

Review

On Exact Solutions and Perturbative Schemes in Higher Spin Theory

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Received: 9 November 2017; Accepted: 12 December 2017; Published: 1 January 2018

Abstract: We review various methods for finding exact solutions of higher spin theory in four dimensions, and survey the known exact solutions of (non)minimal Vasiliev's equations. These include instanton-like and black hole-like solutions in (A)dS and Kleinian spacetimes. A perturbative construction of solutions with the symmetries of a domain wall is also described. Furthermore, we review two proposed perturbative schemes: one based on perturbative treatment of the twistor space field equations followed by inverting Fronsdal kinetic terms using standard Green's functions; and an alternative scheme based on solving the twistor space field equations exactly followed by introducing the spacetime dependence using perturbatively defined gauge functions. Motivated by the need to provide a higher spin invariant characterization of the exact solutions, aspects of a proposal for a geometric description of Vasiliev's equation involving an infinite dimensional generalization of anti de Sitter space are revisited and improved.

Keywords: higher-spin gravity; exact solutions; higher-spin geometry

1. Introduction

Higher spin (HS) theory in four dimensions, in its simplest form and when expanded about its (anti-)de Sitter vacuum solution, describes a self-interacting infinite tower of massless particles of spin $s = 0, 2, 4, \dots$. The full field equations, proposed long ago by Vasiliev [1–3] (for reviews, see [4,5]), are a set of Cartan integrable curvature constraints on master zero-, one- and two-forms living on an extension of spacetime by a non-commutative eight-dimensional twistor space. The latter is fibered over a four-dimensional base, coordinatized by a Grassmann-even $SL(2, \mathbb{C})$ -spinor oscillator $Z^A = (z^\alpha, \bar{z}^{\dot{\alpha}})$, and the fiber is coordinatized by another oscillator $Y^A = (y^\alpha, \bar{y}^{\dot{\alpha}})$; the master fields are horizontal forms on the resulting twelve-dimensional total space, valued an infinite-dimensional associative algebra generated by Y^A , that we shall denote by \mathcal{A} , and subject to boundary conditions on the base manifold.

A key feature of Vasiliev's equations is that they admit asymptotically (anti-)de Sitter solution spaces, obtained by taking the HS algebra \mathcal{A} to be an extension of the Weyl algebra, with its Moyal star product, by involutory chiral delta functions [6,7], referred to as inner Klein operators, relying on a realization of the star product using auxiliary integration variables [4]. Introducing a related class of forms in Z -space, that facilitates a special vacuum two-form in twistor space, the resulting linearized master fields can be brought to a special gauge, referred to as the Vasiliev gauge, in which their symbols defined in a certain normal order are real analytic in twistor space, and the master zero- and one-forms admit Taylor expansions in Y at $Z = 0$ in terms of Fronsdal fields on the mass shell and subject to physical boundary conditions.

Although the Vasiliev equations take a compact and elegant form in the extended space, their analysis in spacetime proceeds in a weak field expansion which takes an increasingly complicated form beyond the leading order. Indeed, they have been determined so far only up to quadratic order. In performing the weak field expansion, a number of challenges emerge. Firstly, obtaining these equations requires boundary conditions in twistor space, referring to the topology of Z space and the classes of functions making up \mathcal{A} [2,8,9]. The proper way of pinning down these aspects remains to be determined. Second, the cosmological constant, Λ , which is necessarily nonvanishing in Vasiliev's theory (as the transvection operators of the isometry algebra are realized in \mathcal{A} as bilinears in Y), appears in the effective equations to its first power via critical mass terms, but also to arbitrary negative powers via non-local interactions [4,10]. Thus, letting ϕ denote a generic Fronsdal field, it follows that $\partial\phi \sim \sqrt{|\Lambda|}\phi$ on-shell, and hence interactions with any number of derivatives are of equal relevance (at a fixed order in weak field amplitudes). This raises the question of just how badly nonlocal are the HS field equations, the attendant problem of divergences arising even at the level of the amplitudes [11,12], and what kind of field redefinitions are admissible. One guide available is the holographic construction of the bulk vertices [13–15]. Clearly, it would be desirable to find the principles that govern the nonlocal interactions, based on the combined boundary conditions in twistor space as well as spacetime, such that an order by order construction of the bulk vertices can proceed from the analysis of Vasiliev equations. The simple and geometrical form of Vasiliev equations, in turn, may pave the way for the construction of an off-shell action that will facilitate the computation of the quantum effects.

In an alternative approach to the construction of HS equations in spacetime, it has been proposed to view Vasiliev's equations as describing stationary points of a topological field theory with a path integral measure based on a Frobenius-Chern-Simons bulk action in nine dimensions augmented by topological boundary terms, which are permitted by the Batalin-Vilkovisky formalism, of which only the latter contribute to the on-shell action [16,17].

This approach combines the virtues of the on-shell approach to amplitudes for massless particles flat spacetime with those of having a background independent action, in the sense that the on-shell action is fixed essentially by gauge symmetries and given on closed form, which together with the background independence of Vasiliev's equations provides a machinery for perturbative quantum computations around general backgrounds.

In this context, it is clearly desirable to explore in more detail how the choice of boundary conditions in the extended space influences the classical moduli space of Vasiliev's equations, with the purpose of spelling out the resulting spaces, computing HS invariant functionals on-shell, and examining how the strongly coupled spacetime nonlocalities are converted into physical amplitudes using the aforementioned auxiliary integral representation of star products in twistor space.

The aim of this article is to review three methods that have been used to find exact solutions of the Vasiliev equations, and to describe two schemes for analyzing perturbations around them. In particular we will describe the gauge function method [18,19] for finding exact solutions and summarize the first such solution found in [20], as well as its generalization to de Sitter spacetime studied in [21] together with the solutions of for a chiral version of the theory with Kleinian $(2,2)$ signature. As we shall see, this method uses the fact that the spacetime dependence of the master fields can be absorbed into gauge functions, upon which the problem of finding exact solutions is cast into a relatively manageable deformed oscillator problem in twistor space. The role of different ordering schemes for star product as well as gauge choices to fix local symmetries in twistor space will also be discussed.

Next, we will describe a refined gauge function method proposed in [22], where the twistor space equations are solved by employing separation of twistor variables and holomorphicity in the Z space in a Weyl ordering scheme and enlarging the Weyl algebra in the fiber Y space by inner Kleinian operators. This approach provides exact solution spaces in a particular gauge, that we refer to as the holomorphic gauge, after which the spacetime dependence is introduced by means of a sequence of large gauge transformations, by first switching on a vacuum gauge function, taking the

solutions to what we refer to as the L -gauge, where the configurations must be real analytic in Z space, which provides an admissibility condition on the initial data in holomorphic gauge. The solutions can then be mapped further to the Vasiliev gauge, where the linearized, or asymptotic, master fields, are real analytic in the full twistor space and obey a particular gauge condition in Z space which ensures that they consist of decoupled Fronsdal fields in a canonical basis; the required gauge transformation, from the L gauge to the Vasiliev gauge, can be constructed in a perturbation scheme, which has so far been implemented mainly at the leading order. We will describe a black hole-like solution in some detail and mention other known solutions obtained by this method so far, including new solutions with six Killing symmetries [23].

