universe

Article

Infinite Spin Fields in $d = 3$ and Beyond

Yurii M. Zinoviev

Institute for High Energy Physics of National Research Center "Kurchatov Institute" Protvino, Moscow Region 142281, Russia; Yurii.Zinoviev@ihep.ru

Received: 27 July 2017; Accepted: 18 August 2017; Published: 30 August 2017

Abstract: In this paper, we consider the frame-like formulation for the so-called infinite (continuous) spin representations of the Poincare algebra. In the three-dimensional case, we give explicit Lagrangian formulation for bosonic and fermionic infinite spin fields (including the complete sets of the gauge-invariant objects and all the necessary extra fields). Moreover, we find the supertransformations for the supermultiplet containing one bosonic and one fermionic field, leaving the sum of their Lagrangians invariant. Properties of such fields and supermultiplets in four and higher dimensions are also briefly discussed.

Keywords: infinite spin fields; massive higher spins; gauge invariance; supersymmetry

1. Introduction

Besides the very well known finite-component massless and massive representations of the Poincare algebra, there are rather exotic so-called infinite (or continuous) spin representations (see e.g., [1,2]). In dimensions $d \geq 4$, they have an infinite number of physical degrees of freedom and so may be of some interest for the higher spins theory. Indeed, they attracted some attention recently [3–8]. One of the reasons is that, contrary to the finite-component massless fields, such representations are characterized by a dimensionful parameter (that can play the same role as the cosmological constant for the massless theories and the mass for the massive ones) and so they may provide an interesting alternative for the massless higher spin theory in the flat space. Note also that such representations can appear in the tensionless limit of the string theory.

It has been noted several times that such infinite spin representations may be considered as a limit of massive higher spin ones where spin goes to infinity and mass goes to zero while the product remains fixed. Moreover, recently, Metsaev has shown that the metric-like Lagrangian formulation for the bosonic [9] and fermionic [10] fields in AdS_d spaces with $d \geq 4$ can be constructed using exactly the same technique as was previously used for the gauge-invariant formulation of massive higher spin bosonic [11] and fermionic [12] fields.

The current paper is devoted to the frame-like formulation for such infinite spin fields. In the first (and main) section, we construct gauge-invariant Lagrangian formulation for bosonic and fermionic cases in $d = 3$. We also elaborate on the whole set of the gauge invariant objects (introducing all necessary extra fields) and rewrite our Lagrangians in the explicitly gauge-invariant form. Moreover, we managed to find supertransformations for the supermultiplet containing one bosonic and one fermionic infinite spin field that leaves the sum of their Lagrangians invariant. For this, we heavily use our previous results on the gauge-invariant formulation for massive bosonic and fermionic higher spin fields in $d = 3$ [13,14] (see also [15–17]) as well as results on the massive higher spin supermultiplets [18–20]. In the last two sections, we briefly discuss the properties of such fields and supermultiplets in $d = 4$ and $d \geq 5$ dimensions, concluding with explicit details on the forthcoming publication.

Notations and conventions We will work in the frame-like multispinor formalism (mostly the same as in [20] but we restrict ourselves to the flat Minkowski space). In this formalism, all objects are forms (3, 2, 1, 0-forms) that have totally symmetric local spinor indices. To simplify the expressions, we will use the condensed notations for the spinor indices such that, e.g.,

$$\Omega^{\alpha(2k)} = \Omega^{(\alpha_1\alpha_2\ldots\alpha_{2k})}$$

Also, we will always assume that spinor indices denoted by the same letters and placed on the same level are symmetrized, e.g.,

$$\Omega^{\alpha(2k)}\zeta^{\alpha} = \Omega^{(\alpha_1\ldots\alpha_{2k}}\zeta^{\alpha_{2k+1})}$$

where symmetrization uses the minimal number of terms necessary without any normalization factor. The coordinate-free description of the three-dimensional flat Minkowski space will use the background frame (one-form) $e^{\alpha(2)}$ and external derivative d

$$d \wedge d = 0$$

Basic elements of 1,2,3-form space are respectively $e^{\alpha(2)}$, $E^{\alpha(2)}$, and E where the last two are defined as the double and triple product of $e^{\alpha(2)}$:

$$e^{\alpha\alpha} \wedge e^{\beta\beta} = \varepsilon^{\alpha\beta}E^{\alpha\beta}, \qquad E^{\alpha\alpha} \wedge e^{\beta\beta} = \varepsilon^{\alpha\beta}\varepsilon^{\alpha\beta}E$$

Further on, the wedge product sign $\wedge$ will be omitted.

2. Infinite Spin Fields in $d = 3$

In this section, we develop the frame-like formalism for the massless infinite spin bosonic and fermionic fields as well as for the supermultiplet containing such fields.

2.1. Infinite Spin Boson

As we have already noted, there is a tight connection between the gauge invariant description for the massive finite spin fields and the one for the massless infinite spin fields. Recall that the general idea of such a description is to begin with the appropriate set of massless (finite component) fields and then glue them together in such a way that keeps all their gauge symmetries. This, in turn, guarantees the correct number of physical degrees of freedom. Thus, we will follow the same approach as in [13] but this time without restriction on the number of components. So, we introduce an infinite set of physical and auxiliary one-forms $\Omega^{\alpha(2k)}$, $\Phi^{\alpha(2k)}$, $1 \leq k \leq \infty$ as well as one-form A and zero-forms $B^{\alpha(2)}$, $\pi^{\alpha(2)}$ and φ [1]. We begin with the sum of kinetic terms for all these fields (recall that the Lagrangians are three-forms in our formalism):

$$\begin{aligned}\mathcal{L}_0 \;=\; & \sum_{k=1}^{\infty}(-1)^{k+1}[k\Omega_{\alpha(2k-1)\beta}e^{\beta}{}_{\gamma}\Omega^{\alpha(2k-1)\gamma} + \Omega_{\alpha(2k)}d\Phi^{\alpha(2k)}] \\ & + EB_{\alpha(2)}B^{\alpha(2)} - B_{\alpha(2)}e^{\alpha(2)}dA - E\pi_{\alpha(2)}\pi^{\alpha(2)} + \pi_{\alpha(2)}E^{\alpha(2)}d\varphi\end{aligned} \tag{1}$$

as well as their initial gauge transformations:

$$\delta_0\Omega^{\alpha(2k)} = d\eta^{\alpha(2k)}, \qquad \delta_0\Phi^{\alpha(2k)} = d\xi^{\alpha(2k)} + e^{\alpha}{}_{\beta}\eta^{\alpha(2k-1)\beta}, \qquad \delta_0 A = d\xi \tag{2}$$

---

1 Note that in three dimensions, such an infinite spin bosonic field (as any massive higher spin boson) has just two physical degrees of freedom, while an infinite spin fermionic field (as any massive higher spin fermion) has just one. However it is impossible to realize such representations using a finite number of components (see e.g., [6]).

