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Review

The Spectral Condition, Plane Waves, and Harmonic Analysis in de Sitter and Anti-de Sitter Quantum Field Theories

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Abstract: We review the role of the spectral condition as a characteristic of Minkowski, de Sitter, and anti-de Sitter quantum field theories. We also discuss the role of plane waves that are compatible with the relevant analyticity domains linked to the spectral condition(s) and discuss harmonic analysis in terms of them.

Keywords: de Sitter; anti-de Sitter; spectral condition

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1. The Birth of the de Sitter Model

After writing his equations for the geometry of spacetime in December 1915, Einstein turned his attention to cosmology and tried to apply the equations to the entire Universe, creating an entirely new science—modern scientific cosmology—whose founding idea is that a global exact solution of Einstein's equations corresponds somehow to a model for the Universe.

Einstein's concern was, at first, epistemological: the metric structure of the Universe must be entirely determined by the material content—this is more or less the so-called Mach principle. But general relativity still retains a remnant of absolute space in the boundary conditions that must be specified at spacelike infinity to determine the spacetime geometry. To solve this problem, or rather, to dispose of it, Einstein's "crazy idea" was to let the Universe be spherical, with spherical spatial sections.

A curved sphere should be imagined as a three-dimensional spherical hypersurface embedded in a Euclidean space of dimension four:

$$S_3 = \{x_1^2 + x_2^2 + x_3^2 + x_4^2 = r^2\}. \tag{1}$$

It obviously has no centre, or rather, it has its centre everywhere[1], and any point is equivalent to any other point. It has no boundary either; therefore, there are no boundary conditions to consider.

There was also a second guiding principle in Einstein's cosmological research: The Universe had to be static and its geometry should not change over time. In 1917, the visible Universe still coincided with the Milky Way, the nebulae enigma had not yet been solved, and the hypothesis of a static Universe was perfectly reasonable. But, Einstein's General Theory of Relativity of 1915 does not allow for spherical static solutions.

Then came an idea that would be remembered as his *biggest blunder:* to add to his equations a constant term Λ that acts repulsively and counteracts the gravitational attraction. This was, to quote Einstein,

> *an* extension of the equations which is not justified by our real knowledge of gravitation [...] *this term is necessary only for the purpose of making possible a quasi-static distribution of matter as required by the low speed of stars* [1].

This commentary indicates that in his 1917 paper, Einstein was already aware of the fact that his original equations of 1915 implied a dynamical Universe, but that he had set aside this possibility. He kept adding Λ and found a perfectly Machian static spherical solution—his static model of 1917.

Shortly after the publication of Einstein's paper, de Sitter published a second solution of the new cosmological equations of 1917: an otherwise empty Universe made only by the cosmological constant. The astronomer's model elaborated on the boundary condition problem. According to de Sitter, Einstein's solution still retained a trace of absolute space; a four-dimensional (complex) sphere could solve the problem in a more convincingly covariant way. As for the sphere (1), the de Sitter model can be better visualised as an embedded surface: it is the four-dimensional one-sheeted hyperboloid embedded in a five-dimensional Minkowski spacetime M_5

$$dS_4 = \{x_0^2 - x_1^2 - x_2^2 - x_3^2 - x_4^2 = -R^2\}. \tag{2}$$

Einstein was very unhappy with the new solution, but all his attempts to demonstrate that de Sitter's calculations were faulty consistently failed. Einstein finally surrendered: the de Sitter Universe was indeed a regular solution of his cosmological equations without matter, but, he said, it was nevertheless without physical interest because it was not globally static. Einstein rejected the possibility of a dynamical Universe, but other scientists simply did not know: until the early 1930s, the fundamental articles published in 1922 [2] and 1924 [3] by Friedmann, who made use of the original equations of general relativity to describe expanding universes, were substantially ignored. Lemaître's independent work in 1927 [4], based on the cosmological equations of 1917, was also ignored.

On a trip to Pasadena, Einstein learned of Hubble's latest observations and was persuaded of the advantages of dynamic models to describe the Universe. In two articles published shortly afterwards, Einstein asserted that the original reasons for introducing the cosmological constant no longer existed. Farewell to the cosmological constant.

In 1947, Einstein wrote to Lemaître:

> *The introduction of such a constant implies a considerable renunciation of the logical simplicity of the theory... Since I introduced this term, I had always a bad conscience ... I am unable to believe that such an ugly thing should be realized in nature.*

Lemaître's answer in 1949 sounds like a prophecy.

> *The history of science provides many instances of discoveries which have been made for reasons which are no longer considered satisfactory. It may be that the discovery of the cosmological constant is such a case.*

In fact, Einstein himself had been prophetic in 1917 in a letter to de Sitter.

> *In any case, one thing is clear. The theory of general relativity allows adding the term Λ in the equations. One day, our real knowledge of the composition of the sky of fixed stars, the apparent motions of the fixed stars and the position of spectral lines as a function of distance, will probably be sufficient to decide empirically whether or not Λ is equal to zero. Conviction is a good motive, but a bad judge.*

In 1997, exactly eighty years after its discovery, the cosmological constant was observed [5,6]; or maybe, it was something similar that we now call "dark energy". These observations have upturned consolidated and rooted ideas, indicating that the gravitational effect of the greatest part of the energy of the Universe consists of producing an accelerated expansion, as in the case of Einstein's cosmological constant. Nowadays, almost every physicist believes that the dark component constitutes about seventy percent of the energy of the Universe and that its proportion, according to the standard cosmological ΛCDM (cold dark matter) model, is destined to increase. In the end, only the cosmological constant will remain, and the Universe will become a perfect de Sitter spacetime.

The de Sitter geometry, therefore, seems to assume the role of the reference geometry of the Universe. In other words, it is de Sitter's geometry, not Minkowski's, that describes spacetime when matter and radiation are absent.

Beyond the acceleration of the Universe at late times, the idea of inflation consists of a phase of accelerated quasi-exponential expansion, approximately described by de Sitter's geometry in the primordial Universe. A theoretical understanding of the structure of the Universe, which is observable today, is based on quantum field theory on de Sitter spacetime: quantum fluctuations of the vacuum at the epoch of inflation are thought to be responsible for the primordial density inhomogeneities that are at the origin of the structures existing in the Universe today.

Actually, once one admits that a cosmological constant may exist, it might also be negative, correct? The model of the Universe with a negative cosmological constant and nothing else is termed anti-de Sitter. It is a curious coincidence that in the very same year, 1997, the negative cosmological constant also took centre stage in theoretical physics [7] with the formulation of the by-now famous AdS/CFT (Anti-de Sitter/Conformal Field Theory) correspondence, a conjectured duality between two different physical theories—1997, the year of the two cosmological constants!

2. Quantum Field Theory: The Spectral Condition

The de Sitter and anti-de Sitter spacetimes thus have great importance in contemporary theoretical physics and cosmology, and both dS and AdS quantum field theory (QFT) also play a major role. The dS and AdS manifolds share the properties of having constant curvature and being maximally symmetric manifolds. Actually, in the general d-dimensional case, they are just different real submanifolds of one and the same complex manifold: the complex d-dimensional sphere

$$S_d^{(c)} = \{z \in \mathbf{C}^{d+1};\ z_0^2 + z_1^2 + \ldots + z_d^2 = R^2\}. \tag{3}$$

However, their geometries are radically different from each other. In particular, the (real) de Sitter manifold has no global timelike Killing vector field, while the (real) anti-de Sitter manifold is not globally hyperbolic and has closed timelike curves. One can eliminate these closed curves by moving to the universal covering of the real AdS manifold (even though this move might be just an illusion), but the universal covering remains not globally hyperbolic.

Global hyperbolicity is a basic property of quantum field theory on curved spacetimes, as it is usually formulated. Its absence renders AdS QFT a little more demanding from a technical viewpoint. But, as we will see, this is not a major difficulty since in AdS, there exists the possibility of identifying a global energy operator. It is precisely the lack of a global energy operator, which is a consequence of the absence of a global timelike Killing vector field, which renders dS QFT more difficult.

There is, however, a unifying characteristic that makes dS and AdS QFTs similar to each other and similar to the standard zero-temperature Minkowski QFT: this is the analyticity of the correlation functions in suitable domains of the respective complexified manifolds. This unifying viewpoint is discussed in the following sections.

Here, to prepare the groundwork, we start by recalling that the fundamental theorem of Stone and Von Neumann, which states the uniqueness of the Hilbert space representation of the canonical commutation relations (CCRs), fails for infinite quantum systems. The distinction between observables and states, which is of no consequence for finitely many degrees of freedom, now becomes crucial, and there exist infinitely many Hilbert space realisations of the same algebra of the observables. In other words, knowing the Lagrangian of a quantum field theory is not enough. The Lagrangian just provides the commutation rules, but there are infinitely many inequivalent solutions of the field equations sharing the same commutation rules; one needs to specify some extra information to find the physically relevant ones. Only after this step has been taken can transition amplitudes be computed and comparisons with the outcomes of the experiments be performed.

3. States and Two-Point Functions

This non-uniqueness is true at the level of free fields. What is unique is the commutator: on a globally hyperbolic manifold $(\mathcal{M}, g)$, the Klein–Gordon Lagrangian uniquely selects the (covariant) commutator $C(x_1, x_2)$, which is an *antisymmetric* bi-distribution solving the Klein–Gordon equation in each variable

$$(\Box_{x_1} + m^2)C(x_1, x_2) = (\Box_{x_2} + m^2)C(x_1, x_2) = 0 \tag{4}$$

with the precise initial condition given by the equal-time canonical commutation relations. The equal-time CCRs, in turn, imply that $C(x_1, x_2) = 0$ for any two events x_1, x_2 of $\mathcal{M}$, which are spacelike-separated with respect to the notion of locality inherent to $\mathcal{M}$.