We shall also outline a third method, in which the HS equations are directly tackled without employing gauge functions. In this method, solving the deformed oscillators in twistor space also employs the projector formalism, though the computation of the gauge potentials does not rely on the gauge function method. The black hole-like solution found in this way in [24] will be summarized.

We shall also review two approaches to the perturbative treatment of Vasiliev's equations. One of them, which we refer to as the normal ordered scheme, is based on a weak field expansion around (anti-)de Sitter spacetime [3,4,25]. It entails nested parametric integrals, introduced via a homotopy contraction of the de Rham differential in Z space used to solve the curvature constraints that have at least one form index in Z space, followed by inserting the resulting perturbatively defined master fields into the remaining curvature constraints with all form indices in spacetime. In an alternative scheme, the equations are instead solved exactly in the aforementioned L -gauge, and a perturbatively realized large HS gauge transformation is then performed to achieve interpretation in terms of Fronsdal fields in asymptotically (anti-)de Sitter spacetimes in Vasiliev gauge [7]. The advantages of the latter approach in describing the fluctuations around more general HS backgrounds will be explained.

A word of caution is in order concerning the usage of 'black hole' terminology in describing certain types of exact solutions to HS equations. This terminology is, in fact, misleading in some respects since the notion of a line interval associated with a metric field is not HS invariant. Indeed, the apparent singular behaviour at the origin may in principle be a gauge artifact. This point is discussed in more detail in Section 4.2. Moreover, given the nonlocal nature of the HS interactions, the formulation of causality, which is crucial in describing the horizon of a black hole, is a challenging problem without any proposal for a solution yet in sight; in fact, a more natural physical interpretation of the black hole-like solutions may turn out to be as smooth black-hole microstates [7,26]. Another aspect of the known black hole-like solution in HS theory is that they activate fields of all possible spins, and apparently it is not possible to switch off all spins except one even in the asymptotically AdS region.

So, what is meant by a black hole solution in HS theory? Firstly, the $SO(3) \times SO(2)$ symmetry of the solution (which is part of an infinite dimensional extended symmetry forming a subgroup of the HS symmetry group) is in common with the symmetry group arising in the asymptotically AdS BH solution of ordinary AdS gravity. Second, the solution contains a spin-two Weyl tensor field which takes the standard Petrov type D form, with a singularity at the origin; more generally, the spin- s Weyl tensors are of a generalized Petrov type D form, given essentially by the s -fold direct products of a spin-one curvature of the Petrov type D form. The BH terminology is thus used in the context of HS theory with the understanding that it is meant to convey these properties, albeit they do not constitute a rigorous definition of a black hole in HS theory.

The use of HS invariants for exact solutions to capture their physical characteristics has been considered and in some cases they have been computed successfully. These particular invariants alone do not, however, furnish an answer to the question of whether it makes any sense to think about event horizons in HS theory at all, and if so, how to define them; in fact, their existence rather supports the aforementioned microstate proposal, wherein the HS invariants can be interpreted as extensive charges defining HS ensembles.

Motivated by the quest for giving a physical interpretation of the exact solution in the context of underlying HS symmetries, a geometrical approach to HS equations was proposed in [27]. We shall

summarize this proposal in which the HS geometry is based on an identification of an infinite dimensional structure group in a fibre bundle setting, and the related soldering phenomenon that leads to a HS covariant definition of classes of (non-unique) generalized vielbeins and related metrics, and as such an infinite dimensional generalization of *AdS* geometry. In doing so, we will improve the formulation of [27] by dispensing with the embedding of the relevant infinite dimensional coset space into a larger one that involves the extended HS algebra that includes the twistor space oscillators.

Finally, we are not aware of any exact solutions of HS theories in dimensions $D > 4$ [28,29], while in $D = 3$, assuming that the scalar field is coupled to HS fields, we can refer to [8,30,31] for the known solutions. Purely topological HS theory, which has no dynamical degrees of freedom, and which allows a more rigorous definition of black holes, is known to admit many exact solutions whose description goes beyond the scope of this review.

2. Vasiliev Equations

2.1. Bosonic Model in (A)dS

Vasiliev’s theory is formulated in terms of horizontal forms on a non-commutative fibered space \mathcal{C} with four-dimensional non-commutative symplectic fibers and eight-dimensional base manifold equipped with a non-commutative differential Poisson structure. On the total space, the differential form algebra $\Omega(\mathcal{C})$ is assumed to be equipped with an associative degree preserving product \star , a differential d , and an Hermitian conjugation operation \dagger , that are assumed to be mutually compatible. The base manifold is assumed to be the direct product of a commuting real four-manifold \mathcal{X}_4 with coordinates x^μ , and a non-commutative real four-manifold \mathcal{Z}_4 with coordinates Z^A ; the fiber space and its coordinates are denoted by \mathcal{Y}_4 and Y^A , respectively. The non-commutative coordinates are assumed to obey

$$[Y^A, Y^B]_\star = 2iC^{AB}, \quad [Z^A, Z^B]_\star = -2iC^{AB}, \quad [Y^A, Z^B]_\star = 0, \quad (2.1)$$

where C_{AB} is a real constant antisymmetric matrix. The non-commutative space is furthermore assumed to have a compatible complex structure, such that

$$Y^A = (y^\alpha, \bar{y}^{\dot{\alpha}}), \quad Z^A = (z^\alpha, \bar{z}^{\dot{\alpha}}), \quad (2.2)$$

$$(y^\alpha)^\dagger = \bar{y}^{\dot{\alpha}}, \quad (z^\alpha)^\dagger = -\bar{z}^{\dot{\alpha}}, \quad (2.3)$$

where the complex doublets obey $[y^\alpha, y^\beta]_\star = 2i\epsilon^{\alpha\beta}$ and $[z^\alpha, z^\beta]_\star = -2i\epsilon^{\alpha\beta}$. The horizontal forms can be represented as sets of locally defined forms on $\mathcal{X}_4 \times \mathcal{Z}_4$ valued in oscillator algebras $\mathcal{A}(\mathcal{Y}_4)$ generated by the fiber coordinates glued together by transition functions, that we shall assume are defined locally on \mathcal{X}_4 , resulting in a bundle over \mathcal{X}_4 with fibers given by $\Omega(\mathcal{Z}_4) \otimes \mathcal{A}(\mathcal{Y}_4)$. The algebra $\mathcal{A}(\mathcal{Y}_4)$ can be given in various bases; we shall use the Weyl ordered basis, and the normal ordered basis consisting of monomials in $a_\pm = Y \pm Z$ with a_+ and a_- oscillators standing to the left and right, respectively. We assume that the elements in $\Omega(\mathcal{Z}_4) \otimes \mathcal{A}(\mathcal{Y}_4)$ have well-defined symbols in both Weyl and normal order. The normal order reduces to Weyl order for elements that are independent of either Y or Z , and in the cases where depend on both Y and Z , they can be composed using the Fourier transformed twisted convolution formula in normal ordered scheme as

$$(f \star g)(Z; Y) = \frac{1}{(2\pi)^4} \int_{\mathbb{R}^4 \times \mathbb{R}^4} d^4U d^4V f(Z + U; Y + U) g(Z - V; Y + V) e^{iV^A U_A}. \quad (2.4)$$