Then, following a general scheme, we add to the Lagrangian a set of cross terms gluing all these components together:

$$\begin{aligned}\mathcal{L}_1 &= \sum_{k=1}^{\infty}(-1)^{k+1}[\tilde{a}_k\Omega_{\alpha(2k)\beta(2)}e^{\beta(2)}\Phi^{\alpha(2k)} + a_k\Omega_{\alpha(2k)}e_{\beta(2)}\Phi^{\alpha(2k)\beta(2)}] \\ &\quad +\tilde{a}_0\Omega_{\alpha(2)}e^{\alpha(2)}A - a_0\Phi_{\alpha\beta}E^{\beta}{}_{\gamma}B^{\alpha\gamma} + \hat{a}_0\pi_{\alpha(2)}E^{\alpha(2)}A \end{aligned} \tag{3}$$

and introduce appropriate corrections for the gauge transformations:

$$\begin{aligned}\delta_1\Omega^{\alpha(2k)} &= \frac{(k+2)}{k}a_k e_{\beta(2)}\eta^{\alpha(2k)\beta(2)} + \frac{a_{k-1}}{k(2k-1)}e^{\alpha(2)}\eta^{\alpha(2k-2)} \\ \delta_1\Phi^{\alpha(2k)} &= a_k e_{\beta(2)}\xi^{\alpha(2k)\beta(2)} + \frac{(k+1)a_{k-1}}{k(k-1)(2k-1)}e^{\alpha(2)}\xi^{\alpha(2k-2)} \\ \delta_1\Omega^{\alpha(2)} &= 3a_1e_{\beta(2)}\eta^{\alpha(2)\beta(2)}, \qquad \delta_1\Phi^{\alpha(2)} = a_1e_{\beta(2)}\xi^{\alpha(2)\beta(2)} + 2a_0e^{\alpha(2)}\xi \\ \delta_1 B^{\alpha(2)} &= 2a_0\eta^{\alpha(2)}, \qquad \delta_1 A = \frac{a_0}{4}e_{\alpha(2)}\xi^{\alpha(2)}, \qquad \delta_1\varphi = -\hat{a}_0\xi \end{aligned} \tag{4}$$

Here, consistency of the gauge transformations with the Lagrangian requires:

$$\tilde{a}_k = -\frac{(k+2)}{k}a_k, \qquad \tilde{a}_0 = 2a_0$$

At last, we introduce mass-like terms for all components and appropriate corrections to the gauge transformations:

$$\mathcal{L}_2 = \sum_{k=1}^{\infty}(-1)^{k+1}b_k\Phi_{\alpha(2k-1)\beta}e^{\beta}{}_{\gamma}\Phi^{\alpha(2k-1)\gamma} + b_0\Phi_{\alpha(2)}E^{\alpha(2)}\varphi + \tilde{b}_0E\varphi^2 \tag{5}$$

$$\delta_2\Omega^{\alpha(2k)} = \frac{b_k}{k}e^{\alpha}{}_{\beta}\xi^{\alpha(2k-1)\beta}, \qquad \delta_2\pi^{\alpha(2)} = b_0\xi^{\alpha(2)} \tag{6}$$

Now, we require that the whole Lagrangian $\mathcal{L} = \mathcal{L}_0 + \mathcal{L}_1 + \mathcal{L}_2$ will be invariant under the gauge transformations $\delta = \delta_0 + \delta_1 + \delta_2$. This produces the following general relations on the parameters:

$$(k+2)^2b_{k+1} = k(k+1)b_k \tag{7}$$

$$\frac{2(k+2)(2k+3)}{(k+1)(2k+1)}{a_k}^2 - \frac{2(k+1)}{(k-1)}{a_{k-1}}^2 + 4b_k = 0 \tag{8}$$

as well as some relations for the lower components:

$$5{a_1}^2 - {a_0}^2 + 4b_1 = 0$$

$${\hat{a}_0}^2 = 64b_1, \qquad b_0 = \frac{\hat{a}_0a_0}{4}, \qquad \tilde{b}_0 = \frac{3{a_0}^2}{2}$$

The general solution of all these relations has two free parameters. In the massive finite spin case, it is just the mass and spin but in our case we choose a_0 and b_1 as the main parameters. Then, all other parameters can be expressed as follows:

$$b_k = \frac{4b_1}{k(k+1)^2} \tag{9}$$

$${a_k}^2 = \frac{k}{(2k+3)}[\frac{3(k+1)}{2(k+2)}{a_0}^2 - \frac{8k}{(k+1)}b_1] \tag{10}$$

Now, we are ready to analyze the solution obtained. Let us begin with the case $a_0{}^2 < 16b_1$. In general, it means that starting from some value of k, all $a_k{}^2$ become negative so that we obtain non-unitary theory. The only exceptions happen when one adjusts the values of $a_0{}^2$ and b_1 so that at some k_0 we obtain $a_{k_0} = 0$. In this case, we obtain unitary theory with the finite number of components and this case corresponds to the gauge-invariant description for the massive bosonic field with the spin $k_0 + 1$. Let us turn to the case $a_0{}^2 = 16b_1$ (this corresponds to the case $\mu_0 = 0$ in [9]). In this case, we obtain:

$$a_k{}^2 = \frac{3k}{2(k+1)(k+2)(2k+3)} a_0{}^2 \tag{11}$$

so we get a unitary theory with an infinite number of components. Note that for the case $a_0{}^2 > 16b_1$ we also obtain unitary theory but as it was shown by Metsaev [9] it corresponds to the tachyonic infinite spin field. Thus, in what follows, we will restrict ourselves to the case $a_0{}^2 = 16b_1$ only.