For free fields, the smeared commutator is a multiple of the identity element of the field algebra (a *c-number*). Given two test functions f and g belonging to a suitable test function space $\mathcal{T}(\mathcal{M})$,

$$[\phi(f), \phi(g)] = C(f, g) \; = \int_{\mathcal{M}\times\mathcal{M}} C(x_1, x_2) f(x_1) g(x_2) \sqrt{-g(x_1)} dx_1 \sqrt{-g(x_2)} dx_2. \tag{5}$$

A quantisation is accomplished when the commutation relations (5) are represented by an operator-valued distribution in a Hilbert space $\mathcal{H}$. One should determine a linear map

$$\phi(f) \longrightarrow \widehat{\phi}(f) \in Op(\mathcal{H}) \tag{6}$$

preserving the algebraic structures such that

$$[\widehat{\phi}(f), \widehat{\phi}(g)] = C(f, g)\, \mathbf{1}. \tag{7}$$

As we stated, the Stone–Von Neumann theorem fails, and there are infinitely many solutions to this problem. How can we construct (at least some of) them?

A possible solution is completely encoded in the knowledge of a two-point function, i.e., a two-point distribution $\mathcal{W} \in \mathcal{T}'(\mathcal{M} \times \mathcal{M})$ that solves the Klein–Gordon equation in each variable

$$\left(\Box_{x_1} + m^2\right) \mathcal{W}(x_1, x_2) = \left(\Box_{x_2} + m^2\right) \mathcal{W}(x_1, x_2) = 0\,. \tag{8}$$

Because of Equation (7), $\mathcal{W}(x_1, x_2)$ is also required to be a solution of the functional equation

$$\mathcal{W}(x_1, x_2) - \mathcal{W}(x_2, x_1) = C(x_1, x_2) \tag{9}$$

in the sense of distributions.

Starting from $\mathcal{W}(x_1, x_2)$, the Hilbert space of the theory $\mathcal{H}$ can be constructed using standard techniques [8]. The first step consists of giving a norm to the one-particle state Ψ_f corresponding to a given test function $f \in \mathcal{T}(\mathcal{M})$. The norm is computed using the two-point function:

$$||\Psi_f||^2 = \int_{\mathcal{M}\times\mathcal{M}} \mathcal{W}(x_1, x_2) f^*(x_1) f(x_2) \sqrt{-g(x_1)} dx_1 \sqrt{-g(x_2)} dx_2. \tag{10}$$

The squared norm (10) is positive (as it should be) if $\mathcal{W}(x_1, x_2)$ satisfies the *positive-definiteness condition*, which is simply the nonnegativity of the right-hand side of Equation (10). We assume that it does.

The norm (10) actually comes from a pre-Hilbert scalar product whose interpretation is that of providing the quantum transition amplitudes between two one-particle states:

$$\langle \Psi_f, \Psi_g \rangle = \int_{M_d} \mathcal{W}(x_1, x_2) f^*(x_1) g(x_2) \sqrt{-g(x_1)} dx_1, \sqrt{-g(x_2)} dx_2. \tag{11}$$

The one-particle Hilbert space $\mathcal{H}^{(1)}$ is obtained by quotienting out the subspace of zero-norm states and by taking the Hilbert completion. The full Hilbert space is the symmetric Fock space

$$\mathcal{H} = F_s(\mathcal{H}^{(1)}) = \mathcal{H}_0 \oplus [\oplus_n Sym(\mathcal{H}_1)^{\otimes n}]$$

(with Sym denoting the symmetrisation operation and $\mathcal{H}_0 = \{\lambda 1, \lambda \in \mathbf{C}\}$). In the final step, one introduces the field operator $\widehat{\phi}(f)$ decomposed into its "creation" and "annihilation" parts

$$\widehat{\phi}(f) = \widehat{\phi}^+(f) + \widehat{\phi}^-(f); \tag{12}$$

the latter are defined by their action on the dense subspace $\mathcal{H}^{(0)}$ of vectors having finitely many non-vanishing components $\Psi = (\Psi_0, \Psi_1, \ldots \Psi_k, \ldots, 0, 0, 0, \ldots)$:

$$\left(\widehat{\phi}^-(f)\Psi\right)_n = \sqrt{n+1}\int \mathcal{W}(x,x')f(x)\Psi_{n+1}(x',x_1,\ldots,x_n)\sqrt{-g(x)}dx\sqrt{-g(x')}dx', \tag{13}$$

$$\left(\widehat{\phi}^+(f)\Psi\right)_n = \frac{1}{\sqrt{n}}\sum_{j=1}^{n} f(x_j)\Psi_{n-1}(x_1,\ldots,\cancel{x}_j,\ldots,x_n). \tag{14}$$

Equation (9) shows that these formulae imply the commutation relations (7) and that

$$\mathcal{W}(x,x') = \langle \Psi_0, \widehat{\phi}(x)\widehat{\phi}(x')\Psi_0 \rangle \tag{15}$$

where

$$\Psi_0 = (1, 0, 0, \ldots,) \tag{16}$$

is the cyclic reference state of the representation.

In the end, either in flat or curved spacetime, quantizing a free-field theory amounts to specifying its two-point function, which carries all the information about the Hilbert space and the field operators. Furthermore, the knowledge of the two-point function and the commutator allows us to determine the Green's functions, modulo the necessary renormalisations; thus, the two-point function encodes not only the dynamics of the free field but also the possibility of studying interactions perturbatively.

But, how do we specify a criterion to choose among the infinitely many existing possibilities? Here, the spectral condition comes into play.

4. Prelude: The Spectral Condition in Minkowski Space

This section contains material that can be found in (good) textbooks. The reason to recall it here is to better appreciate and understand the role of the spectral condition and plane waves in the de Sitter and anti-de Sitter contexts.

On page 97 of the classic book by R. Streater and A.S. Wightman, the following basic assumption about a relativistic quantum field theory is declared:

> *Axiom 0. Assumptions of Relativistic Quantum Theory.*
>
> The states of the theory are described by unit rays in a separable Hilbert space $\mathcal{H}$. The relativistic transformation law of the states is given by a continuous unitary representation of the inhomogeneous Lorentz group $\{a, A\} \to U(a, A)$. Since $U(a, 1)$ is unitary, it can be written as $U(a, 1) = \exp(ia_\mu P^\mu)$ where P^μ is an unbounded operator interpreted as the energy momentum operator of the theory. The eigenvalues of P^μ is lie in or on the forward cone (*spectral condition*). There is an invariant state Ψ_0, $\;U(a, 1)\Psi_0 = \Psi_0$ unique up to a constant phase factor (*uniqueness of the vacuum*).

Stated more succinctly:

> The joint spectrum of the infinitesimal generators of $U(a, 1)$ lies in the closed forward cone $\overline{V}_+$.

This is the spectral condition of standard (zero-temperature) QFT. It is its most important and characteristic feature. All the other axioms are of a kinematical character[2].

Here, we consider a general d-dimensional Minkowski spacetime M_d with metric

$$\eta_{\mu\nu} = diag(+, -, \ldots, -) \tag{17}$$

and one scalar field. The open future cone of the origin (also called the forward cone) is the set

$$V_+ \quad = \quad \{x \in M_d \ : \ x \cdot x > 0, \quad x^0 > 0\}. \tag{18}$$

Given the n-point vacuum expectation values of the field (in short, the n-point functions):

$$\mathcal{W}_n(x_1, \ldots x_n) = \langle \Psi_0, \widehat{\phi}(x_1) \ldots \widehat{\phi}(x_n)\Psi_0 \rangle, \tag{19}$$

the spectral condition is immediately translated into a property of the support of their Fourier transforms $\widetilde{W}_n(p_1, \ldots, p_n)$. The distribution

$$\widetilde{\mathcal{W}_n}(p_1, \ldots, p_n) = \int e^{ip_1 \cdot x_1 + \ldots + ip_n \cdot x_n} \mathcal{W}_n(x_1, \ldots, x_n) dx_1 \ldots dx_n \tag{20}$$

vanishes unless all momenta are in the energy-momentum spectrum of the states

$$p_1 \in \overline{V}_+, \quad p_1 + p_2 \in \overline{V}_+, \ldots \quad p_1 + p_2 + \ldots + p_n \in \overline{V}_+. \tag{21}$$

By Fourier–Laplace transform, support properties in one space give rise to analyticity properties in the dual space [8]. A fundamental theorem of this category shows that the n-point distributions are boundary values of n-point functions holomorphic in tubular domains of the complex Minkowski spacetime.

Theorem 1 (A.S. Wightman)**.** *The distribution $\mathcal{W}_n(x_1, \ldots x_n)$ is the boundary value of a function $W_n(z_1, \ldots z_n)$ holomorphic in the tube*

$$T_n = \{(z_1, \ldots z_n) \ : \ \mathrm{Im}\,(z_{j+1} - z_j) \in \overline{V}_+\}. \tag{22}$$

Wightman's reconstruction theorem [8] finally states the *equivalence of the analyticity of the n-point function in the tubes T_n and the spectral condition*: starting from a set of Wightman functions with such analyticity properties, it is possible to reconstruct the Hilbert space of the theory, the representation of the inhomogeneous Lorentz group, and the infinitesimal generators of the translation group and verify that their joint spectrum is contained in the closed forward cone.

The above analyticity properties and the spectral condition, therefore, have the same precise *physical meaning*: they assert that the states of the theory have positive energy in every Lorentz frame.

Focusing now on two-point functions, the spectral condition is equivalent to the following simpler property.

Corollary 1 (Normal analyticity property)**.** *$\mathcal{W}(x_1, x_2)$ is the boundary value of a function $W(z_1, z_2)$ holomorphic in the tube $T_{12} = T_- \times T_+$:*

$$\mathcal{W}(x_1, x_2) = \langle \Psi_0, \widehat{\phi}(x_2)\widehat{\phi}(x_1)\Psi_0 \rangle = \underset{\substack{T_- \ni z_1 \to x_1 \\ T_+ \ni z_2 \to x_2}}{b.v.} W(z_1, z_2) \tag{23}$$

where

$$T_\pm = \{(z = x + iy : \quad \pm y \in \overline{V}_+\} \tag{24}$$

are the past and future tubes.