The model is formulated in terms of a zero-form Φ , a one-form

$$A = dx^\mu W_\mu + dz^\alpha V_\alpha + d\bar{z}^{\dot{\alpha}} \bar{V}_{\dot{\alpha}}, \quad (2.5)$$

and a non-dynamical holomorphic two-form

$$J := -\frac{ib}{4} dz^\alpha \wedge dz_\alpha \kappa, \tag{2.6}$$

with Hermitian conjugate $\bar{J} = (J)^\dagger$, where b is a complex parameter and

$$\kappa := \kappa_y \star \kappa_z, \quad \kappa_y := 2\pi\delta^2(y), \quad \kappa_z := 2\pi\delta^2(z), \tag{2.7}$$

are inner Klein operators obeying $\kappa_y \star f \star \kappa_y = \pi_y(f)$ and $\kappa_z \star f \star \kappa_z = \pi_z(f)$ for any zero-form f , where π_y and π_z are the automorphisms of $\Omega(\mathcal{Z}_4) \otimes \mathcal{A}(\mathcal{Y}_4)$ defined in Weyl order by

$$\pi_y(x; z, \bar{z}; y, \bar{y}) = (x; z, \bar{z}; -y, \bar{y}), \quad \pi_z(x; z, \bar{z}; y, \bar{y}) = (x; -z, \bar{z}; y, \bar{y}). \tag{2.8}$$

It follows that $dJ = 0$, $J \star f = \pi(f) \star J$ and $\pi(J) = J$, *idem* \bar{J} , with

$$\pi := \pi_y \circ \pi_z, \quad \bar{\pi} := \pi_{\bar{y}} \circ \pi_{\bar{z}}. \tag{2.9}$$

It is useful to note that the inner Kleinian takes the following forms in different ordering schemes:

$$\kappa = \begin{cases} e^{iy^\alpha z_\alpha} & \text{in normal ordering scheme} \\ (2\pi)^2 \delta^2(y) \delta^2(z) & \text{in Weyl ordering scheme} \end{cases} \tag{2.10}$$

The nonminimal and minimal models with all integer spins and only even spins, respectively, are obtained by imposing the conditions

$$\text{Non-minimal model } (s = 0, 1, 2, 3, \dots) : \pi \circ \bar{\pi}(A) = A, \quad \pi \circ \bar{\pi}(B) = B, \tag{2.11}$$

$$\text{Minimal model } (s = 0, 2, 4, \dots) : \tau(A) = -A, \quad \tau(B) = \bar{\pi}(B), \tag{2.12}$$

where τ is the anti-automorphism

$$\tau(x^\mu; Y^A, Z^A) = f(x^\mu; iY^A, -iZ^A), \quad \tau(f \star g) = \tau(g) \star \tau(f), \tag{2.13}$$

It follows that $\tau(J, \bar{J}) = (-J, -\bar{J})$. Models in Lorentzian spacetimes with cosmological constants Λ are obtained by imposing reality conditions as follows [21]:

$$\rho(B^\dagger) = \pi(B), \quad \rho(A^\dagger) = -A, \quad \rho := \begin{cases} \pi, & \Lambda > 0 \\ \text{Id}, & \Lambda < 0 \end{cases} \tag{2.14}$$

Basic building blocks for Vasiliev equations are the curvature and twisted-adjoint covariant derivative defined by

$$F := dA + A \star A, \quad DB := dB + [A, B]_\pi, \tag{2.15}$$

respectively, where the π -twisted star commutators is defined as

$$[f, g]_\pi := f \star g - (-1)^{|f||g|} g \star \pi(f), \tag{2.16}$$

and

$$d := d_x + d_z, \quad d_x = dx^\mu \partial_\mu, \quad d_z = dz^\alpha \partial_\alpha + d\bar{z}^{\dot{\alpha}} \bar{\partial}_{\dot{\alpha}}. \tag{2.17}$$

Vasiliev equations of motion are given by

$$F + B \star (J - \bar{J}) = 0, \quad DB = 0, \tag{2.18}$$

which are compatible with the kinematic conditions and the Bianchi identities, implying that the classical solution space is invariant under the following infinitesimal gauge transformations:

$$\delta A = D\epsilon := d\epsilon + [A, \epsilon]_\star, \quad \delta B = -[\epsilon, B]_\pi, \tag{2.19}$$

for parameters obeying the same kinematic conditions as the connection.

It remains a challenging problem to determine if these equations can be derived from a suitable tensionless, or critical tension, limit followed by a consistent truncation of string field theory on a background involving AdS_4 . It will be very interesting to also determine if these equations follow from a consistent quantization of a topological string field theory. For a more detailed discussion and progress in this direction, see [32,33]. The component fields of V_A do not transform properly under the Lorentz transformations generated by $(\frac{1}{4i}(y^\alpha y^\beta - z^\alpha z^\beta) - h.c.)$. To remedy this problem and achieve manifest Lorentz covariance, one introduces the field-dependent Lorentz generators [4,25]

$$M_{\alpha\beta} = M_{\alpha\beta}^{(0)} + S_{(\alpha} \star S_{\beta)}, \quad M_{\dot{\alpha}\dot{\beta}}^{(0)} := y_{(\alpha} \star y_{\beta)} - z_{(\alpha} \star z_{\beta)}, \tag{2.20}$$

and their complex conjugates, where

$$S_\alpha = z_\alpha - 2iV_\alpha, \quad \bar{S}_{\dot{\alpha}} = \bar{z}_{\dot{\alpha}} - 2i\bar{V}_{\dot{\alpha}}. \tag{2.21}$$

Next one defines

$$W'_\mu = W_\mu - \frac{1}{4i} \left(\omega_\mu^{\alpha\beta} M_{\alpha\beta} + \bar{\omega}_\mu^{\dot{\alpha}\dot{\beta}} \bar{M}_{\dot{\alpha}\dot{\beta}} \right), \tag{2.22}$$

where $(\omega_\mu^{\alpha\beta}, \bar{\omega}_\mu^{\dot{\alpha}\dot{\beta}})$ is the canonical Lorentz connection.