Naturally, all the physical properties of the solutions obtained are the same as in the metric-like formulation by Metsaev because metric-like and frame-like formalisms are equivalent and so, for the free theories, which one to use is just a matter of preference. However, for the investigation of possible interactions, the frame-like formalism may provide some advantages. In-particular, one of the nice and general features of the frame-like formalism is that for each field (physical or auxiliary) one can construct a corresponding gauge-invariant object. For the case at hand, we will follow the massive case in [17,20]. For almost all fields, corresponding gauge-invariant objects can be directly constructed from the known form for the gauge transformations given above (here, for convenience, we changed the normalization for the zero-forms $B^{\alpha(2)} \Rightarrow 2a_0 B^{\alpha(2)}$, $\pi^{\alpha(2)} \Rightarrow b_0 \pi^{\alpha(2)}$):

$$\begin{aligned}
\mathcal{R}^{\alpha(2k)} &= d\Omega^{\alpha(2k)} + \frac{b_k}{k} e^\alpha{}_\beta \Phi^{\alpha(2k-1)\beta} + \frac{(k+2)}{k} a_k e_{\beta(2)} \Omega^{\alpha(2k)\beta(2)} + \frac{a_{k-1}}{k(2k-1)} e^{\alpha(2)} \Omega^{\alpha(2k-2)} \\
\mathcal{T}^{\alpha(2k)} &= d\Phi^{\alpha(2k)} + e^\alpha{}_\beta \Omega^{\alpha(2k-1)\beta} + a_k e_{\beta(2)} \Phi^{\alpha(2k)\beta(2)} + \frac{(k+1)a_{k-1}}{k(k-1)(2k-1)} e^{\alpha(2)} \Phi^{\alpha(2k-2)} \\
\mathcal{R}^{\alpha(2)} &= d\Omega^{\alpha(2)} + b_1 e^\alpha{}_\beta \Phi^{\alpha\beta} + 3a_1 e_{\beta(2)} \Omega^{\alpha(2)\beta(2)} - a_0{}^2 E^\alpha{}_\beta B^{\alpha\beta} + b_0 E^{\alpha(2)} \varphi \\
\mathcal{T}^{\alpha(2)} &= d\Phi^{\alpha(2)} + e^\alpha{}_\beta \Omega^{\alpha\beta} + a_1 e_{\beta(2)} \Phi^{\alpha(2)\beta(2)} + 2a_0 e^{\alpha(2)} A \\
\mathcal{A} &= dA - 2a_0 E_{\alpha(2)} B^{\alpha(2)} + \frac{a_0}{4} e_{\alpha(2)} \Phi^{\alpha(2)} \\
\Phi &= d\varphi - \frac{\sqrt{3}}{2} a_0{}^2 e_{\alpha(2)} \pi^{\alpha(2)} + 2\sqrt{3} a_0 A
\end{aligned} \tag{12}$$

However, to construct gauge-invariant objects for $B^{\alpha(2)}$ and $\pi^{\alpha(2)}$, one must introduce a first pair of the so-called extra fields [2] $B^{\alpha(4)}$ and $\pi^{\alpha(4)}$:

$$\begin{aligned}
\mathcal{B}^{\alpha(2)} &= dB^{\alpha(2)} - \Omega^{\alpha(2)} + b_1 e^\alpha{}_\beta \pi^{\alpha\beta} + 3a_1 e_{\beta(2)} B^{\alpha(2)\beta(2)} \\
\Pi^{\alpha(2)} &= d\pi^{\alpha(2)} + e^\alpha{}_\beta B^{\alpha\beta} - \Phi^{\alpha(2)} - \frac{1}{\sqrt{3}} e^{\alpha(2)} \varphi + a_1 e_{\beta(2)} \pi^{\alpha(2)\beta(2)}
\end{aligned} \tag{13}$$

which transform as follows:

$$\delta B^{\alpha(4)} = \eta^{\alpha(4)}, \qquad \delta\pi^{\alpha(4)} = \zeta^{\alpha(4)}$$

---

2 Recall that extra fields are the fields that do not enter the free Lagrangian but are necessary for the construction of the whole set of gauge-invariant objects. Moreover, such fields play an important role in the construction of the interactions.

However, to construct gauge-invariant objects for these new fields, one must introduce the next pair of extra fields and so on. This results in the infinite chain of zero forms $B^{\alpha(2k)}$ and $\pi^{\alpha(2k)}$, $1 \le k \le \infty$ with the following set of gauge-invariant objects:

$$\begin{aligned}
\mathcal{B}^{\alpha(2k)} &= dB^{\alpha(2k)} - \Omega^{\alpha(2k)} + \frac{b_k}{k} e^\alpha{}_\beta \pi^{\alpha(2k-1)\beta} + \frac{(k+2)}{k} a_k e_{\beta(2)} B^{\alpha(2k)\beta(2)} \\
&\quad + \frac{a_{k-1}}{k(2k-1)} e^{\alpha(2)} B^{\alpha(2k-2)} \\
\Pi^{\alpha(2k)} &= d\pi^{\alpha(2k)} - \Phi^{\alpha(2k)} + e^\alpha{}_\beta B^{\alpha(2k-1)\beta} + a_k e_{\beta(2)} \pi^{\alpha(2k)\beta(2)} \\
&\quad + \frac{(k+1)a_{k-1}}{k(k-1)(2k-1)} e^{\alpha(2)} \pi^{\alpha(2k-2)}
\end{aligned} \tag{14}$$

Here:

$$\delta B^{\alpha(2k)} = \eta^{\alpha(2k)}, \qquad \delta \pi^{\alpha(2k)} = \xi^{\alpha(2k)}$$