The tubes $T_\pm$ are invariant under the action of the real inhomogeneous Lorentz group. Acting with the complex group, one discovers that every Lorentz invariant two-point function satisfying the spectral condition enjoys a much larger analyticity domain.

Theorem 2 (Maximal analyticity property)**.** *1. The two-point function $W(z_1, z_2)$ depends only on the Lorentz-invariant variable $\lambda = (z_1 - z_2)^2$.*

2. *$W(z_1, z_2)$ can be continued to a function $\mathfrak{W}(z_1, z_2)$ analytic in the cut domain*

$$\Delta_0 = \{(z_1, z_2);\ \ (z_1 - z_2)^2 \neq \rho,\ \ \rho \geq 0\} \tag{25}$$

that contains all pairs of complex events with the exception of all pairs of real events that are causally connected (the causal cut).

3. *$\mathfrak{W}(z_1, z_2)$ is invariant in Δ_0 under the action of the complex inhomogeneous Lorentz group.*
4. *The permuted two-point function is the boundary value of $\mathfrak{W}(z_1, z_2)$ from the opposite tube $T_{21} = T_+ \times T_-$:*

$$\mathcal{W}(x_2, x_1) = \langle \Psi_0, \widehat{\phi}(x_2)\widehat{\phi}(x_1)\Psi_0 \rangle = \underset{\substack{T_+ \ni z_1 \to x_1 \\ T_- \ni z_2 \to x_2}}{b.v.} \mathfrak{W}(z_1, z_2). \tag{26}$$

5. *The cut domain Δ_0 contains all pairs of non-coinciding Euclidean points*

$$\dot{\mathcal{E}} = \{z_1, z_2 \in \Delta,\ \ \mathrm{Re}\, z_1^0 = \mathrm{Re}\, z_2^0 = 0,\ \mathrm{Im}\, z_1^i = \mathrm{Im}\, z_2^i = 0,\ i = 1, \ldots, d-1,\ \ z_1 \neq z_2\}. \tag{27}$$

The Schwinger function S (also called the Euclidean propagator) is the restriction of $\mathfrak{W}(z_1, z_2)$ to the non-coincident Euclidean points $\dot{\mathcal{E}}$. S is analytic in $\dot{\mathcal{E}}$ and can be extended as a distribution to the whole Euclidean space $\mathcal{E}$, including the coinciding points.

Klein–Gordon Fields

Now, let us see how the spectral condition works in practice for Klein–Gordon fields. We begin by identifying a suitable basis of solutions of the Klein–Gordon operator. In flat space, the exponential plane waves are almost always the convenient choice since they are also characters of the translation group:

$$\psi^{(\pm)}_{\vec{p}}(x) = \frac{1}{2\sqrt{(2\pi)^{d-1}\omega}} \exp(\pm ipx), \quad p^0 = \omega = \sqrt{|\vec{p}|^2 + m^2}. \tag{28}$$

The above plane waves extend to functions that are holomorphic in the whole complex Minkowski spacetime $M^{(c)}_d$. The important point to be noted is described below.

Remark 1. *Positive frequency waves $\psi^{(-)}_{\vec{p}}(z)$ are exponentially decreasing in the past tube T_-; negative frequency waves $\psi^{(-)}_{\vec{p}}(z)$ are exponentially decreasing in the future tube T_+.*

Let us now examine the two-point function. By translation invariance, it may depend only on the difference variable $\xi = x_1 - x_2$. Taking the Fourier transform of the Klein–Gordon equation with respect to ξ gives

$$(p^2 - m^2)\tilde{\mathcal{W}}_m(p) = 0. \tag{29}$$

The most general Lorentz invariant distributional solution has two disconnected components:

$$\tilde{\mathcal{W}}_m(p) = a\theta(p^0)\delta(p^2 - m^2) + b\theta(-p^0)\delta(p^2 - m^2) \tag{30}$$

and the spectral condition imposes $b = 0$. By Fourier anti-transforming, we obtain:

$$\mathcal{W}_m(x_1, x_2) = \frac{1}{2(2\pi)^{d-1}} \int \frac{e^{-i\omega(x_1^0 - x_2^0) + i\vec{p}\cdot(\vec{x_1} - \vec{x_2})}}{\sqrt{|\vec{p}|^2 + m^2}}\, d\vec{p} = \int \psi^{(-)}_{\vec{p}}(x_1)\psi^{(+)}_{\vec{p}}(x_2) d\vec{p}. \tag{31}$$

Remark 1 invites us to move the first point into the past tube T_- and the second point into the forward tube T_+. This move greatly improves the convergence of the integral, as the function

$$W_m(z_1, z_2) = \int \psi^{(-)}_{\vec{p}}(z_1)\psi^{(+)}_{\vec{p}}(z_2)d\vec{p}, \quad z_1 \in T_-, \ z_2 \in T_+ \tag{32}$$

is now an analytic function of $(z_1, z_2) \in T_- \times T_+$. The two-point distribution $\mathcal{W}(x_1, x_2)$ is recovered by taking the boundary value. The normalisation ensures that the CCRs hold with the correct coefficient.

Let us discuss the following elementary massless case in more detail:

$$W((t - is, 0 \ldots, 0), 0) = \int \frac{e^{-i\omega(t-is)}k^{d-3}}{2(2\pi)^{d-1}} e^{-i\omega(t-is)}k^{d-3}\, dk\, d\Omega_{d-2} \tag{33}$$

$$= \frac{1}{(4\pi)^{\frac{d-1}{2}}} \frac{\Gamma(d-2)}{\Gamma\left(\frac{d-1}{2}\right)} \frac{1}{(it+s)^{d-2}}. \tag{34}$$

By restoring in this expression the Lorentz-invariant variable $(z_1 - z_2)^2$, we immediately obtain the maximally analytic two-point function:

$$\mathfrak{W}(z_1, z_2) = \frac{\Gamma\left(\frac{d-2}{2}\right)}{4\pi^{\frac{d}{2}}}[-(z_1 - z_2)^2]^{-\frac{d-2}{2}}. \tag{35}$$

Its boundary values from the relevant tubes give the two-point function $\mathcal{W}(x_1, x_2)$ and the permuted two-point function $\mathcal{W}(x_2, x_1)$. The covariant commutator is their difference (9):

$$C(x, y) = \frac{\Gamma\left(\frac{d-2}{2}\right)}{4\pi^{\frac{d}{2}}}\left([-(x-y)^2 + i\varepsilon(x^0 - y^0)]^{-\frac{d-2}{2}} - [-(x-y)^2 - i\varepsilon(x^0 - y^0)]^{-\frac{d-2}{2}}\right). \tag{36}$$

Using the notations of [9], we obtain

$$C(x, y) = \frac{1}{2\pi i}\frac{1}{\Gamma\left(2 - \frac{d}{2}\right)}\varepsilon(x^0 - y^0)[-(x-y)^2]_-^{-\frac{d-2}{2}}. \tag{37}$$

where $\varepsilon(x) = \theta(x) - \theta(-x)$. When the spacetime dimension d is even, the distribution $[-(x-y)^2]_-^\lambda$ has a simple pole at $\lambda = -\frac{d-2}{2}$ with residue

$$\operatorname{Res}_{\lambda=-\frac{d-2}{2}}[-(x-y)^2]_-^\lambda = \frac{(-1)^{\frac{d}{2}-2}}{\Gamma\left(\frac{d}{2}-2\right)}\delta^{\left(\frac{d}{2}-2\right)}[(x-y)^2] \tag{38}$$

while $1/\Gamma\left(\frac{d}{2}-2\right)$ has a zero, and we obtain that the support of the commutator is the light cone (Huygens principle):

$$C(x, y) = \frac{1}{2\pi i}\varepsilon(x^0 - y^0)\delta^{\left(\frac{d}{2}-2\right)}[(x-y)^2]. \tag{39}$$

In particular, for $d = 4$, we obtain the well-known dominant term of the Pauli–Jordan function.

5. de Sitter

Let us consider now the d-dimensional de Sitter Universe (see Equation (2)):

$$dS_d = \{x \in M_{d+1} \ : \ x \cdot x = -R^2 = -1\}\,. \tag{40}$$

The future cone of the origin of the ambient spacetime in one dimension more is given by

$$V^{+} = \{x \in M_{d+1} \; : \; x^2 > 0, \; x^0 > 0\} \tag{41}$$

which provides the causal ordering of the de Sitter manifold. An event x_2 is in the future of another event x_1 if the vector $(x_2 - x_1)$ belongs to the closed future cone $\overline{V}^{+}$ of the ambient spacetime. Two events $x_1, x_2 \in dS_d$ are spacelike separated if and only if

$$(x_1 - x_2)^2 = -2 - 2x_1 \cdot x_2 < 0. \tag{42}$$

A straightforward adaptation of the spectral condition of Wightman QFT is just not possible because there exists no global energy operator available in the de Sitter case. This is a consequence of the absence of a global timelike Killing vector field on the de Sitter manifold. Timelike Killing vector fields exist only on wedge-like submanifolds bordered by bifurcate Killing horizons, but there is no Killing vector field that remains timelike on the whole manifold.

Still, since the complexification of Minkowski space plays such a crucial role in Minkowski QFT, we may go on and consider the complex de Sitter spacetime, visualised here as a submanifold of the complex $(d+1)$-dimensional Minkowski space:

$$dS_d^{(c)} = \{z \in M_{d+1}^{(c)} \; : \; z \cdot z = -R^2 = -1\} \, . \tag{43}$$

Note that $z = x + iy \in dS_d^{(c)}$ if and only if $x^2 - y^2 = -R^2$ and $x \cdot y = 0$. On the complex manifold, the complex de Sitter group $G^{(c)}$ acts, which is the complexification of the restricted Lorentz group of the ambient space $G = SO_0(1, d)$.