It is defined up to tensorial shifts [27] that can be fixed by requiring that the projection of W' onto $M_{\alpha\beta}^{(0)}$ and its complex conjugate, vanish at $Z = 0$, that is

$$\frac{\partial^2}{\partial y^\alpha \partial y^\beta} W'|_{Y=Z=0} = 0, \quad \frac{\partial^2}{\partial \bar{y}^{\dot{\alpha}} \partial \bar{y}^{\dot{\beta}}} W'|_{Y=Z=0} = 0. \tag{2.23}$$

The above redefinitions ensure that under the Lorentz transformations with parameters

$$\epsilon_L = \frac{1}{4i} \Lambda^{\alpha\beta} M_{\alpha\beta}, \quad \epsilon_L^{(0)} = \frac{1}{4i} \Lambda^{\alpha\beta} M_{\alpha\beta}^{(0)}, \tag{2.24}$$

the master fields transform properly under the Lorentz transformations as [25]

$$\delta_L B = [\epsilon_L^{(0)}, B]_\star, \tag{2.25}$$

$$\delta_L S_\alpha = [\epsilon_L^{(0)}, S_\alpha]_\star + \Lambda_\alpha{}^\beta S_\beta, \quad \text{idem} \quad \bar{S}_{\dot{\alpha}}, \tag{2.26}$$

$$\delta_L W'_\mu = [\epsilon_L^{(0)}, W'_\mu]_\star + \frac{1}{4i} \left(\partial_\mu \Lambda^{\alpha\beta} M_{\alpha\beta} + h.c. \right). \tag{2.27}$$

Using (2.21), the component form of Vasiliev equations reads

$$d_x W + W \star W = 0, \tag{2.28}$$

$$d_x B - [W, B]_\pi = 0, \tag{2.29}$$

$$d_x S_\alpha + [W, S_\alpha]_\star = 0, \quad d_x \bar{S}_{\dot{\alpha}} + [W, \bar{S}_{\dot{\alpha}}]_\star = 0, \tag{2.30}$$

$$[S_\alpha, B]_\pi = 0, \quad [\bar{S}_{\dot{\alpha}}, B]_\pi = 0, \quad [S_\alpha, \bar{S}_{\dot{\beta}}]_\star = 0, \tag{2.31}$$

$$[S^\alpha, S_\alpha]_\star = 4i(1 - bB \star \kappa), \quad [\bar{S}^{\dot{\alpha}}, \bar{S}_{\dot{\alpha}}]_\star = 4i(1 - \bar{b}B \star \bar{\kappa}). \tag{2.32}$$

This is the form of the equations typically used to seek exact solutions as it displays the role of the deformed oscillator algebra in the last two equations. Here one may exploit the technology developed in the study of noncommutative field theories and the construction of projection operators in a suitably defined oscillator space.

2.2. The Nonminimal Chiral Model in Kleinian Space

In this model the spinor oscillators are now representations of $SL(2, \mathbb{R})_L \times SL(2, \mathbb{R})_R$, and as such their hermitian conjugates are now given by

$$(y^\alpha)^\dagger = y^\alpha, \quad (z^\alpha)^\dagger = -z^\alpha, \quad (\bar{y}^{\dot{\alpha}})^\dagger = \bar{y}^{\dot{\alpha}}, \quad (\bar{z}^{\dot{\alpha}})^\dagger = -\bar{z}^{\dot{\alpha}}. \tag{2.33}$$

The field equations are now given by

$$F = J \star B, \quad DB = 0, \quad J := -\frac{i}{4} dz^\alpha \wedge dz_\alpha \kappa, \tag{2.34}$$

with reality conditions $A^\dagger = -A$ and $B^\dagger = \pi(B)$, and kinematical conditions

$$\text{Minimal model } (s = 0, 2, 4, \dots) : \tau(A) = -A, \quad \tau(B) = \bar{\pi}(B), \tag{2.35}$$

$$\text{Non-minimal model } (s = 0, 1, 2, 3, \dots) : \pi \circ \bar{\pi}(A) = A, \quad \pi \circ \bar{\pi}(B) = B. \tag{2.36}$$

The field equations in components take the same form as in (2.28)–(2.32), but now with

$$b = 1, \quad \bar{b} = 0. \tag{2.37}$$

These models are referred to as chiral in view of the half-flatness condition on the twistor space curvature, namely $\bar{F}_{\dot{\alpha}\beta} = 0$. These models admit the coset space $H_{3,2} = SO(3, 2)/SO(2, 2)$ as a vacuum solution, which has the Kleinian signature (2, 2). For a detailed description of these spaces, including the curved Kleinian geometries, see [34]. Our motivation for highlighting this case is due to the fact that the first exact solution of the Vasiliev equation in which all HS fields are nonvanishing was found for this model [21], and that the Kleinian geometry is relevant to $N = 2$ superstring as well as to integrable models.

3. Gauge Function Method and Solutions

3.1. The Method

In order to construct solutions to Vasiliev’s equations, one may consider the approach [19] in which they are homotopy contracted in simply connected spacetime regions U to deformed oscillator algebras in twistor space at a base point $p \in U$; the constraints

$$F_{\mu\nu} = 0, \quad F_{\mu\alpha} = 0, \quad D_\mu B = 0, \tag{3.1}$$

are thus integrated in U using a gauge function $g = g(Z, Y|x)$ obeying

$$g|_p = 1, \tag{3.2}$$

and initial data

$$B' = B|_p, \quad S'_A = S_A|_p, \tag{3.3}$$

subject to

$$[S'_{\alpha}, B']_{\pi} = 0, \quad [\bar{S}'_{\dot{\alpha}}, B']_{\pi} = 0, \quad [S'_{\alpha}, \bar{S}'_{\dot{\beta}}]_{\star} = 0, \tag{3.4a}$$

$$[S'^{\alpha}, S'_{\alpha}]_{\star} = 4i(1 - bB' \star \kappa), \quad [\bar{S}'^{\dot{\alpha}}, \bar{S}'_{\dot{\alpha}}]_{\star} = 4i(1 - \bar{b}B' \star \bar{\kappa}). \tag{3.4b}$$

