_2^2}{2}\hat{\Pi}_{15}^TW_1\hat{\Pi}_{15} \\
& + \frac{r_2^2}{2}\hat{\Pi}_{15}^TW_3\hat{\Pi}_{15} + \hat{\Pi}_{24}^T\hat{X}_{18}\hat{\Pi}_{24} + \hat{\Pi}_{25}^T\hat{X}_{19}\hat{\Pi}_{25} + \hat{\Pi}_{15}^TL_1\hat{\Pi}_{26}
\end{aligned}$$

$$
\begin{aligned}
&+\epsilon_1\eta^2\hat{\Pi}_1^T I\hat{\Pi}_1-\epsilon_2\hat{\Pi}_{27}^T I\hat{\Pi}_{27}+\epsilon_2\rho^2\hat{\Pi}_{28}^T I\hat{\Pi}_{28}-\epsilon_2\hat{\Pi}_{29}^T I\hat{\Pi}_{29},\\
&\hat{\Pi}_i=\Pi_i, i=1,2,\cdots,29, \quad \hat{X}_{1j}=X_{1j}, j=1,2,\cdots,9, \quad \hat{\chi}_k=\chi_k, k=1,2,\cdots,8,\\
&\hat{\Theta}=\Theta, \quad \hat{\Upsilon}=\Upsilon, \quad \text{and } \hat{\varepsilon}_i=\varepsilon_i, i=1,2,\cdots,23.
\end{aligned}
$$

Corollary 1. *For given positive constants h_2, r_2, u, if there exist symmetric positive definite matrices P_1, Z_1, Z_2, Z_3, R_1, R_3, R_7, R_9, W_1, W_3, X_l, $l=2,4,5,6,7,9$, any appropriate dimensional matrices M_1, M_2, R_2, R_8, S, P_i, Q_j, R_k, $i=5,6,7,11,12,13$, $j=1,2,\ldots,6$, and positive constants η, ρ such that the following LMIs hold:*

$$\hat{\Phi}(\alpha,\beta) < 0, \tag{46}$$

$$\begin{bmatrix} Z_2 & S \\ * & Z_2 \end{bmatrix} > 0, \tag{47}$$

holds for $\alpha=\{0,1\}, \dot{r}(t)=\beta=\{-u,u\}$, i.e.,

$$\hat{\Phi}(0,-u) < 0, \tag{48}$$

$$\hat{\Phi}(0,u) < 0, \tag{49}$$

$$\hat{\Phi}(1,-u) < 0, \tag{50}$$

$$\hat{\Phi}(1,u) < 0, \tag{51}$$

then the system (9) is asymptotically stable.

4. Numerical Examples

Example 1. *Consider the system (1) with:*

$$A=\begin{bmatrix} -1.2 & 0.1 \\ -0.1 & -1.0 \end{bmatrix}, B=\begin{bmatrix} -0.6 & 0.7 \\ -1.0 & -0.8 \end{bmatrix}, \eta \geq 0, \rho \geq 0. \tag{52}$$

By using the LMIs Toolbox in **MATLAB** *(with an accuracy of 0.01) for the application of Theorem 1 to System (1) with (52), the maximum upper bounds h_2 for the asymptotic stability of Example 1 are listed in the comparison in Table 1 for different values of h_1 and η. Table 1 shows that the results derived in this research are less conservative than the results in [4,9,14,25,28].*

Table 1. Upper bounds h_2 for different conditions for Example 1 with $r_1=h_1, r_2=h_2$.

h_1	Methods	$u=0.9,\rho=0.1$	$u=0.9,\rho=0.1$
		$\eta=0$	$\eta=0.1$
0.5	Zhang et al. (2010) [25]	1.338	1.245
	Ramakrishnan and Ray (2011) [14]	1.558	1.384
	Hui et al. (2015) [4]	1.824	1.524
	Zhou et al. (2014) [28]	1.8599	1.6622
	Liu (2015) [9]	2.1714	1.9573
	Theorem 1	4.5821	4.3269
1	Zhang et al. (2010) [25]	1.543	1.408
	Ramakrishnan & Ray (2011) [14]	1.760	1.543
	Hui et al. (2015) [4]	1.993	1.638
	Zhou et al. (2014) [28]	2.065	1.8188
	Liu (2015) [9]	2.2749	1.9629
	Theorem 1	4.7418	4.4751