$dS_d^{(c)}$ contains tuboids $\mathcal{T}_\pm$, which are very similar to the past and future tubes of Minkowski space. Actually, they can be described in the simplest way as the intersections of the ambient tubes $T_\pm$ with the complex de Sitter manifold:

$$\mathcal{T}_\pm = dS_d^{(c)} \cap T_\pm = \{x + iy \in dS_d^{(c)} \; : \; y \in \pm V_+\} \, . \tag{44}$$

The set of points with purely imaginary zero component $z^0 = iy^0$ and purely real spatial components $z^i = x^i, \; i = 1, \ldots, d$, forms the Euclidean sphere of the complex de Sitter manifold:

$$S_d = \{z = (iy^0, x^1, \ldots, x^d) \in \mathbf{C}^{1+d} : {y^0}^2 + {x^1}^2 + \ldots + {x^d}^2 = R^2 = 1\}. \tag{45}$$

Now, we come to de Sitter QFT. While it is impossible to formulate a true spectral condition, we may retain its most characteristic consequence: in the case of two-point functions, we can assume [10] that the following holds.

Assumption 1 (Normal analyticity property). *$\mathcal{W}(x_1, x_2)$ is the boundary value of a function $W(z_1, z_2)$ holomorphic in the tube $\mathcal{T}_{12} = \mathcal{T}_- \times \mathcal{T}_+$,*

$$\mathcal{W}(x_1, x_2) = \langle \Psi_0, \widehat{\phi}(x_2)\widehat{\phi}(x_1)\Psi_0 \rangle = \underset{\substack{\mathcal{T}_- \ni z_1 \to x_1 \\ \mathcal{T}_+ \ni z_2 \to x_2}}{b.v.} W(z_1, z_2) \tag{46}$$

where $\mathcal{T}_-$ and $\mathcal{T}_+$ are the de Sitter past and future tubes (see Figure 1).

Of course, the physical interpretation of this property cannot be the positivity of the energy spectrum of the states. It turns out that the correct physical interpretation is thermodynamical [10–12].

The tubes $\mathcal{T}_\pm$ are invariant under the action of the real de Sitter group. By acting with the complex group, a much larger analyticity domain appears, as before. The following theorem [10] is mutatis mutandis identical to Theorem 2 [10].

Theorem 3 (Maximal analyticity property)**.** *1. The two-point function $W(z_1, z_2)$ depends only on the Lorentz-invariant variable $\zeta = z_1 \cdot z_2$.*

2. *$W(z_1, z_2)$ can be continued to a function $\mathfrak{W}(z_1, z_2)$ analytic in the cut domain*

$$\Delta = \{(z_1, z_2);\ \ \zeta \neq \rho,\ \ \rho \leq -1\} \tag{47}$$

that contains all pairs of complex events minus the causal cut (42).

3. *$\mathfrak{W}(z_1, z_2)$ is invariant under the action of the complex de Sitter group.*

4. *The permuted two-point function is the boundary value of $\mathfrak{W}(z_1, z_2)$ from the opposite tube $T_{21} = T_+ \times T_-$:*

$$W(x_2, x_1) = \langle \Psi_0, \widehat{\phi}(x_2)\widehat{\phi}(x_1)\Psi_0\rangle = \underset{\substack{T_+ \ni z_1 \to x_1 \\ T_- \ni z_2 \to x_2}}{b.v.} \mathfrak{W}(z_1, z_2). \tag{48}$$

5. *The cut domain Δ contains all the non-coinciding Euclidean points*

$$\dot{\mathcal{E}} = \{z_1, z_2 \in \Delta,\ \ z_1 \in S_d,\ z_2 \in S_d,\ \ z_1 \neq z_2\}. \tag{49}$$

The Schwinger function S is the restriction of $\mathfrak{W}(z_1, z_2)$ to the non-coincident Euclidean points $\dot{\mathcal{E}}$. S is analytic in $\dot{\mathcal{E}}$ and can be extended as a distribution to the whole Euclidean space $\mathcal{E}$, including the coinciding points.

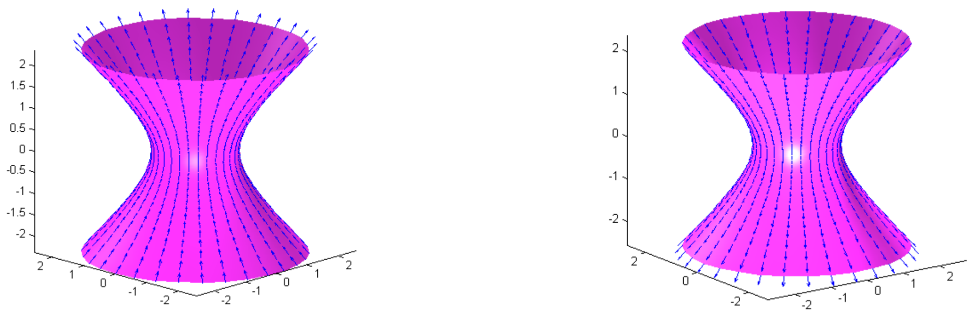

Figure 1. Sections of the forward and backward tubes in the complex dS manifold. The arrows represent the imaginary parts y of the vectors $z = x + iy$, attached at the end points of their real parts x, which belong to the hyperboloid with a radius of $-R^2 + y^2$. This illustration shows a section at a fixed positive value of y^2. Recall that $x \cdot y = 0$.

5.1. Klein–Gordon Fields and Plane Waves

Now we want to construct dS Klein–Gordon quantum fields starting from two-point functions (as in Equation (12)) that are *normal analytic* in the sense of Assumption 1. Following the paradigm of flat space, we should look for wave solutions of the Klein–Gordon equation analytic in the past, and respectively, in the future tube, and write a two-point function similar to Equation (32). When solving the Klein–Gordon equation, the normal strategy is to separate the variables; however, this would not be a good idea if the normal analyticity property has to appear manifestly, as in Equation (32).

One possibility comes from the study of geodesics [13]: a de Sitter timelike geodesic may be parametrised by the choice of two lightlike vectors belonging to the future light cone C_+ of the ambient Minkowski spacetime (see Figure 2), as follows:

$$x^\mu(\tau) = \frac{R}{\sqrt{2\xi \cdot \eta}}(\xi^\mu e^{\frac{\tau}{R}} - \eta^\mu e^{-\frac{\tau}{R}}). \tag{50}$$

The two null vectors parametrising the geodesics point towards its asymptotic directions. In fact, the conformal compactification of the Sitter manifold has a boundary at timelike infinity, and the light cone of the ambient spacetime is, in a precise sense, equivalent to it [14].

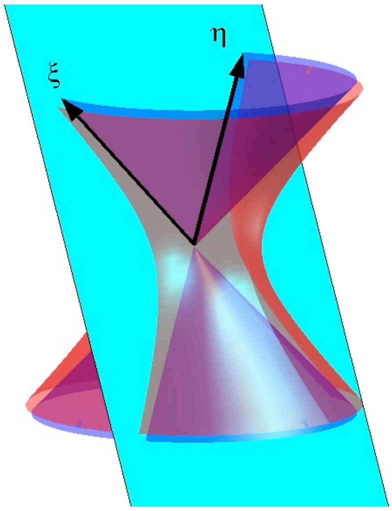

Figure 2. Timelike geodesics can be parametrised by the choice of two null vectors in the ambient space; they have the physical interpretation of asymptotic momentum directions.

A natural basis of the solutions of the de Sitter Klein–Gordon equation

$$\Box_{dS}\psi(z) + m^2\psi(z) = 0 \tag{51}$$

can thus be parametrised by the choice of a lightlike vector $\xi \in C^+$ and a complex number λ, as follows:

$$\psi_\lambda(z,\xi) = (z\cdot\xi)^\lambda = e^{\lambda \log(z\cdot\xi)}. \tag{52}$$

In this definition, the scalar product is in the sense of the ambient spacetime. The functions (52) are plane waves, as their phase is constant on the planes $z\cdot\xi = const$. As required, they are well defined and analytic in each of the tubes $\mathcal{T}^+$ and $\mathcal{T}^-$ [10].

It is useful to introduce a new complex parameter ν with the following definition:

$$\lambda = -\frac{d-1}{2} + i\nu. \tag{53}$$

The parameters λ and ν are related to the complex mass squared and the complex dimension as follows:

$$m^2 = -\lambda(\lambda+d-1) = \frac{(d-1)^2}{4} + \nu^2. \tag{54}$$

Of course, m^2 is real and positive only under the following conditions:

1. ν is real. This corresponds, in a group-theoretical language, to the principal series of unitary representations of the Lorentz group;
2. ν is purely imaginary and $|\nu| < \frac{d-1}{2}$. This corresponds to the complementary series of unitary representations of the Lorentz group.

But, in the de Sitter Universe, there is also room for negative mass squared at certain discrete values [15,16].

5.2. Construction of the Two-Point Function

We can now mimic Equation (32) and consider the two-point function:

$$\int_\gamma (\xi\cdot z_1)^{-\frac{d-1}{2}-i\nu}(\xi\cdot z_2)^{-\frac{d-1}{2}+i\nu}\, d\mu_\gamma, \quad z_1\in\mathcal{T}_-, \quad z_2\in\mathcal{T}_+ \tag{55}$$

where

$$d\mu_\gamma(\xi) = \alpha(\xi)|_\gamma = (\xi^0)^{-1}\sum_{j=1}^{d}(-1)^{j+1}\xi^j\, d\xi^1\ldots\widehat{d\xi^j}\ldots d\xi^d|_\gamma\,. \tag{56}$$

γ denotes any $(d-1)$-dimensional integration cycle in C^+. To fix the ideas, we can integrate over the spherical basis S_{d-1} of the cone C^+ equipped with its canonical orientation:

$$\gamma_0 = S_{d-1} = C^+ \cap \{\xi \,:\, \xi^0 = 1\} = \{\xi \in C^+ : {\xi^1}^2 + \ldots + {\xi^d}^2 = 1\}. \tag{57}$$

In this case, $\alpha(\xi)|_\gamma$ coincides with the rotation-invariant measure $d\mu_{\gamma_0}$ on S_{d-1}, normalised as follows:

$$\omega_d = \int_{\gamma_0} d\mu_{\gamma_0} = \frac{2\pi^{\frac{d}{2}}}{\Gamma\left(\frac{d}{2}\right)}. \tag{58}$$

The following is self-evident.