The fields in U can then be expressed explicitly as

$$W_{\mu} = g^{-1} \star \partial_{\mu} g, \quad S_A = g^{-1} \star S'_A \star g, \quad B = g^{-1} \star B' \star \pi(g), \tag{3.5}$$

after which the Lorentz covariant HS gauge fields can be obtained from (2.22) subject to (2.23), which serves to determine the spin connection $\omega_{\mu ab}$. Thus, the deviations in the spacetime HS gauge fields away from the topological vacuum solution, that is the solution with $W_{\mu} = 0$, thus come from the gauge function g as well as the non-linear shift on the account of achieving manifest Lorentz covariance. The deformed oscillator algebra requires a choice of topology for \mathcal{Z}_4 , initial data for B' and a flat background connection. In what follows, we shall assume that \mathcal{Z}_4 has the topology of \mathbb{R}^4 with suitable fall-off conditions at infinity [7,17], and impose

$$C'(Y) = B'|_{Z=0}, \quad S'_A|_{C'=0} = Z_A; \tag{3.6}$$

for nontrivial flat connections on \mathcal{Z}_4 , that are not pure gauge, see [21]. The gauge function represents a gauge transformation that is large in the sense that it affects the asymptotics of gauge fields so as to introduce additional physical degrees of freedom to the system, over and above those contained in the twistor space initial data and flat connection; strictly speaking, in order to define such transformations, one should first introduce a set of classical observables forming a BRST cohomology modulo a set of boundary conditions on ghosts, after which a large gauge transformation is a gauge transformation that does not preserve all the classical observables. In particular, in order to describe asymptotically maximally symmetric, or Weyl flat, solutions, one may take

$$g|_{B'=0} = L, \tag{3.7}$$

where $L = L(Y|x)$ is a metric vacuum gauge function, to be described below. In order to obtain exact solutions, we shall choose $g = L$ for all C' , that we refer to as the L -gauge. However, in order to extract Fronsdal fields in the asymptotic region, one has to impose a gauge condition in twistor space to the leading order in the weak field expansion in the asymptotic region, which introduces a dressing of the vacuum gauge function by an additional perturbatively determined gauge function; see Section 6.

3.2. Vacuum Solutions

In order to obtain solutions containing locally maximally symmetric asymptotic regions, one may take the gauge function $L(Y|x)$ to be corresponding coset representatives. In what follows, we shall focus on the spaces $AdS_4 = SO(3,2)/SO(3,1)$, $dS_4 = SO(4,1)/SO(3,1)$ and $H_{3,2} := SO(3,2)/SO(2,2)$, which can be realized as the embeddings

$$X^A X^B \eta_{AB} \equiv -(X^0)^2 + (X^1)^2 + (X^2)^2 + (X^3)^2 + \epsilon (X^5)^2 = -\lambda^{-2}, \tag{3.8}$$

where

$$\left(\epsilon, \frac{\lambda^2}{|\lambda^2|} \right) = \begin{cases} (-, +) & \text{for } AdS_4 \\ (+, -) & \text{for } dS_4 \\ (-, -) & \text{for } H_{3,2} \end{cases} \tag{3.9}$$

These spaces can be conveniently described in a unified fashion using the stereographic coordinates x_{\pm}^a ($a = 0, 1, 2, 3$) obtained by means of the parametrization

$$X^M|_{U_{\pm}} \approx \left(\frac{2x_{\pm}^a}{1 - \lambda^2 x_{\pm}^2}, \pm \ell \frac{1 + \lambda^2 x_{\pm}^2}{1 - \lambda^2 x_{\pm}^2} \right), \quad -1 \leq \lambda^2 x_{\pm}^2 < 1, \tag{3.10}$$

$$x_{\pm}^2 := x_{\pm}^a x_{\pm}^b \eta_{ab}, \quad \eta_{ab} = \text{diag}(\epsilon, \lambda^2/|\lambda^2|, +, +), \quad \ell = |\lambda|^{-1}, \tag{3.11}$$

where U_{\pm} denotes the two stereographic coordinates charts, each covering one half of the space (3.9); on the overlap one has $\lambda^2 x_{\pm}^2 = -1$, and the coordinate transition function

$$x_{\pm}^a = R^a(x_{\mp}), \quad \lambda^2 x_{\pm}^2 < 0, \tag{3.12}$$

where the reflection map

$$R^a(v) := -\frac{v^a}{\lambda^2 v^2}. \tag{3.13}$$

The boundary is given by $\lambda^2 x_{\pm}^2 = 1$, which has the topology of $S^2 \times S^1$ in the case of AdS_4 and $H_{3,2}$, and $S^3 \cup S^3$ in the case of dS_4 . Instead of covering the vacuum manifold with two charts, one may extend either one of the charts to $\mathbb{R}^4 \setminus \{x^a : \lambda^2 x^2 = 1\}$, which provides a global cover using a single chart, with the understanding that $\{x^a : \lambda^2 x^2 = 1^-\} \cup \{x^a : \lambda^2 x^2 = 1^+\}$ provides a two-sheeted cover of the boundary. The induced line element $ds_0^2 = dX^A dX^B \eta_{AB}|_{\lambda^2 x^2 = -1}$ is given by

$$ds_0^2 = \frac{4dx^2}{(1 - \lambda^2 x^2)^2}. \tag{3.14}$$

On $|\lambda^2| x^2 < 1$, the corresponding vacuum gauge function

$$L = \frac{2h}{1+h} \exp(-iy^{\alpha} a_{\alpha}{}^{\dot{\alpha}} \bar{y}_{\dot{\alpha}}), \tag{3.15}$$

where

$$a_{\alpha\dot{\alpha}} = \frac{\lambda x_{\alpha\dot{\alpha}}}{1+h}, \quad x_{\alpha\dot{\alpha}} = (\sigma^a)_{\alpha\dot{\alpha}} x_a, \quad h = \sqrt{1 - \lambda^2 x^2}. \tag{3.16}$$

$$W_0 \equiv e_0 + \omega_0 = L^{-1} \star dL = \frac{1}{4i} \left[\omega_0^{\alpha\beta} y_{\alpha} y_{\beta} + \bar{\omega}_0^{\dot{\alpha}\dot{\beta}} \bar{y}_{\dot{\alpha}} \bar{y}_{\dot{\beta}} + 2e_0^{\alpha\dot{\alpha}} y_{\alpha} \bar{y}_{\dot{\alpha}} \right], \tag{3.17}$$

where

$$e_0^{\alpha\dot{\alpha}} = -\frac{\lambda(\sigma^a)^{\alpha\dot{\alpha}} dx_a}{h^2}, \quad \omega_0^{\alpha\beta} = -\frac{\lambda^2(\sigma^{ab})^{\alpha\beta} dx_a dx_b}{h^2}. \tag{3.18}$$

A global description can be obtained using two gauge functions $L_{\pm} = L(Y|x_{\pm})$ defined on U_{\pm} ; the Z_2 -symmetry implies that if $\Phi_{\pm}|_{p_{\pm}} = C'$, where $p_{\pm} := x_{\pm}^{-1}(0)$, then the two locally defined solutions on U_{\pm} can be glued together using the gauge transition function $T_{-}^{+} := L_{+}^{-1} \star L_{-} = 1$ defined on the overlap region where $\lambda^2 x_{\pm}^2 = -1$.