Example 2. *Consider the system (6) with:*

$$A=\begin{bmatrix} 0.0 & 1.0 \\ -1.0 & -2.0 \end{bmatrix}, B=\begin{bmatrix} 0.0 & 0.0 \\ -1.0 & 1.0 \end{bmatrix}. \tag{53}$$

Table 2 shows some calculation results obtained from the application of Corollary 1 to System (6) with (53). The values in Table 2 are the maximum upper with (53). The values in Table 2 are the maximum upper bounds on the delay h_2 under different values of u than those in [6–8,15,22,24,26].

Table 2. Upper bounds of time-delay h_2 for different conditions for Example 2 with $r_1 = h_1 = 0$, $r_2 = h_2$, $\eta = 0$, $\rho = 0$.

Methods	$u = 0.1$	$u = 0.2$	$u = 0.5$	$u = 0.8$
[15]	6.590	3.672	1.411	1.275
[6]	7.125	4.413	2.243	1.662
[24]	7.148	4.466	2.352	1.768
[7]	7.167	4.517	2.415	1.838
[26]	7.230	4.556	2.509	1.940
[8]	7.297	4.625	2.264	2.038
[22]	10.095	6.808	3.676	2.615
Corollary 1	11.018	7.612	4.340	3.289

Example 3. *Consider the system (6) with:*

$$A = \begin{bmatrix} -2.0 & 0.0 \\ 0.0 & -0.9 \end{bmatrix}, B = \begin{bmatrix} -1.0 & 0.0 \\ -1.0 & -1.0 \end{bmatrix}. \tag{54}$$

For different u, Table 3 presents the allowable upper bound of $h(t)$, which guarantees the stability of System (6). Table 3 shows that our method produces a larger upper bound h_2 than those in [5,8,15,16,24,26].

Table 3. Upper bounds of time-delay h_2 for different conditions for Example 3 with $r_1 = h_1 = 0$, $r_2 = h_2$, $\eta = 0$, $\rho = 0$.

Methods	$u = 0.1$	$u = 0.2$	$u = 0.5$	$u = 0.8$
[15]	4.703	3.834	2.420	2.137
[5]	4.753	-	2.429	2.183
[24]	4.788	4.060	3.055	2.615
[16]	4.930	4.220	3.090	2.660
[26]	4.910	-	3.233	2.789
[8]	4.996	4.308	3.251	2.867
[22]	5.650	4.913	3.793	3.251
Corollary 1	6.014	5.371	4.105	3.561

Example 4. *Consider the system (9) with:*

$$A = \begin{bmatrix} -0.9 & 0.2 \\ 0.1 & -0.9 \end{bmatrix}, B = \begin{bmatrix} -1.1 & -0.2 \\ -0.1 & -1.1 \end{bmatrix}, C = \begin{bmatrix} -0.2 & 0 \\ 0.2 & -0.1 \end{bmatrix}, \eta \geq 0, \rho \geq 0. \tag{55}$$

By using the LMI Toolbox in MATLAB for Theorem 1 to (55), one can obtain the maximum upper bounds of the time-delay h_2 for the asymptotic stability of Example 4, which are listed in Table 4 for different values of u and r_2.

Table 4. Upper bounds of time-delay h_2 for different conditions for Example 4.

Theorem 1	$h_1 = 0.1, r_1 = 0.1, \eta = 0.1, \rho = 0.05$	
	$r_2 = 5$	$r_2 = 9$
$u = 0.1$	0.7200	0.5213
$u = 0.5$	0.7021	0.4617
$u = 0.9$	0.6732	0.4275

5. Conclusions

In this paper, we focus on the problem of asymptotic stability criteria for linear systems with distributed interval time-varying delays and nonlinear perturbations without using the model transformation and delay-decomposition approach. Firstly, we obtain the new asymptotic stability criteria for the uncertain linear systems by using a suitable Lyapunov–Krasovskii functional, an improved Peng–Par integral inequality, and a novel triple integral inequality. Finally, we demonstrate numerical examples that are less conservative than other literature examples.

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