Property 1. *The two-point function (55) solves the Klein–Gordon equation by construction in each variable and is manifestly holomorphic in $\mathcal{T}_- \times \mathcal{T}_+$.*

Since the integrand is a homogeneous function of ξ of degree $(1-d)$, the integral (55) is actually the integral of a closed differential form and, as such, does not depend on the integration cycle. This immediately implies the following property.

Property 2. *The two-point function (55) is de Sitter-invariant and depends only on the invariant $\zeta = z_1 \cdot z_2$.*

To compute it explicitly, we may, therefore, choose the two arbitrary points z_1 in $\mathcal{T}^-$ and z_2 in $\mathcal{T}^+$ in the way that most pleases us. Interestingly, different choices produce different integral representations of the same function. A useful choice is

$$z_1 = (-i, 0, \ldots, 0, 0), \quad z_2(s) = (\sinh(is), 0, \ldots, 0, \cosh(is)), \quad \xi = (1, \vec{n}\sin\theta, \cos\theta),$$

so that

$$\zeta = z_1 \cdot z_2(s) = \sin(s), \;\; (\zeta^2 - 1)^{\frac{1}{2}} = i\cos s. \tag{59}$$

The condition z_2 in $\mathcal{T}^+$ means $0 < s < \pi$. We obtain [17], Equation (7), p. 156

$$\begin{aligned} &\int_{S_{d-1}} (\xi \cdot z_1)^{-\frac{d-1}{2} - i\nu} (\xi \cdot z_2)^{-\frac{d-1}{2} + i\nu} \, d\xi = \\ = \;\; & \omega_{d-1} \int_0^\pi e^{-\frac{i\pi}{2}\left(-\frac{d-1}{2} - i\nu\right)} (i \sin s - \cos s \cos\theta)^{-\frac{d-1}{2} + i\nu} \sin^{d-2}\theta d\theta \\ = \;\; & (2\pi)^{\frac{d}{2}} e^{-\pi\nu} (\zeta^2 - 1)^{-\frac{d-2}{4}} P^{-\frac{d-2}{2}}_{-\frac{1}{2} + i\nu}(\zeta). \end{aligned} \tag{60}$$

Imposing the normalisation of the CCRs gives the plane-wave expansion of the two-point function, valid for any complex value of ν that is not a pole of $\Gamma\left(\frac{d-1}{2} + i\nu\right)\Gamma\left(\frac{d-1}{2} - i\nu\right)$.

Main formula: The canonically normalised (so-called Bunch–Davis) Wightman function of a Klein–Gordon de Sitter scalar field has the following expressions:

$$W_\nu(z_1, z_2) = w_\nu(z_1 \cdot z_2)$$

$$= \frac{\Gamma\left(\frac{d-1}{2} + i\nu\right)\Gamma\left(\frac{d-1}{2} - i\nu\right)e^{\pi\nu}}{2^{d+1}\pi^d} \int_\gamma (\xi \cdot z_1)^{-\frac{d-1}{2} - i\nu} (\xi \cdot z_2)^{-\frac{d-1}{2} + i\nu} \alpha(\xi) \tag{61}$$

$$= \frac{\Gamma\left(\frac{d-1}{2} + i\nu\right)\Gamma\left(\frac{d-1}{2} - i\nu\right)}{2(2\pi)^{d/2}} (\zeta^2 - 1)^{-\frac{d-2}{4}} P^{-\frac{d-2}{2}}_{-\frac{1}{2} + i\nu}(\zeta). \tag{62}$$

Equation (61) is only valid in the normal domain of analyticity, with z_1 in $\mathcal{T}^-$ and z_2 in $\mathcal{T}^+$. On the other hand, the right-hand side of Equation (62) is maximally analytic, that is, entire in the cut plane Δ.

The discontinuity of the Wightman function on the cut provides the commutator. The retarded propagator function is obtained by (carefully) multiplying the commutator with the relevant step function:

$$C_\nu(x_1, x_2) = \mathcal{W}_\nu(x_1, x_2) - \mathcal{W}_\nu(x_2, x_1), \tag{63}$$

$$R_\nu(x_1, x_2) = i\theta(x_2, x_1)C_\nu(x_1, x_2). \tag{64}$$

To compute the retarded propagator, let us choose x_2 in the future cone of the origin:

$$x_0 = (0, 0, \ldots, 0, 1) \quad x_2(t) = (\sinh t, 0, \ldots, 0 \cosh t), \quad t > 0, \quad \zeta = -\cosh t.$$

The retarded discontinuity $(x_2 > x_1)$ is, therefore,

$$\begin{aligned} r_\nu(u) &= \frac{i\,\Gamma\left(\frac{d-1}{2} + i\nu\right)\Gamma\left(\frac{d-1}{2} - i\nu\right)}{2(2\pi)^{d/2}}(u^2 - 1)^{-\frac{d-2}{4}}\left(P^{-\frac{d-2}{2}}_{-\frac{1}{2}+i\nu}(\zeta - i\epsilon) - P^{-\frac{d-2}{2}}_{-\frac{1}{2}+i\nu}(\zeta + i\epsilon)\right) \\ &= \cosh(\pi\nu)\frac{\Gamma\left(\frac{d-1}{2} + i\nu\right)\Gamma\left(\frac{d-1}{2} - i\nu\right)}{(2\pi)^{d/2}}(\zeta^2 - 1)^{-\frac{d-2}{4}}\,P^{-\frac{d-2}{2}}_{-\frac{1}{2}+i\nu}(-\zeta). \end{aligned} \tag{65}$$

The Schwinger function is the restriction of the maximally analytic two-point function to the Euclidean sphere. Given any two points of the Euclidean sphere, their invariant product can be parametrised as follows: $z_1 \cdot z_2 = -\cos(s)$. The choice of sign is because at coincident points, $z^2 = -1$. Thus,

$$G_\nu(-\cos s) = \frac{\Gamma(\frac{d-1}{2} + i\nu)\Gamma(\frac{d-1}{2} - i\nu)}{2(2\pi)^{d/2}}(\sin s)^{-\frac{d-2}{2}}\,e^{\frac{i\pi}{2}(d-2)}P^{-\frac{d-2}{2}}_{-\frac{1}{2}+i\nu}(-\cos s). \tag{66}$$

At this point, we are fully equipped to begin studying perturbative quantum field theory on the de Sitter Universe. Of course, we do not do it here, but we want to discuss one remarkable success of the above formalism.

5.3. Linearisation and the Källén–Lehmann Representation

In Minkowski space, any scalar two-point function $W(z_1, z_2)$ satisfying the properties described in Section 4 admits a Källén–Lehmann representation of the form

$$W(z_1, z_2) = \int_0^\infty \rho(m^2)\, W_m(z_1, z_2)\, dm^2 \tag{67}$$

where $W_m(z_1, z_2)$ is given in Equation (32) and the weight $\rho(m^2)$ is a positive measure of tempered growth. In particular, given two masses m_1 and m_2, computing the weight for the bubble

$$W_{m_1}(z_1, z_2)W_{m_2}(z_1, z_2) = \int_{(m_1+m_2)^2}^\infty \rho(m^2 : m_1, m_2)W_m(z_1, z_2)dm^2 \tag{68}$$

is an easy exercise of Fourier transformation.

The corresponding de Sitter case is much more difficult. To obtain the Källén–Lehmann weight of the corresponding integral

$$W_\lambda(z_1, z_2)W_\nu(z_1, z_2) = \int_{-\infty}^\infty \rho(\lambda, \nu, \kappa)W_\kappa(z_1, z_2)\kappa d\kappa, \tag{69}$$

one should compute the Mehler–Fock transform of $W_\lambda(\zeta)W_\nu(\zeta)$. This amounts to the following integral:

$$h_d(\lambda,\nu,\kappa) = \int_1^\infty P^{-\frac{d-2}{2}}_{-\frac{1}{2}+i\lambda}(u)P^{-\frac{d-2}{2}}_{-\frac{1}{2}+i\nu}(u)P^{-\frac{d-2}{2}}_{-\frac{1}{2}+i\kappa}(u)\,(u^2-1)^{-\frac{d-2}{4}}\,du \tag{70}$$

and the Källén–Lehmann weight is

$$\rho(\lambda,\nu,\kappa) = \frac{\Gamma\left(\frac{d-1}{2}+i\nu\right)\Gamma\left(\frac{d-1}{2}-i\nu\right)\Gamma\left(\frac{d-1}{2}+i\lambda\right)\Gamma\left(\frac{d-1}{2}-i\lambda\right)\sinh(\pi\kappa)\,h_d(\lambda,\nu,\kappa)}{2(2\pi)^{1+\frac{d}{2}}}. \tag{71}$$

The evaluation of $h_d(\lambda,\nu,\kappa)$ is very far from obvious. In the particular case where the two masses are equal, $h_d(\lambda,\lambda,\kappa)$ may be evaluated by Mellin transform techniques, used for the first time in the de Sitter context in [18,19]. The same idea of using Mellin techniques was used a few years later to compute the Källén–Lehmann weight in the case of two equal masses [20] in AdS QFT[3].

The general case of two independent masses cannot be dealt with by Mellin transformation techniques, and something more similar to the Fourier transform of flat space is needed. It is precisely at this point that the plane-wave representation (61) makes a substantial difference.