Now, we have an infinite set of gauge one-forms as well as an infinite set of Stueckelberg zero-forms. As in the massive finite spin case [17,20], this allows us to rewrite the Lagrangian in the explicitly gauge-invariant form:

$$\mathcal{L} = -\frac{1}{2} \sum_{k=1}^{\infty} (-1)^{k+1} [\mathcal{R}_{\alpha(2k)} \Pi^{\alpha(2k)} + \mathcal{T}_{\alpha(2k)} \mathcal{B}^{\alpha(2k)}] + \frac{1}{2} e_{\alpha(2)} \mathcal{B}^{\alpha(2)} \Phi \tag{15}$$

By construction, each term here is separately gauge-invariant and the explicit values for all coefficients are determined by the so-called extra field decoupling conditions:

$$\frac{\delta \mathcal{L}}{\delta B^{\alpha(2k)}} = 0, \qquad \frac{\delta \mathcal{L}}{\delta \pi^{\alpha(2k)}} = 0, \qquad 2 \le k \le \infty$$

2.2. Fermionic Case

In this case, we will also follow the construction for the massive finite spin field [14] but this time for the infinite set of components. So, we introduce a set of one-forms $\Psi^{\alpha(2k+1)}$, $0 \le k \le \infty$ and a zero-form ψ^α. Once again, we begin with the sum of kinetic terms for all fields:

$$\frac{1}{i} \mathcal{L}_0 = \sum_{k=0}^{\infty} \frac{(-1)^{k+1}}{2} \Psi_{\alpha(2k+1)} d\Psi^{\alpha(2k+1)} + \frac{1}{2} \psi_\alpha E^\alpha{}_\beta d\psi^\beta \tag{16}$$

as well as with their initial gauge transformations:

$$\delta_0 \Psi^{\alpha(2k+1)} = d\zeta^{\alpha(2k+1)} \tag{17}$$

Now we add a set of cross terms gluing them together

$$\frac{1}{i} \mathcal{L}_1 = \sum_{k=1}^{\infty} (-1)^{k+1} c_k \Psi_{\alpha(2k-1)\beta(2)} e^{\beta(2)} \Psi^{\alpha(2k-1)} + c_0 \Psi_\alpha E^\alpha{}_\beta \psi^\beta \tag{18}$$

and corresponding corrections to the gauge transformations:

$$\begin{aligned}
\delta_1 \Psi^{\alpha(2k+1)} &= c_{k+1} e_{\beta(2)} \zeta^{\alpha(2k+1)\beta(2)} + \frac{c_k}{k(2k+1)} e^{\alpha(2)} \zeta^{\alpha(2k-1)}, \\
\delta_1 \psi^\alpha &= c_0 \zeta^\alpha
\end{aligned} \tag{19}$$

At last, we add the mass-like terms for all fields and appropriate corrections to the gauge transformations:

$$\frac{1}{i}\mathcal{L}_2 = \sum_{k=0}^{\infty}(-1)^{k+1}\frac{d_k}{2}\Psi_{\alpha(2k)\beta}e^{\beta}{}_{\gamma}\Psi^{\alpha(2k)\gamma} - \frac{m_0}{2}E\psi_\alpha\psi^\alpha \tag{20}$$

$$\delta_2\Psi^{\alpha(2k+1)} = \frac{d_k}{(2k+1)}e^{\alpha}{}_{\beta}\zeta^{\alpha(2k)\beta} \tag{21}$$

Now, we require that the whole Lagrangian $\mathcal{L} = \mathcal{L}_0 + \mathcal{L}_1 + \mathcal{L}_2$ will be invariant under the gauge transformations $\delta = \delta_0 + \delta_1 + \delta_2$. This produces a number of general relations on the parameters

$$(2k+5)d_{k+1} = (2k+3)d_k \tag{22}$$

$$\frac{(k+2)(2k+1)}{(k+1)(2k+3)}{c_{k+1}}^2 - {c_k}^2 + \frac{{d_k}^2}{(2k+1)} = 0 \tag{23}$$

as well as

$$\frac{8}{3}{c_1}^2 - {c_0}^2 + 4{d_0}^2 = 0, \qquad d_0 = \frac{m_0}{3}$$

As in the bosonic case, the general solution for all these relations has two free parameters and we choose c_0 and m_0 this time. Then, all other coefficients can be expressed as follows:

$$d_k = \frac{m_0}{(2k+3)} \tag{24}$$

$${c_k}^2 = \frac{(2k+1)}{4(k+1)}{c_0}^2 - \frac{k}{2(2k+1)}{m_0}^2 \tag{25}$$

The properties of this solution appear to be the same as in the bosonic case. Namely, for the case ${m_0}^2 > 2{c_0}^2$, in general, we obtain non-unitary theory. The only exceptions appear if one adjusts these parameters so that at some k_0 we get $c_{k_0} = 0$. In this case, we obtain unitary theory with a finite number of components which corresponds to the gauge-invariant description for a massive fermionic field with spin $k_0 + 3/2$. For the ${m_0}^2 = 2{c_0}^2$ (this corresponds to $\mu_0 = 0$ in [10]), we obtain

$${c_k}^2 = \frac{{c_0}^2}{4(k+1)(2k+1)} \tag{26}$$

that corresponds to the unitary massless infinite spin field while for the ${m_0}^2 < {c_0}^2$, we again obtain tachyonic infinite spin case. As in the bosonic case, in what follows, we will restrict ourselves to the case ${m_0}^2 = 2{c_0}^2$ only.

Now, we proceed with the construction of the full set of gauge-invariant objects. For all one-forms, the construction is pretty straightforward (again, for convenience, we changed the normalization for the zero-form $\psi^\alpha \Rightarrow c_0\psi^\alpha$):

$$\begin{aligned} \mathcal{F}^{\alpha(2k+1)} &= d\Psi^{\alpha(2k+1)} + \frac{d_k}{(2k+1)}e^{\alpha}{}_{\beta}\Psi^{\alpha(2k)\beta} + c_{k+1}e_{\beta(2)}\Psi^{\alpha(2k+1)\beta(2)} \\ &\quad + \frac{c_k}{k(2k+1)}e^{\alpha(2)}\Psi^{\alpha(2k-1)} \\ \mathcal{F}^{\alpha} &= D\Psi^{\alpha} + d_0e^{\alpha}{}_{\beta}\Psi^{\beta} + c_1e_{\beta(2)}\Psi^{\alpha\beta(2)} - {c_0}^2E^{\alpha}{}_{\beta}\psi^{\beta} \end{aligned} \tag{27}$$