For later purposes, it is convenient to introduce alternative coordinate systems which are defined by the embeddings (with $|\lambda|^2 = 1$)

$$AdS_4 : \quad x^2 > 0 : \quad X^0 = \sinh \tau \sinh \psi, \quad X^i = n^i \cosh \tau \sinh \psi, \quad X^5 = \cosh \psi, \tag{3.19}$$

$$x^2 < 0 : \quad X^0 = \sin \tau \cosh \psi, \quad X^i = n^i \sin \tau \sinh \psi, \quad X^5 = \cos \tau,$$

$$dS_4 : \quad x^2 > 0 : \quad X^0 = \sinh \tau \sin \psi, \quad X^i = n^i \cosh \tau \sin \psi, \quad X^5 = \cos \psi, \tag{3.20}$$

$$x^2 < 0 : \quad X^0 = \sinh \tau \cosh \psi, \quad X^i = n^i \sinh \tau \sinh \psi, \quad X^5 = \cosh \tau,$$

The metrics for (A)dS in these coordinate systems are given in (3.41)–(3.44). In the case of $H_{3,2}$, we will find the following coordinate system to be useful (with $|\lambda^2| = 1$)

$$X^0 = r \sin t, \quad X^i = n^i \sqrt{1 + r^2}, \quad X_5 = r \cos t, \tag{3.21}$$

where $n^i n^j \delta_{ij} = 1, 0 \leq r < \infty$ and $0 \leq t \leq 2\pi$.

3.3. Instanton Solutions of Minimal Model in (anti) de Sitter Space

Having obtained a vacuum gauge function, the next task is to solve the deformed oscillator problem (3.4) subject to the initial data in twistor space. To this end, it is helpful to constrain the primed configurations further by assuming that they preserve a nontrivial amount of HS symmetries. This can be achieved by imposing symmetry conditions by seeking a subspace \mathfrak{k}' of HS gauge parameters ϵ' that are unbroken, i.e.,

$$\delta_{\epsilon'} B' = 0, \quad \delta_{\epsilon'} S'_\alpha = 0, \tag{3.22}$$

for all $\epsilon' \in \mathfrak{k}'$; upon switching on the gauge function g , the resulting full solution is invariant under gauge parameters in the space $\mathfrak{k} = g^{-1} \star \mathfrak{k}' \star g$. For example, one may require that an n -dimensional subalgebra \mathfrak{g}_n of the maximal finite dimensional subalgebra \mathfrak{g}_{10} of the HS algebra remains unbroken, which implies that \mathfrak{k}' is given by the intersection of $\text{Env}(\mathfrak{g}_n)$ and the HS algebra. In particular, taking $n = 10$ yields the vacuum solution

$$W' = W_0, \quad S_A = z_A, \quad B = 0, \tag{3.23}$$

which preserves the HS algebra itself. Taking $n < 10$, the first distinct cases with nontrivial Weyl zero-form arise for $n = 6$; the space \mathfrak{k}' is then given exactly, as we shall describe below for a particular realization of \mathfrak{g}_6 , or perturbatively. In the latter case the \mathfrak{g}_6 will be realized as a subalgebra of the HS algebra in the leading order.

In [20], asymptotically anti-de Sitter solutions with $\mathfrak{g}_6 = \mathfrak{o}(1, 3)$ were constructed by taking \mathfrak{k} to be generated by the full Lorentz generators $M_{\alpha\beta}$ from (2.20). Thus, in the primed basis, the corresponding symmetry conditions read

$$[M'_{\alpha\beta}, B']_\pi = 0, \quad [M'_{\alpha\beta}, S'_\gamma]_\star = 0, \quad [M'_{\alpha\beta}, \bar{S}'_\gamma]_\star = 0, \tag{3.24}$$

and complex conjugates, where $M'_{\alpha\beta} = M_{\alpha\beta}^{(0)} + S'_{(\alpha} \star S'_{\beta)}$, which are given by $y_\alpha y_\beta$ plus perturbative corrections, and whose star product commutators close modulo Lorentz transformations acting on the component fields; thus, consistency of the invariance conditions implies that all canonical Lorentz tensors that are not singlets must vanish. Alternatively, it is possible to use other embeddings of $\mathfrak{o}(1, 3)$ into the algebra of primed HS gauge transformations; for example, one can simply take $y_\alpha y_\beta$, as we shall comment on in Section 5, though it remains an open problem whether the resulting solutions are gauge equivalent to those that will be presented below. Taking \mathfrak{g}_n to be generated by unperturbed functions of Y is useful, however, in considering unbroken symmetry algebra involving transvection operators, as we shall spell out in further detail in Section 5 in the case of domain walls and related time dependent solutions.

Turning to (3.24), the simplest possible ansatz for B' is a constant, viz.,

$$B' = \nu, \tag{3.25}$$

which leads to a deformed oscillator problem with exact solutions closely related to those of the 3D HS theory constructed by Prokushkin and Vasiliev [8]. Adapting to the 4D Type A model, for which $b = 1$, the following solution for the twistor space connection was found in [20]

$$S'_\alpha = z_\alpha + z_\alpha \int_{-1}^1 dt q(t) \exp\left(\frac{i}{2}(1+t)u\right), \quad u := y^\alpha z_\alpha, \tag{3.26}$$

$$q(t) = -\frac{\nu}{4} \left({}_1F_1\left[\frac{1}{2}; 2; \frac{\nu}{2} \log \frac{1}{t^2}\right] + t {}_1F_1\left[\frac{1}{2}; 2; -\frac{\nu}{2} \log \frac{1}{t^2}\right] \right). \tag{3.27}$$

Expanding $\exp(itu/2)$ results in integrals of the degenerate hypergeometric functions times positive algebraic powers of t , which improve the convergence at $t = 0$. Thus V_α is a power series expansion in u with coefficients that are functions of ν that are well-behaved provided this is the case for the coefficient of u^0 . This is the case for ν in some finite region around $\nu = 0$, as discussed in detail in [20,35]. Indeed, as we shall see below after carrying out the integration over t , ν must lie in the interval $-3 \leq \nu \leq 1$.

The solution in spacetime is obtained in the two regions $\lambda^2 x^2 < 1$ and $\lambda^2 \tilde{x}^2 < 1$ using the stereographic gauge functions $L \equiv L(x|Y)$ and $\tilde{L} \equiv L(\tilde{x}|Y)$ where x and \tilde{x} are related by the reflection map in the overlap region where $\lambda^2 x^2 < 0$ and $\lambda^2 \tilde{x}^2 < 0$. From (3.16), one finds [20]

$$B = \nu(1 - \lambda^2 x^2) \exp[-i\lambda x^{\dot{\alpha}} y_\alpha \bar{y}_{\dot{\alpha}}]. \tag{3.28}$$

This shows that the physical scalar field is given in the x^a -coordinate chart by

$$\phi(x) = B|_{Y=Z=0} = \nu(1 - \lambda^2 x^2), \quad \lambda^2 x^2 < 1, \tag{3.29}$$

while the Weyl tensors for spin $s = 2, 4, \dots$ vanish. Using instead \tilde{L} , the physical scalar field in the \tilde{x}^a -coordinate chart is given by

$$\tilde{\phi} = \nu(1 - \lambda^2 \tilde{x}^2), \quad \lambda^2 \tilde{x}^2 < 1. \tag{3.30}$$

As a result, the two scalar fields are related by a duality transformation in the overlap region

$$\tilde{\phi}(\tilde{x}) = \frac{\nu\phi(x)}{\phi(x) - \nu}, \quad \lambda^2 x^2 = (\lambda^2 \tilde{x}^2)^{-1} < 0. \tag{3.31}$$

Thus, if the transition takes place at $\lambda^2 x^2 = \lambda^2 \tilde{x}^2 = -1$, then the amplitude of the physical scalar never exceeds 2ν .