An especially important Fourier-type representation is obtained by evaluating (61) at the purely imaginary events in the tubes [22]: $z = -iy \in \mathcal{T}^-$ and $z = +iy' \in \mathcal{T}^+$. y and y' can be visualised as points belonging to a Lobachevsky space, modelled as the upper sheet of a two-sheeted hyperboloid:

$$H_d = \{y \in \mathbb{M}_{1,d} :\ y^2 = y\cdot y = R^2,\ \ y^0 > 0\}. \tag{72}$$

It follows that

$$\begin{aligned} w_\nu(-iy,iy') &= \frac{\Gamma(\frac{d-1}{2}+i\nu)\Gamma(\frac{d-1}{2}-i\nu)}{2^{d+1}\pi^d}\int_\gamma (y\cdot\xi)^{-\frac{d-1}{2}+i\nu}(\xi\cdot y')^{-\frac{d-1}{2}-i\nu}d\mu_\gamma(\xi) \\ &= \frac{\Gamma\left(\frac{d-1}{2}+i\nu\right)\Gamma\left(\frac{d-1}{2}-i\nu\right)}{2(2\pi)^{\frac{d}{2}}}\left((y\cdot y')^2-1\right)^{-\frac{d-2}{4}}P^{-\frac{d-2}{2}}_{-\frac{1}{2}+i\nu}(y\cdot y'). \end{aligned} \tag{73}$$

By choosing, in particular, $\gamma = \gamma_0$ and $y' = (1,0,\ldots,0)$ so that $y\cdot y' = y^0 = u \ge 1$, we then obtain the following integral representation:

$$\left(u^2-1\right)^{-\frac{d-2}{4}}P^{-\frac{d-2}{2}}_{-\frac{1}{2}+i\nu}(u) = \frac{1}{(2\pi)^{\frac{d}{2}}}\int_{\gamma_0}(y\cdot\xi)^{-\frac{d-1}{2}-i\nu}d\mu_{\gamma_0}. \tag{74}$$

This formula is of crucial importance for computing $h_d(\lambda,\nu,\kappa)$: it allows us to rewrite the one-dimensional integral (70) as the following multiple integrals over the manifold $H_d \times S_{d-1}\times S_{d-1}\times S_{d-1}$:

$$h_d(\lambda,\nu,\kappa) = \frac{(2\pi)^{-\frac{3d}{2}}}{\omega_{d-1}}\int_{\gamma_0}\int_{\gamma_0}\int_{\gamma_0}\int_{H_d}(y\cdot\xi_1)^{-\frac{d-1}{2}-i\lambda}(y\cdot\xi_2)^{-\frac{d-1}{2}-i\nu}(y\cdot\xi_3)^{-\frac{d-1}{2}-i\kappa}dy\,d\mu_{\gamma_0}d\mu_{\gamma_0}d\mu_{\gamma_0} \tag{75}$$

where dy is the Lorentz-invariant measure on H_d. The above integral may be computed, and the final result is [22]

$$\rho(\lambda, \nu, \kappa) = \frac{1}{2^d \pi^{\frac{d-1}{2}} \kappa \Gamma\left(\frac{d-1}{2}\right)} \frac{\prod_{\epsilon,\epsilon',\epsilon''=\pm 1} \Gamma\left(\frac{d-1}{4} + \frac{i\epsilon\lambda + i\epsilon'\nu + i\epsilon''\kappa}{2}\right)}{\prod_{\epsilon=\pm 1} \Gamma\left(\frac{i\epsilon\kappa}{2}\right)\Gamma\left(\frac{1}{2} + \frac{i\epsilon\kappa}{2}\right)\Gamma\left(\frac{d-1}{4} + \frac{i\epsilon\kappa}{2}\right)\Gamma\left(\frac{d+1}{4} + \frac{i\epsilon\kappa}{2}\right)}. \tag{76}$$

The application of this formula and its AdS twin to loop calculations are discussed in [23,24].

6. Anti-de Sitter

The anti-de Sitter spacetime can also be visualised as a hypersurface embedded in an ambient flat space, which is $\mathbf{R}^{d+1}$ with two timelike directions and metric mostly minus, as follows:

$$AdS_d = \{x \in \mathbf{R}^{d+1} \,:\, x^2 = x \cdot x = R^2\}, \tag{77}$$

$$x \cdot y = x^0 y^0 - x^1 y^1 - \ldots - x^{d-1} y^{d-1} + x^d y^d. \tag{78}$$

The AdS manifold has a boundary at spacelike infinity and, therefore, is not globally hyperbolic. This feature gives AdS QFT a little extra complication with respect to the standard globally hyperbolic case.

We also have to consider the complexification of the AdS manifold, which is defined as before by an embedding:

$$AdS_d^{(c)} = \{z = x + iy \in \mathbf{C}^{d+1} \,:\, z^2 = R^2\}. \tag{79}$$

While $AdS_d^{(c)}$ is simply connected, the real manifold AdS_d is not and admits a nontrivial universal covering space $\widetilde{AdS}_d$. Here, we focus mainly on the uncovered manifold AdS_d.

The symmetry group of the anti-de Sitter spacetime is the pseudo-orthogonal group of the ambient space $SO(2, d-1)$. This group may also be regarded as the conformal group of transformations of the boundary, represented as the null cone of the ambient space

$$C_d = \{\xi \in \mathbf{R}^{d+1} \,:\, \xi^2 = \xi \cdot \xi = 0\}. \tag{80}$$

This simple geometrical fact lies at the basis of the AdS/CFT correspondence. The null cone of the ambient space also plays the role of giving a causal order to the AdS spacetime, which is, however, only *local* due to the existence of closed timelike curves. Two events are spacelike separated if

$$(x_1 - x_2)^2 = 2 - 2x_1 \cdot x_2 < 0. \tag{81}$$

The covering manifold is globally causal but remains non-globally hyperbolic (see Figures 3 and 4).

It is possible to identify in the complex manifold $AdS_d^{(c)}$ an analogue of the Euclidean subspace of the complex Minkowski spacetime: it is the real submanifold H_d of $AdS_d^{(c)}$ defined by

$$H_d = \{z \in \mathbf{C}^{d+1} \,:\, z \cdot z = R^2,\ z(y) = (y^0, \ldots, y^{d-1}, iy^d),\ y^\mu \in \mathbf{R},\ y^0 > 0\}. \tag{82}$$

This is indeed the same Lobachevsky space we met before in (72) at the end of the de Sitter tubes, but it has, of course, a different interpretation in the AdS context, and, more importantly, AdS correlation functions have singularities at coincident Euclidean points of H_d, while dS correlation functions do not.

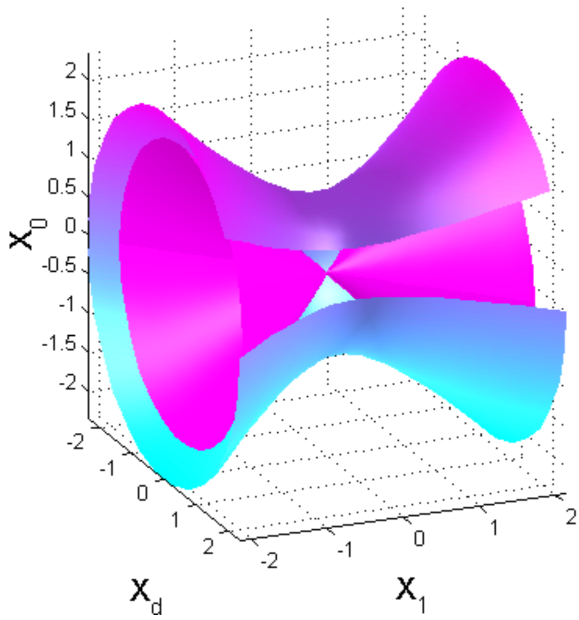

Figure 3. The AdS manifold and the null cone of the ambient space, which models its boundary at spacelike infinity.

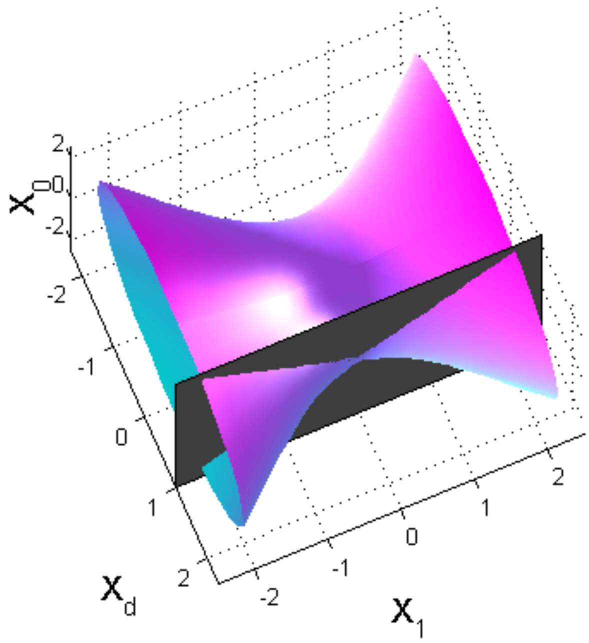

Figure 4. The null cone of the ambient space induces a causal order on the AdS manifold, which is only local.

6.1. The Analytic Structure of Two-Point Functions

The status of AdS QFT is more similar to that of Minkowski space, and it is possible to formulate a true spectral condition. This question was studied in full generality in [25]. A simplified account can be found in [26]. The point is that the parameter of the group of rotations in the $(0, d)-$plane may be interpreted as a global time variable: the AdS spectral condition is thus formulated by requiring that the corresponding generator M_{0d} be represented in the Hilbert space of the theory by a self-adjoint operator whose spectrum is positive. As in flat space, this requirement is equivalent to the precise analyticity properties of the n-point functions [25].

In particular, there are two distinguished complex domains of $AdS_d^{(c)}$, invariant under real AdS transformations [25,26], defined as follows (see Figure 5)

$$\mathcal{Z}_+ = \{z = x + iy \in AdS_d^{(c)};\ y^2 > 0,\ \epsilon(z) = +1\}, \tag{83}$$

$$\mathcal{Z}_- = \{z = x + iy \in AdS_d^{(c)};\ y^2 > 0,\ \epsilon(z) = -1\}, \tag{84}$$

where

$$\epsilon(z) = \mathrm{sign}(y^0 x^d - x^0 y^d). \tag{85}$$

The tubes $\mathcal{Z}_+$ and $\mathcal{Z}_-$ have a definite chirality and wrap the real AdS manifold in opposite directions. The spaces $\mathcal{Z}_+$, $\mathcal{Z}_-$, and AdS_d have the same homotopy type. Their universal coverings are denoted as $\widetilde{\mathcal{Z}}_+$ and $\widetilde{\mathcal{Z}}_-$.