However, to construct a gauge-invariant object for the zero-form, one must introduce a first extra field:

$$\mathcal{C}^{\alpha} = d\psi^{\alpha} - \Psi^{\alpha} + d_0e^{\alpha}{}_{\beta}\psi^{\beta} + c_1e_{\beta(2)}\psi^{\alpha\beta(2)}, \qquad \delta\psi^{\alpha(3)} = \zeta^{\alpha(3)} \tag{28}$$

Then, to construct a gauge-invariant object for this field, one must introduce the second one and so on. This results in the infinite set of zero-forms with the corresponding gauge-invariant objects:

$$\begin{aligned} \mathcal{C}^{\alpha(2k+1)} &= d\psi^{\alpha(2k+1)} - \Psi^{\alpha(2k+1)} + \frac{d_k}{(2k+1)} e^\alpha{}_\beta \psi^{\alpha(2k)\beta} + c_{k+1} e_{\beta(2)} \psi^{\alpha(2k+1)\beta(2)} \\ &+ \frac{c_k}{k(2k+1)} e^{\alpha(2)} \psi^{\alpha(2k-1)} \end{aligned} \tag{29}$$

where

$$\delta\psi^{\alpha(2k+1)} = \zeta^{\alpha(2k+1)}$$

Now, we have an infinite set of one-form and zero-form fields and their gauge-invariant two and one forms. This allows us to rewrite the Lagrangian in the explicitly gauge-invariant form:

$$\mathcal{L} = -\frac{i}{2} \sum_{k=0}^{\infty} (-1)^{k+1} \mathcal{F}_{\alpha(2k+1)} \mathcal{C}^{\alpha(2k+1)} \tag{30}$$

As in the bosonic case, each term is separately gauge-invariant while the specific values of all coefficients are determined by the extra field decoupling condition:

$$\frac{\delta\mathcal{L}}{\delta\psi^{\alpha(2k+1)}} = 0, \qquad 1 \le k \le \infty$$

2.3. Infinite Spin Supermultiplet

It is interesting (see e.g., [1]) that, similarly to the usual massless and massive fields, such massless infinite spin fields can also form supermultiplets. In $d = 3$, the minimal supermultiplets contain just one bosonic and one fermionic field. Due to the tight relation with gauge-invariant formulation for the massive higher spin fields and supermultiplets, here we will heavily use the results of our recent paper [20]. The main difference (besides the infinite set of components) is the essentially different expressions for the coefficients a_k and c_k.

The general strategy will be to find the explicit form of the supertransformations for all fields such that all gauge-invariant two and one forms transform covariantly and to check the invariance of the Lagrangian. Let us begin with the bosonic fields. For the general case $k \ge 2$, we will use the following ansatz:

$$\begin{aligned} \delta\Omega^{\alpha(2k)} &= i\rho_k \Psi^{\alpha(2k-1)} \zeta^\alpha + i\sigma_k \Psi^{\alpha(2k)\beta} \zeta_\beta \\ \delta\Phi^{\alpha(2k)} &= i\alpha_k \Psi^{\alpha(2k-1)} \zeta^\alpha + i\beta_k \Psi^{\alpha(2k)\beta} \zeta_\beta \end{aligned} \tag{31}$$

and require that the corresponding two-form transform covariantly:

$$\begin{aligned} \delta\mathcal{R}^{\alpha(2k)} &= i\rho_k \mathcal{F}^{\alpha(2k-1)} \zeta^\alpha + i\sigma_k \mathcal{F}^{\alpha(2k)\beta} \zeta_\beta \\ \delta\mathcal{T}^{\alpha(2k)} &= i\alpha_k \mathcal{F}^{\alpha(2k-1)} \zeta^\alpha + i\beta_k \mathcal{F}^{\alpha(2k)\beta} \zeta_\beta \end{aligned} \tag{32}$$

First of all, this gives us an important relation

$$c_0{}^2 = 6a_0{}^2 \tag{33}$$

Recall that the parameters a_0 and c_0 are the main dimension-full parameters that determine the whole construction for the bosonic and fermionic fields. So this relation plays the same role as the requirement that masses of bosonic and fermionic fields in the supermultiplet must be equal. Further, we obtain explicit expressions for all parameters

$$\alpha_k{}^2 = k\hat{\alpha}^2, \qquad \beta_k{}^2 = \frac{(k+1)}{2k(2k+1)}\hat{\alpha}^2$$

$$\sigma_k{}^2 = \frac{3a_0{}^2}{4k(k+1)^2}\hat{\alpha}^2, \qquad \rho_k{}^2 = \frac{3a_0{}^2}{8k^3(k+1)(2k+1)}\hat{\alpha}^2$$

where $\hat{\alpha}$ is an arbitrary parameter that can be fixed by the normalization of the superalgebra.

For the three bosonic components that require separate consideration, we obtain:

$$\begin{aligned} \delta\Omega^{\alpha(2)} &= i\rho_1\Psi^{\alpha}\zeta^{\alpha} + i\sigma_1\Psi^{\alpha(2)\beta}\zeta_{\beta} - \frac{i\sqrt{3}a_0{}^2}{4}\hat{\alpha}e^{\alpha(2)}\psi^{\beta}\zeta_{\beta} \\ \delta A &= \frac{i\hat{\alpha}}{2}\Psi^{\alpha}\zeta_{\alpha} + \frac{i\sqrt{3}a_0}{2}\hat{\alpha}\psi_{\alpha}e^{\alpha\beta}\zeta_{\beta}, \qquad \delta\varphi = -i\sqrt{3}a_0\hat{\alpha}\psi^{\alpha}\zeta_{\alpha} \end{aligned} \tag{34}$$

At last, the supertransformations for the zero-forms look like:

$$\begin{aligned} \delta B^{\alpha(2k)} &= i\sigma_k\psi^{\alpha(2k)\beta}\zeta_{\beta} + i\rho_k\psi^{\alpha(2k-1)}\zeta^{\alpha} \\ \delta\pi^{\alpha(2k)} &= i\beta_k\psi^{\alpha(2k)\beta}\zeta_{\beta} + i\alpha_k\psi^{\alpha(2k-1)}\zeta^{\alpha} \end{aligned} \tag{35}$$

where all coefficients α_k, β_k, ρ_k and σ_k are the same as above.