The master fields S_A and W'_μ are obtained from (3.5) and (2.22). The generating functional for the spacetime gauge fields is given by [20]¹

$$W'|_{Z=0} = W_0 - \frac{1}{4i} Q \omega_\mu^{\alpha\beta} \left[(1+a^2)^2 y_\alpha y_\beta + 4(1+a^2) a_\alpha^{\dot{\alpha}} y_\beta \bar{y}_{\dot{\alpha}} + 4 a_\alpha^{\dot{\alpha}} a_\beta^{\dot{\beta}} \bar{y}_{\dot{\alpha}} \bar{y}_{\dot{\beta}} \right] - \text{h.c.}, \tag{3.32}$$

¹ An Euclidean version of this solution has been obtained in [21], and as the spin connection plays an eminent role in this solution and assuming that the action, as proposed in [17], is finite on this solution, we use the terminology of “instanton solution”.

subject to (2.23), which serve to determine the spin connection, and where $a_{\alpha\dot{\alpha}}$ is defined in (3.16), and

$$a^2 := a^{\alpha\dot{\alpha}}a_{\alpha\dot{\alpha}} = \frac{1 - \sqrt{1 - \lambda^2 x^2}}{1 + \sqrt{1 - \lambda^2 x^2}}, \quad -1 \leq a^2 \leq 1, \tag{3.33}$$

$$Q = -\frac{1}{4}(1 - a^2)^2 \int_{-1}^1 dt \int_{-1}^1 dt' \frac{q(t)q(t')(1+t)(1+t')}{(1 - tt'a^2)^4}. \tag{3.34}$$

This gives [20]

$$Q = -\frac{(1 - a^2)^2}{4} \sum_{p=0}^{\infty} \left[\binom{2p+3}{2p} \left(\sqrt{1 - \frac{\nu}{2p+1}} - \sqrt{1 + \frac{\nu}{2p+3}} \right)^2 a^{4p} - \binom{2p+4}{2p+1} \left(\sqrt{1 - \frac{\nu}{2p+3}} - \sqrt{1 + \frac{\nu}{2p+3}} \right)^2 a^{4p+2} \right], \tag{3.35}$$

exhibiting branch cuts for $\text{Re } \nu \leq -3$ and $\text{Re } \nu \geq 1^2$. For $\nu \ll 1$, and in the interval $-1 \leq a^2 \leq 1$, this function can be approximated by [20]

$$Q \simeq \frac{\nu^2(1 - a^2)^2}{48a^4} \left[1 - \frac{2a^2}{(1 - a^2)^2} + \frac{(1 - a^2)^2}{2a^2} \log \frac{1 - a^2}{1 + a^2} \right]. \tag{3.36}$$

Using (2.23), one determines $\omega^{\alpha\beta}$ and $e^{\alpha\dot{\alpha}}$ from (3.32), while the HS Fronsdal potentials $\phi_{\mu a_1 \dots a_s}$ vanish for $s > 2$:

$$\phi_{\mu}{}^{a_1 \dots a_{s-1}} = \frac{\partial^{2s-2}}{\partial y^{\alpha_1} \dots \partial y^{\alpha_{s-1}} \bar{\partial} \bar{y}^{\dot{\alpha}_1} \dots \bar{\partial} \bar{y}^{\dot{\alpha}_{s-1}}} W' |_{Z=Y=0} = 0, \quad s > 2. \tag{3.37}$$

Even though the HS fields vanish, it is to be noted that the solution of the metric and scalar field constitute a solution of a highly nonlinear system of equations in which all higher derivatives play a role. One can reverse engineer a two derivative action describing the coupling of gravity to scalar field that admits the same solution [20] but such an action is clearly of limited use in the context of HS theory.

An advantage of presenting the solutions in stereographic coordinates is that it facilitates their unified description for (A)dS. In these coordinates the solution for the scalar field is given by (3.29) and the metric by

$$ds^2 = \frac{4\Omega^2(d(g_1x))^2}{(1 - \lambda^2 g_1^2 x^2)^2}, \tag{3.38}$$

$$\Omega = \frac{(1 - \lambda^2 g_1^2 x^2)f_1}{2g_1}, \quad g_1 = \exp \left(\frac{1}{2} \int_1^{x^2} \frac{f_2(t) dt}{f_1(t)} \right), \tag{3.39}$$

where

$$f_1(x^2) = \frac{2f}{h^2} \left[1 + (1 - a^2)^2 Q \right], \quad f_2(x^2) = \frac{16Qf}{h^2(1+h)^2},$$

$$f(x^2) = \left[1 + (1 + 6a^2 + a^4)Q \right]^{-1}, \tag{3.40}$$

² The unitary representations of Wigner’s deformed oscillator algebra can be obtained starting from the standard Fock space and factoring out ideals that depend on integer part of $(1 + \nu)/2$, that is, the ideal jumps for odd values of ν [36–39]. It would be interesting to examine to what extent it is possible to extend the solution to general ν properly taking into account the branch points in Q at odd ν .

and it is understood that the integration variable in (3.39) is $t = x^2$.

It is also convenient to give the result in the coordinate system defined in (3.19). In these coordinates, the solution takes the form [35]

$$AdS_4 : \quad x^2 > 0 \quad : \quad ds^2 = d\psi^2 + \eta^2 \sinh^2 \psi \left(-d\tau^2 + \cosh^2 \tau d\Omega_2 \right) , \quad (3.41)$$

$$\phi = \nu \operatorname{sech}^2 \frac{\psi}{2} ,$$

$$x^2 < 0 \quad : \quad ds^2 = -d\tau^2 + \eta^2 \sin^2 \tau \left(d\psi^2 + \sinh^2 \psi d\Omega_2 \right) , \quad (3.42)$$

$$\phi = \nu \sec^2 \frac{\tau}{2} ,$$

$$dS_4 : \quad x^2 > 0 \quad : \quad ds^2 = d\psi^2 + \eta^2 \sin^2 \psi \left(-d\tau^2 + \cosh^2 \tau d\Omega_2 \right) , \quad (3.43)$$

$$\phi = \nu \sec^2 \frac{\psi}{2} ,$$

$$x^2 < 0 \quad : \quad ds^2 = -d\tau^2 + \eta^2 \sinh^2 \tau \left(d\psi^2 + \sinh^2 \psi d\Omega_2 \right) , \quad (3.44)$$

$$\phi = \nu \operatorname{sech}^2 \frac{\tau}{2} ,$$

where we have set $|\lambda|^2 = 1$ and

$$\eta = \frac{f_1 h^2}{2} = \frac{1 + (1 - a^2)^2 Q}{1 + (1 + 6a^2 + a^4) Q} . \quad (3.45)$$

$$AdS_4 : \quad a^2 = \begin{cases} \tanh^2 \frac{\psi}{4} & \text{for } x^2 > 0 \\ -\tan^2 \frac{\tau}{4} & \text{for } x^2 < 0 \end{cases} \quad dS_4 : \quad a^2 = \begin{cases} -\tanh^2 \frac{\psi}{4} & \text{for } x^2 > 0 \\ \tanh^2 \frac{\tau}{4} & \text{for } x^2 < 0 \end{cases} \quad (3.46)$$