The AdS spectral condition implies that a general two-point function satisfies the following property [25].

Normal analyticity property: *$\mathcal{W}(x_1, x_2)$ is the boundary value of a function $W(z_1, z_2)$ holomorphic in the domain $\mathcal{Z}_- \times \mathcal{Z}_+$*

$$\mathcal{W}(x_1, x_2) = \langle \Psi_0, \widehat{\phi}(x_1)\widehat{\phi}(x_2)\Psi_0 \rangle = \underset{\substack{\mathcal{Z}_- \ni z_1 \to x_1 \\ \mathcal{Z}_+ \ni z_2 \to x_2}}{b.v.} W(z_1, z_2). \tag{86}$$

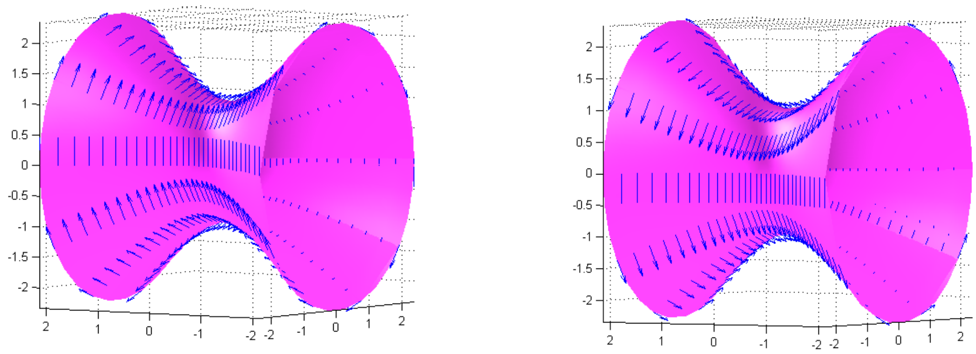

Figure 5. Sections of the backward and forward tubes in the complex AdS manifold. The tubes wrap the real manifold in opposite directions. The arrows represent the imaginary parts y of the vectors $z = x + iy$ attached at the end of the real parts x, which vary on the hyperboloid with a radius of $R^2 + y^2$. This illustration shows a section at a fixed positive value of y^2. Recall that $x \cdot y = 0$.

AdS invariance and normal analyticity imply the following.

Theorem 4 (Maximal analyticity property)**.** *1. The two-point function $W(z_1, z_2)$ depends only on the AdS-invariant variable $\zeta = z_1 \cdot z_2$.*

2. *$W(z_1, z_2)$ can be continued to a function $\mathfrak{W}(z_1, z_2)$ analytic in the cut domain*

$$\Delta_1 = \{\mathbf{C} \setminus [-1, 1]\}. \tag{87}$$

3. *$\mathfrak{W}(z_1, z_2)$ is invariant under the action of the complex de Sitter group.*
4. *The permuted two-point function is the boundary value of $\mathfrak{W}(z_1, z_2)$ from the opposite tube $\mathcal{Z}_{21} = \mathcal{Z}_+ \times \mathcal{Z}_-$.*
5. *The cut domain Δ_1 contains all the non-coinciding Euclidean points*

$$\dot{\mathcal{E}} = \{z_1, z_2 \in \Delta,\ \ z_1 \in S_d,\ z_2 \in H_d,\ \ z_1 \neq z_2\}. \tag{88}$$

The Schwinger function S is the restriction of $\mathfrak{W}(z_1, z_2)$ to the non-coincident Euclidean points $\dot{\mathcal{E}}$. S is analytic in $\dot{\mathcal{E}}$ and can be extended as a distribution to the whole Euclidean space $\mathcal{E}$, including the coinciding points.

Regarding the global hyperbolicity issue, the maximal analytic structure completely determines the two-point functions for Klein–Gordon fields and, consequently, selects the boundary behaviour of the modes.

6.2. Klein–Gordon Fields and Plane Waves

Klein–Gordon fields display the simplest example of the previous analytic structure. For a given mass m, the two-point function $\mathcal{W}(x_1, x_2)$ must satisfy the equation

$$(\Box_{x_i} + m^2)\mathcal{W}(x_1, x_2) = 0, \;\; i = 1, 2, \tag{89}$$

with respect to both variables, where $\Box_{x_i}$ is the Laplace–Beltrami operator relative to the AdS metric. The two-point functions are labelled by the (complex) dimension d and a (complex) parameter ν as follows:

$$W_\nu^d(z_1, z_2) = w_\nu^d(\zeta) = \frac{1}{(2\pi)^{\frac{d}{2}}}(\zeta^2 - 1)^{-\frac{d-2}{4}} e^{-i\pi\frac{d-2}{2}} Q^{\frac{d-2}{2}}_{-\frac{1}{2}+\nu}(\zeta) = \tag{90}$$

$$= \frac{\Gamma\left(\frac{d-1}{2} + \nu\right)}{2\pi^{\frac{d-1}{2}}(2\zeta)^{\frac{d-1}{2}+\nu}\Gamma(\nu + 1)} \, {}_2F_1\left(\frac{d-1}{4} + \frac{\nu}{2}, \frac{d+1}{4} + \frac{\nu}{2}; \nu + 1; \frac{1}{\zeta^2}\right) \tag{91}$$

where the various parameters are related as follows:

$$m^2 = \nu^2 - \frac{(d-1)^2}{4}. \tag{92}$$

There are two possible cases:

1. For $\nu > 1$, the spectrum condition uniquely selects one field theory for each given value of mass parameter ν;
2. For $|\nu| < 1$, there are two acceptable theories for each given mass.

The difference between the two theories lies in their large distance behaviour. More precisely, in view of [17], Equation (3.3.1.4), one has

$$w_{-\nu}^d(\zeta) = w_\nu^d(\zeta) + \frac{\sin \pi\nu \, \Gamma\left(\frac{d-1}{2} - \nu\right)\Gamma\left(\frac{d-1}{2} + \nu\right)}{(2\pi)^{\frac{d}{2}}}(\zeta^2 - 1)^{-\frac{d-2}{4}} P^{-\frac{d-2}{2}}_{-\frac{1}{2}-\nu}(\zeta). \tag{93}$$

The last term in this relation is *regular on the cut* $\zeta \in [-1, 1]$ and, therefore, does not contribute to the commutator. Consequently, the two theories represent the same algebra of local observables at short distances. But since the second term at the right-hand side grows faster as $|\nu|$ increases (see [17], Equation (3.9.2)), the two theories have drastically different long-range behaviours.

Let us now proceed to the harmonic analysis in plane waves for the AdS correlation functions. Here, to keep the discussion as simple as possible, we limit ourselves to the two-dimensional uncovered anti-de Sitter spacetime AdS_2 [27]. A full analysis will be presented elsewhere.

As for the de Sitter case, the wave solutions of the anti-de Sitter Klein–Gordon equation may also be parameterised by the choice of a null vector $\xi \in C_2$ and a complex number λ, as follows:

$$\psi_\lambda(z, \xi) = (z \cdot \xi)^\lambda = e^{\lambda \log(z\cdot\xi)}. \tag{94}$$

Since we are considering the uncovered manifold, the parameter λ must be an integer:

$$\lambda = \ell. \tag{95}$$

Now, we observe that while Equation (50) maintains its validity in the present context, for real ξ, η belonging to the null cone C_2, it describes spacelike geodesics. Timelike geodesics would correspond to vectors ξ belonging to the complex cone

$$C_2^{(c)} = \{\xi \in \mathbf{C}^3 \ : \ \xi^2 = \xi \cdot \xi = 0\}. \tag{96}$$

This suggests that the harmonic analysis of the AdS correlation function should also be made in terms of waves parametrised by null complex vectors ξ.

The complex cone $C_2^{(c)}$ admits the partition $C_2^{(c)} = C_2 \cup C_{2+} \cup C_{2-}$, where

$$C_{2+} = \{\xi \in C^{(c)};\ \epsilon(\xi) = +\}, \tag{97}$$

$$C_{2-} = \{\xi \in C^{(c)};\ \epsilon(\xi) = -\}; \tag{98}$$

and as before,

$$\epsilon(z) = \mathrm{sign}[(\mathrm{Im}\,\xi^0)(\mathrm{Re}\,\xi^2) - (\mathrm{Re}\,\xi^2)(\mathrm{Im}\,\xi^2)]. \tag{99}$$

The bases for the cones C_{2+} and C_{2-} are

$$(\gamma_0^{(c)})_+ = \{\xi = \xi(\Phi) = (\sin\Phi, 1, \cos\Phi);\ \Phi = \phi + i\eta,\ \eta > 0\}, \tag{100}$$

$$(\gamma_0^{(c)})_- = \{\xi = \xi(\Phi) = (\sin\Phi, 1, \cos\Phi);\ \Phi = \phi + i\eta,\ \eta < 0\}. \tag{101}$$

Let us now consider the integral

$$\int_{\gamma(z_1)} [z_1 \cdot \xi]^\ell \ [\xi \cdot z_2]^{-\ell-1} \mathrm{d}\mu_{\gamma^{(c)}}(\xi), \quad z_1\, in \mathcal{Z}_-, \quad z_2\, in \mathcal{Z}_+. \tag{102}$$

For each z_1 in $\mathcal{Z}_-$, $\gamma(z_1)$ is a relative cycle in $H^1(C_2^{(c)}, \{\xi;\ [z \cdot \xi] = 0\})$ with support contained in C_{2-} and end points belonging, respectively, to the two linear generatrices of the cone $C_2^{(c)}$ defined by $[z_1 \cdot \xi] = 0$. Being the integral of a closed differential form, (102) does not depend on the choice of $\gamma(z_1)$ inside its homology class.