Now, let us turn to the fermionic components. For the general case $k \geq 1$, we will consider the following ansatz:

$$\begin{aligned} \delta\Psi^{\alpha(2k+1)} &= \frac{\alpha_k}{(2k+1)}\Omega^{\alpha(2k)}\zeta^{\alpha} + 2(k+1)\beta_{k+1}\Omega^{\alpha(2k+1)\beta}\zeta_{\beta} \\ &\quad +\gamma_k\Phi^{\alpha(2k)}\zeta^{\alpha} + \delta_k\Phi^{\alpha(2k+1)\beta}\zeta_{\beta} \end{aligned} \tag{36}$$

Then, the requirement that the corresponding two-forms transform covariantly:

$$\begin{aligned} \delta\mathcal{F}^{\alpha(2k+1)} &= \frac{\alpha_k}{(2k+1)}\mathcal{R}^{\alpha(2k)}\zeta^{\alpha} + 2(k+1)\beta_{k+1}\mathcal{R}^{\alpha(2k+1)\beta}\zeta_{\beta} \\ &\quad +\gamma_k\mathcal{T}^{\alpha(2k)}\zeta^{\alpha} + \delta_k\mathcal{T}^{\alpha(2k+1)\beta}\zeta_{\beta} \end{aligned} \tag{37}$$

gives us the same relation on the parameters a_0 and c_0 as before and also gives:

$$\gamma_k{}^2 = \frac{3a_0{}^2}{4k(k+1)^2(2k+1)^2}\hat{\alpha}^2$$

$$\delta_k{}^2 = \frac{3a_0{}^2}{2(k+1)(k+2)(2k+3)}\hat{\alpha}^2$$

Again, there are a couple of components that need to be considered separately:

$$\begin{aligned} \delta\Psi^{\alpha} &= 2\beta_1\Omega^{\alpha\beta}\zeta_{\beta} + \delta_0\Phi^{\alpha\beta}\zeta_{\beta} + a_0\hat{\alpha}e_{\beta(2)}B^{\beta(2)}\zeta^{\alpha} + \sqrt{3}a_0\hat{\alpha}A\zeta^{\alpha} - \frac{\sqrt{3}a_0}{2}\hat{\alpha}\varphi e^{\alpha}{}_{\beta}\zeta^{\beta} \\ \delta\psi^{\alpha} &= \frac{2\sqrt{3}}{3}\hat{\alpha}B^{\alpha\beta}\zeta_{\beta} + \frac{a_0}{2}\hat{\alpha}\pi^{\alpha\beta}\zeta_{\beta} + \frac{\hat{\alpha}}{2}\varphi\zeta^{\alpha} \end{aligned} \tag{38}$$

At last, for the Stueckelberg zero-forms, we obtain:

$$\begin{aligned} \delta\psi^{\alpha(2k+1)} &= \frac{\alpha_k}{(2k+1)}B^{\alpha(2k)}\zeta^\alpha + 2(k+1)\beta_{k+1}B^{\alpha(2k+1)\beta}\zeta_\beta \\ &+\gamma_k\pi^{\alpha(2k)}\zeta^\alpha + \delta_k\pi^{\alpha(2k+1)\beta}\zeta_\beta \end{aligned} \tag{39}$$

where all parameters α_k, β_k, γ_k and δ_k are the same as before.

We have explicitly checked that the sum of the bosonic and fermionic Lagrangians is invariant under these supertransformations up to the terms proportional to the auxiliary fields $B^{\alpha(2)}$ and $\pi^{\alpha(2)}$ equations in the same way as in the case of massive higher spin supermultiplets [20].

3. Infinite Spin Fields in $d = 4$

Similarly to the three-dimensional case in $d = 4$, there exist just one bosonic and one fermionic infinite spin representation corresponding to the completely symmetric (spin-)tensors. Metric-like gauge-invariant Lagrangian formulation (valid also in $d > 4$) has been constructed recently [9,10]. Frame-like Lagrangian formulation can be straightforwardly obtained from the frame-like gauge-invariant formalism for the massive completely symmetric (spin-)tensors developed in [21]. These results will be presented elsewhere.

The complete set of the gauge-invariant objects for the massive bosonic higher spin fields in $d \geq 4$ has been constructed in [22]. It requires the following three sets of fields:

$$\Phi_\mu{}^{a(k),b(l)}, \quad S^{a(k),b(l)} \quad 0 \leq k \leq s-1, \quad 0 \leq l \leq k$$

$$W^{a(k),b(l)} \quad k \geq s, \quad 0 \leq l \leq s-1$$

where notation $\Phi_\mu{}^{a(k),b(l)}$ means that local indices correspond to the Young tableau with two rows. Thus, we have two finite sets of gauge one-forms and Stueckelberg zero-forms as well as an infinite number of gauge-invariant zero-forms. As in the three-dimensional case, one can try to consider the limit where spin goes to infinity and mass goes to zero, but in $d > 3$ it appears to be a rather involved task. As for the analogous formulation for the massive fermionic higher spin fields, to the best of our knowledge, it still remains to be elaborated.

As is quite well known, in $d = 4$, there exist two types of massive higher spin $N = 1$ supermultiplets corresponding to the integer or half-integer superspins:

$$\begin{pmatrix} & s+\frac{1}{2} & \\ s & & s' \\ & s-\frac{1}{2} & \end{pmatrix} \qquad \begin{pmatrix} & s+1 & \\ s+\frac{1}{2} & & s+\frac{1}{2} \\ & s & \end{pmatrix}$$

Their explicit Lagrangian description was constructed in [23] using gauge-invariant description for massive bosonic and fermionic higher spin fields. The main idea was that the massive supermultiplet can be constructed out of the appropriately chosen set of massless supermultiplets. The decomposition of these two massive supermultiplets into the massless one is as follows:

$$\begin{pmatrix} & \Psi_{s+\frac{1}{2}} & \\ A_s & & B_s \\ & \Psi_{s-\frac{1}{2}} & \end{pmatrix} \quad \Rightarrow \quad \sum_{k=1}^{s} \begin{pmatrix} & \Phi_{k+\frac{1}{2}} & \\ A_k & & B_k \\ & \Psi_{k-\frac{1}{2}} & \end{pmatrix} \oplus \begin{pmatrix} \Phi_{\frac{1}{2}} \\ z \end{pmatrix}$$

$$\begin{pmatrix} & A_{s+1} & \\ \Phi_{s+\frac{1}{2}} & & \Psi_{s+\frac{1}{2}} \\ & B_s & \end{pmatrix} \quad \Rightarrow \quad \begin{pmatrix} A_{s+1} \\ \Psi_{s+\frac{1}{2}} \end{pmatrix} \oplus \sum_{k=1}^{s} \begin{pmatrix} & \Phi_{k+\frac{1}{2}} & \\ A_k & & B_k \\ & \Psi_{k-\frac{1}{2}} & \end{pmatrix} \oplus \begin{pmatrix} \Phi_{\frac{1}{2}} \\ z \end{pmatrix}$$

It was crucial for the whole construction that each pair of bosonic fields with equal spins must have opposite parities and one has to consider a kind of duality mixing between these fields. Moreover, such mixing arises already at the massless supermultiplets level so that even in the massless infinite spin limit these pairs do not decouple and we still have two infinite spin bosonic and two infinite spin fermionic components. It is still possible that by abandoning parity one can construct the supermultiplet containing just one bosonic and one fermionic field but it remains to be checked.

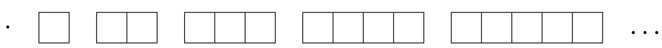
The mixing angles for the bosonic components take rather different values for the two types of supermultiplets but as can be seen from their explicit expressions in [23], in the infinite spin limit, they all become equal. At the same time, the main structural difference between them—the presence of the left most multiplet $(A_{s+1}, \Phi_{s+1/2})$—in the infinite spin limit disappears, so both types of massive supermultiplets produce the same result (up to some field re-definitions).

4. Infinite Spin Fields in $d \geq 5$

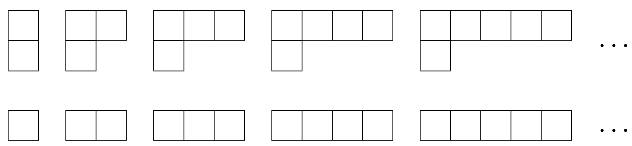
Contrary to the three- and four-dimensional cases in $d \geq 5$, there exists an infinite number of such infinite spin representations. Let us briefly reiterate how their classification arises [1]. For the massless fields, we have ${p_\mu}^2 = 0$ and by the Lorentz transformations one can always bring this vector to the canonical form $p_\mu = (1, 0, \dots, 0, 1)$. This leads to the so-called little group (i.e., group of transformations leaving this vector intact) that, besides the group $SO(d-2)$, contains pseudo translations T_i, $i = 1, 2, \dots, d-2$ that are specific combinations of spatial rotations and Lorentz boosts. Usual finite helicity massless representations correspond to the case where all $T_i = 0$ while to construct infinite spin representations one can follow the same root as for the Poincare group itself. Namely, one can consider eigen vectors for these pseudo translations $T_i|\xi_i >= \xi_i|\xi_i >$, ${\xi_i}^2$ being invariant. By using $SO(d-2)$ transformations, one can always bring such a vector to the form $(1, 0, \dots, 0)$ and this, in turn, leads to the so-called short little group $SO(d-3)$, leaving this vector intact. Thus, infinite spin representations are determined by the corresponding representations of this short little group.

Now, it is clear that for the $d = 3$ and $d = 4$, this short little group is trivial; that is why we have just one bosonic and one fermionic representation while in $d \geq 5$ there exists an infinite number of them. For example, in $d = 5$ and $d = 6$, such representations can be labeled by the parameter l taking integer $l = 0, 1, 2, \dots$ or half integer $l = \frac{1}{2}, \frac{3}{2}, \dots$ values for the bosonic and fermionic cases correspondingly. Lagrangian formulation for such representations can be obtained from the frame-like gauge-invariant formulation for the massive mixed symmetry bosonic and fermionic fields corresponding to the Young tableau $Y(k, l)$ with two rows developed in [24–26]. Namely, one has to consider a limit where mass goes to zero, k goes to infinity while l remains fixed. This construction will be presented in the forthcoming publication, so here let us just illustrate how the spectrum of such representations appears (by the spectrum, we mean a collection of usual massless fields that we have to combine to obtain an infinite spin one).

The completely symmetric case considered before corresponds to the $l = 0$ and has the following spectrum (dot stands for the scalar field):

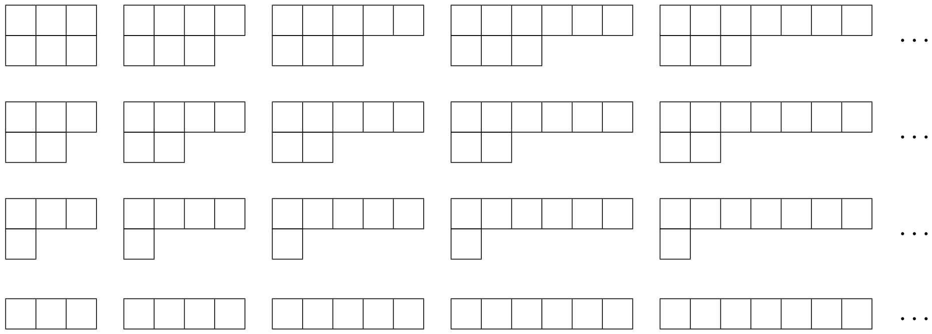


For the first non-trivial case $l = 1$, we will have two infinite chains of components:



The first line begins with the anti-symmetric second rank tensor, then it contains a hook and the whole set of long hooks, while in the second line we again have completely symmetric tensors starting with the vector field this time.

Let us give here one more concrete example for $l = 3$:



Hopefully, the general pattern is clear now. In general, in the upper left corner, we have a rectangular diagram with length l. Moving to the right, we add one box to the first row, while moving down we cut one box from the second row until we end again with the completely symmetric tensors in the bottom line.