There is no loss of generality in defining the integration cycle $\gamma(z_1)$ only for points of the form

$$z_1 = z_v = (i \sinh v, 0, \cosh v), \ \ v < 0. \tag{103}$$

We choose the path

$$\phi \to \xi(\phi + iv) = (\sin(\phi + iv), 1, \cos(\phi + iv)), \ -\frac{\pi}{2} < \phi < \frac{\pi}{2}. \tag{104}$$

The support of $\gamma(z_v)$ does belong to C_{2-}, and $[z_v \cdot \xi(\phi + iv)] = \cos(\phi)$ vanishes, as required, at the boundaries of the cycle.

Since $z_2 \in \mathcal{Z}_+$, the factor $[\xi \cdot z_2]^{-\ell-1}$ never becomes singular on the integration cycle $\gamma(z_v)$. This can be seen by explicitly giving coordinates to $\mathcal{Z}_+$ [27]. This suffices to show the AdS invariance of the integral (102), which is, therefore, a function of the invariant variable $\zeta = z_1 \cdot z_2$, holomorphic in the cut domain Δ_1.

To compute (102), we choose the second point at the origin $x_0 = (0, 0, 1)$ so that $z_1 \cdot z_2 = z_v \cdot x_0 = \cosh v$. With a few self-explanatory changes of variables, we obtain [17], Equation (2), p. 155:

$$\begin{aligned}
&\int_{\gamma(z_v)} [z_v \cdot \xi]^\ell \ [\xi \cdot x_0]^{-\ell-1} \mathrm{d}\mu_\gamma(\xi) = \\
&= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} cos^\ell\phi \ \cos^{-(\ell+1)}(\phi + iv) \ \mathrm{d}\phi = \int_{-\infty}^{\infty} (\cosh v - it \sinh v)^{-(\ell+1)} \frac{\mathrm{d}t}{(1+t^2)^{\frac{1}{2}}} \\
&= 2\int_0^\infty (\cosh v + \cosh u \sinh v)^{-(\ell+1)} du = 2Q_l(\cosh v).
\end{aligned} \tag{105}$$

Therefore, we can write the following plane-wave expansion of the two-point function:

$$W^{2}_{l+\frac{1}{2}}(z_1, z_2) = \frac{1}{\pi}\int_{\gamma(z_v)} [z_v \cdot \xi]^{\ell}\, [\xi \cdot x_0]^{-\ell-1} \mathrm{d}\mu_{\gamma}(\xi). \tag{106}$$

7. Conclusions and Outlook

There is a unifying feature that connects Minkowski, de Sitter, and anti-de Sitter quantum field theories: the analyticity properties of the correlation functions of the quantum fields in the relevant tubular domains of the corresponding complex manifolds [10,12,25] These analyticity properties are, in the Minkowski and the anti-de Sitter cases, a consequence of the spectral condition, i.e., a consequence of the requirement that the Hamiltonian operator has a positive spectrum in every Lorentz frame. The reconstruction procedures show that the analyticity properties are equivalent to the spectral condition [8,25]. In the de Sitter case, the analyticity of the correlation functions may be taken as a replacement of the spectral condition since there exists no globally defined conserved energy operator in the de Sitter geometry. Taking the analyticity properties of correlation functions seriously has produced valuable nontrivial results, which, at the moment, seem to be out of reach of other methods, such as the Källén–Lehmann representations of two-point functions [10,21,22] and the calculation of one- and two-loop Feynman diagrams for both the de Sitter and anti-de Sitter cases. There are many opportunities created by this formalism and there is much work to be done. This review is also an invitation to readers interested in either dS or AdS quantum field theory to join the effort.

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Notes

[1] *Sphaera infinita cuius centrum est ubique, circumferentia tamen nullibi* is the second definition of God that can be read in the Liber XXIV philosophorum, an anonymous medieval treatise attributed to Hermes Trismegistus. Nicolas de Cues applied this definition to the Universe: The world machine has, so to speak, its centre everywhere and its circumference nowhere (La Docte Ignorance, 1440). Giordano Bruno later adopted this definition in various works. Einstein's novelty was that his sphere was not infinite but rather finite and curved.

[2] Apart from the nonlinear (and hard to verify) positivity condition of the correlation functions necessary to reconstruct a Hilbert space.

[3] At the very same time, however, a general Källén-Lehmann formula for AdS fields with two different masses was for the first time published and available [21]. But many subsequent authors seem to have ignored it.

References

1. Einstein, A. *Cosmological Considerations in the General Theory of Relativity*; Sitzungsber. Preuss. Akad. Wiss. Berlin (Math. Phys.); 1917; Volume 6, pp. 142–152.
2. Friedmann, A.A. Über die Krümmung des Raumes. *Z. Phys.* **1922**, *10*, 377–386. [CrossRef]
3. Friedmann, A.A. Über die Möglichkeit einer Welt mit konstanter negativer Krümmung des Raumes. *Z. Phys.* **1924**, *21*, 326–332. [CrossRef]
4. Lemaître, G. Un univers homogène de masse constante et de rayon croissant, rendant compte de la vitesse radiale des nébuleuses extra-galactiques. *Ann. Soc. Sci. Brux. A* **1927**, *47*, 49–59.
5. Riess, A.G.; Filippenko, A.V.; Challis, P.; Clocchiatti, A.; Diercks, A.; Garnavich, P.M.; Gilliland, R.L.; Hogan, C.J.; Jha, S.; Kirshner, R.P.; et al. Observational Evidence from Supernovae for an Accelerating Universe and a Cosmological Constant. *Astron. J.* **1998**, *116*, 1009–1038. [CrossRef]
6. Perlmutter, S.; Aldering, G.; Goldhaber, G.; Knop, R.A.; Nugent, P.; Castro, P.G.; Deustua, S.; Fabbro, S.; Goobar, A.; Groom, D.E.; et al. Measurements of Ω and Λ from 42 High-Redshift Supernovae. *Astron. J.* **1999**, *517*, 565. [CrossRef]
7. Maldacena, J.M. The Large N limit of superconformal field theories and supergravity. *Adv. Theor. Math. Phys.* **1998**, *2*, 231–252. [CrossRef]

8. Streater, R.F.; Wightman, A.S. *PCT, Spin and Statistics, and All That*; Benjamin: New York, NY, USA, 1964.
9. Gelfand, I.M.; Shilov, G.E. *Generalized Functions*; Academic Press: New York, NY, USA, 1964.
10. Bros, J.; Moschella, U. Two point functions and quantum fields in de Sitter universe. *Rev. Math. Phys.* **1996**, *8*, 327–392. [CrossRef]
11. Gibbons, G.W.; Hawking, S.W. Cosmological event horizons, thermodynamics, and particle creation. *Phys. Rev. D* **1977**, *10*, 2738–2751. [CrossRef]
12. Bros, J.; Epstein, H.; Moschella, U. Analyticity properties and thermal effects for general quantum field theory on de Sitter space-time. *Commun. Math. Phys.* **1998**, *196*, 535–570. [CrossRef]
13. Cacciatori, S.; Gorini, V.; Kamenshchik, A.; Moschella, U. Conservation laws and scattering for de Sitter classical particles. *Class. Quant. Grav.* **2008**, *25*, 075008. [CrossRef]
14. Bros, J.; Epstein, H.; Moschella, U. The Asymptotic symmetry of de Sitter space-time. *Phys. Rev. D* **2002**, *65*, 084012. [CrossRef]
15. Bros, J.; Epstein, H.; Moschella, U. Scalar tachyons in the de Sitter universe. *Lett. Math. Phys.* **2010**, *93*, 203–211. [CrossRef]
16. Epstein, H.; Moschella, U. de Sitter tachyons and related topics. *Commun. Math. Phys.* **2015**, *336*, 381–430. [CrossRef]
17. Erdélyi, A. (Ed.) *The Bateman Project: Higher Transcendental Functions*; McGraw-Hill Book Company: New York, NY, USA, 1953; Volume I.
18. Bros, J.; Epstein, H.; Moschella, U. Lifetime of a massive particle in a de Sitter universe *J. Cosmol. Astropart. Phys.* **2008**, *2008*, 003. [CrossRef]
19. Bros, J.; Epstein, H.; Moschella, U. Particle decays and stability on the de Sitter universe. *Ann. Henri Poincare* **2010**, *11*, 611–658. [CrossRef]
20. Fitzpatrick, A.L.; Kaplan, J. Analyticity and the Holographic S-Matrix. *J. High Energy Phys.* **2012**, *10*, 127. [CrossRef]
21. Bros, J.; Epstein, H.; Gaudin, M.; Moschella, U.; Pasquier, V. Anti de Sitter quantum field theory and a new class of hypergeometric identities. *Commun. Math. Phys.* **2012**, *309*, 255–291. [CrossRef]
22. Bros, J.; Epstein, H.; Gaudin, M.; Moschella, U.; Pasquier, V. Triangular invariants, three-point functions and particle stability on the de Sitter universe. *Commun. Math. Phys.* **2010**, *295*, 261–288. [CrossRef]
23. Cacciatori, S.L.; Epstein, H.; Moschella, U. Loops in de Sitter space. *arXiv* **2024**, arXiv:2403.13145.
24. Cacciatori, S.L.; Epstein, H.; Moschella, U. Loops in Anti de Sitter space. *arXiv* **2024**, arXiv:2403.13142.
25. Bros, J.; Epstein, H.; Moschella, U. Towards a general theory of quantized fields on the anti-de Sitter space-time. *Commun. Math. Phys.* **2002**, *231*, 481–528. [CrossRef]
26. Bertola, M.; Bros, J.; Moschella, U.; Schaeffer, R. A general construction of conformal field theories from scalar anti-de Sitter quantum field theories. *Nucl. Phys.* **2000**, *587*, 619–644. [CrossRef]
27. Bros, J.; Moschella, U. Fourier analysis and holomorphic decomposition on the one-sheeted hyperboloid. In *Géomtrie Complexe II. Aspects Contemporains dans les Mathématiques et la Physique*; Norguet, F., et Ofman, S., Szczeciniarz, J.-J., Eds.; Hermann: Paris, France, 2003; Volume II, pp. 100–145.