5. Conclusions

Thus, we have seen that the same frame-like gauge-invariant formalism that has been developed for the description of massive higher spin fields can be successfully applied to the massless infinite spin case as well providing an explicit realization for the general idea that massless infinite spin representations can be obtained as an appropriate limit from the massive ones. As we have already noted, the presence of the dimensionful parameter gives hope that it may be possible to consider interactions for such fields directly in the flat space without any need to go to the anti de Sitter space. A close relationship between the frame-like gauge-invariant description for the massive higher spin fields and massless infinite spin fields means that we can try to use the same technique for the construction of possible interactions, as in the massive case. At the same time, it means that we must be ready to face the same technical difficulties as we have seen in our attempts to work with the massive high spin fields.

Acknowledgments: The author is grateful to the Joseph L. Buchbinder and Timofey V. Snegirev for collaboration. The author is also grateful to the organizers of the "Workshop on higher spin gauge theories", 26–28 April 2017, UMONS, Mons, Belgium for the kind hospitality during the workshop.

Conflicts of Interest: The author declares no conflict of interest.

References

1. Brink, L.; Khan, A.M.; Ramond, P.; Xiong, X. Continuous Spin Representations of the Poincare and Super-Poincare Groups. *J. Math. Phys.* **2002**, *43*, 6279–6295.
2. Bekaert, X.; Boulanger, N. The unitary representations of the Poincare group in any spacetime dimension. *arXiv* **2006**, arXiv:hep-th/0611263. [arXiv:hep-th/0611263].
3. Bekaert, X.; Mourad, J. The continuous spin limit of higher spin field equations. *J. High Energy Phys.* **2006**, *2006*, 115.
4. Bengtsson, A.K.H. BRST Theory for Continuous Spin. *J. High Energy Phys.* **2013**, *2013*, 108.
5. Schuster, P.; Toro, N. Continuous-spin particle field theory with helicity correspondence. *Phys. Rev. D* **2015**, *91*, 025023.
6. Schuster, P.; Toro, N. A New Class of Particle in 2 + 1 Dimensions. *Phys. Lett. B* **2015**, *743*, 224–227
7. Rivelles, V.O. Gauge Theory Formulations for Continuous and Higher Spin Fields. *Phys. Rev. D* **2015**, *91*, 125035.

8. Bekaert, X.; Najafizadeh, M.; Setare, M.R. A gauge field theory of fermionic Continuous-Spin Particles. *Phys. Lett. B* **2016**, *760*, 320–323.
9. Metsaev, R. Continuous spin gauge field in (A)dS space. *Phys. Lett. B* **2017**, *767*, 458–464.
10. Metsaev, R. Fermionic continuous spin gauge field in (A)dS space. *arXiv* **2017**, arXiv:1703.05780. [arXiv:1703.05780]
11. Zinoviev, Y.M. On Massive High Spin Particles in (A)dS. *arXiv* **2001**, arXiv:hep-th/0108192. [arXiv:hep-th/0108192]
12. Metsaev, R.R. Gauge invariant formulation of massive totally symmetric fermionic fields in (A)dS space. *Phys. Lett. B* **2006**, *643*, 205–212.
13. Buchbinder, I.L.; Snegirev, T.V.; Zinoviev, Y.M. Gauge invariant Lagrangian formulation of massive higher spin fields in (A)dS$_3$ space. *Phys. Lett. B* **2012**, *716*, 243–248.
14. Buchbinder, I.L.; Snegirev, T.V.; Zinoviev, Y.M. Frame-like gauge invariant Lagrangian formulation of massive fermionic higher spin fields in AdS$_3$ space. *Phys. Lett. B* **2014**, *738*, 258–262.
15. Boulanger, N.; Ponomarev, D.; Sezgin, E.; Sundell, P. New Unfolded Higher Spin Systems in AdS$_3$. *Class. Quantum Gravity* **2015**, *32*, 155002.
16. Zinoviev, Y.M. Massive higher spins in d = 3 unfolded. *J. Phys. A Math. Theor.* **2016**, *49*, 095401.
17. Zinoviev, Y.M. Towards the Fradkin-Vasiliev formalism in three dimensions. *Nucl. Phys. B* **2016**, *910*, 550–567.
18. Buchbinder, I.L.; Snegirev, T.V.; Zinoviev, Y.M. Lagrangian formulation of the massive higher spin supermultiplets in three dimensional space-time. *J. High Energy Phys.* **2015**, *2015*, 148.
19. Buchbinder, I.L.; Snegirev, T.V.; Zinoviev, Y.M. Unfolded equations for massive higher spin supermultiplets in AdS$_3$. *J. High Energy Phys.* **2016**, *2016*, 075.
20. Buchbinder, I.L.; Snegirev, T.V.; Zinoviev, Y.M. Lagrangian description of massive higher spin supermultiplets in AdS$_3$ space. *arXiv* **2017**, arXiv:1705.06163. [arXiv:1705.06163]
21. Zinoviev, Y.M. Frame-like gauge invariant formulation for massive high spin particles. *Nucl. Phys. B* **2009**, *808*, 185–204.
22. Ponomarev, D.S.; Vasiliev, M.A. Frame-Like Action and Unfolded Formulation for Massive Higher-Spin Fields. *Nucl. Phys. B* **2010**, *839*, 466–498.
23. Zinoviev, Y.M. Massive N = 1 supermultiplets with arbitrary superspins. *Nucl. Phys. B* **2007**, *785*, 98–114.
24. Zinoviev, Y.M. Towards frame-like gauge invariant formulation for massive mixed symmetry bosonic fields. *Nucl. Phys. B* **2009**, *812*, 46–63.
25. Zinoviev, Y.M. Frame-like gauge invariant formulation for mixed symmetry fermionic fields. *Nucl. Phys. B* **2009**, *821*, 21–47.
26. Zinoviev, Y.M. Towards frame-like gauge invariant formulation for massive mixed symmetry bosonic fields. II. General Young tableau with two rows. *Nucl. Phys. B* **2010**, *826*, 490–510.