In addition to the $SO(3,1)$ symmetry generated by $M'_{\alpha\beta}$, the solution is also left invariant by additional transformations with rigid HS parameters

$$\epsilon' = \sum_{\ell=0}^{\infty} \epsilon'_\ell , \quad (3.47)$$

where the ℓ' th level is given by [20]

$$\epsilon'_\ell = \sum_{m+n=2\ell+1} \Lambda^{\alpha_1 \dots \alpha_{2m}, \dot{\alpha}_1 \dots \dot{\alpha}_{2n}} M'_{\alpha_1 \alpha_2} \star \dots \star M'_{\alpha_{2m-1} \alpha_{2m}} \star \bar{M}'_{\dot{\alpha}_1 \dot{\alpha}_2} \star \dots \star \bar{M}'_{\dot{\alpha}_{2n-1} \dot{\alpha}_{2n}} - \text{h.c.} , \quad (3.48)$$

with constant $\Lambda^{\alpha_1 \dots \alpha_{2m}, \dot{\alpha}_1 \dots \dot{\alpha}_{2n}}$. The full symmetry algebra is thus a higher-spin extension of $SO(3,1) \simeq SL(2, \mathbb{C})$, that we shall denote by

$$hsl(2, \mathbb{C}; \nu) \supset sl(2, \mathbb{C}) , \quad (3.49)$$

where $sl(2, \mathbb{C})$ is generated by $M'_{\alpha\beta}$ and its hermitian conjugate, and we have indicated that in general the structure coefficients may depend on the deformation parameter ν .

Turning back to the solutions given above, a holographic and cosmological interpretation of (3.41) has been discussed in [35], where a bouncing cosmology scenario was observed, and its comparison with a similar phenomenon occurring supergravities [40,41] was made. In this context, it is useful to examine the behaviour of the solutions, both with AdS and dS asymptotics, near the boundary as well as distant future.

For $|\nu| \ll 1$ and near the boundary, where $\lambda^2 x^2 \rightarrow 1$, or equivalently $a^2 \rightarrow 1$, the scale factors η and Ω behaves as³

$$\lambda^2 x^2 \rightarrow 1 \implies \eta \rightarrow \frac{1}{1 - \frac{\nu^2}{3}}, \quad \Omega \rightarrow 1, \tag{3.50}$$

which means that the solutions are asymptotically maximally symmetric spacetimes with undeformed radius.

Another interesting limit to consider is $\eta \rightarrow 0$; for $|\nu| \ll 1$ this takes place for $a^2 + 1 \ll 1$, that is, for a^2 close but not equal to -1 , which corresponds to $\tau \rightarrow \pm \tau_{\text{crit}}$ in the AdS case, and $\psi \rightarrow \psi_{\text{crit}}$ in the dS case. In the former case, we have⁴

$$AdS_4: \quad \eta \simeq \frac{\nu^2}{6} \left[\exp\left(\frac{3}{2\nu^2}\right) \right] (\tau_{\text{crit}} - \tau), \quad \tau_{\text{crit}} \simeq \sin^{-1} \left[2 \exp\left(\frac{-3}{2\nu^2}\right) \right]. \tag{3.51}$$

In the Einstein frame⁵, it takes infinite proper time to reach the critical surface, which means that one may interpret the future region of the solution as a singularity free $SO(3, 1)$ invariant cosmology with a finite asymptotic scalar field, as

$$\phi \rightarrow \phi_{\text{crit}} \simeq 4\nu \exp\left(\frac{3}{\nu^2}\right). \tag{3.52}$$

In the dS case, one may instead interpret the critical limit as a domain wall at an infinite proper space-like distance from the center of the solution.

3.4. Solutions of the Non-Minimal Chiral Model in Kleinian Space

In obtaining the solutions described above, symmetries on the master fields were imposed. In [21] projection operators were used as well. In the case of the non-minimal chiral model⁶ in Kleinian space, it is possible to use projectors to build solutions with non-vanishing Weyl zero-form and HS fields. They are

$$B' = (1 - P) \star \kappa, \quad S'_\alpha = z_\alpha \star P, \quad \bar{S}'_{\dot{\alpha}} = \bar{z}_{\dot{\alpha}} \star (1 - 2\bar{P}), \tag{3.53}$$

where

$$P \star P = P, \quad \bar{P} \star \bar{P} = \bar{P}, \quad [P, \bar{P}]_\star = 0, \tag{3.54}$$

and satisfy the conditions $\pi \circ \bar{\pi}(P, \bar{P}) = (P, \bar{P})$. The projectors P and \bar{P} are independent. Consider the simplest case in which

$$P = 2e^{-2uv} = 2e^{yby}, \quad \bar{P} = 0, \tag{3.55}$$

where, in terms of constant spinors $(\lambda_\alpha, \mu_\alpha)$ we have defined

$$u = \lambda^\alpha y_\alpha, \quad v = \mu^\alpha y_\alpha, \quad b_{\alpha\beta} = 2\lambda_{(\alpha} \mu_{\beta)}, \quad \lambda^\alpha \mu_\alpha = \frac{i}{2}. \tag{3.56}$$

³ The result for the limits of η and Ω given here correct Equations (4.67) and (4.68) in [20].

⁴ The expression for η and τ_{crit} corrects a factor of two in [35].

⁵ This is the (torsion-free) frame obtained by rescaling the vielbein as $e^a \rightarrow \eta^{-1} e^a$.

⁶ The solution can be constructed for the minimal model as well by working with a convenient integral presentation of the projection operators.

Upon the L -dressing, and expanding the result in Y -oscillators, the following component results are found in [21]:

$$\phi = -1, \quad C_{\alpha_1 \dots \alpha_{2s}} = 0, \tag{3.57}$$

while the anti-self-dual Weyl tensors take the form

$$\bar{C}_{\dot{\alpha}_1 \dots \dot{\alpha}_{2s}} = -2^{2s+1} (2s - 1)!! \left(\frac{h^2 - 1}{h^2} \right)^s \bar{u}_{(\dot{\alpha}_1} \dots \bar{u}_{\dot{\alpha}_s} \bar{v}_{\dot{\alpha}_{s+1}} \dots \bar{v}_{\dot{\alpha}_{2s}}), \tag{3.58}$$

where

$$\bar{u}_{\dot{\alpha}} = \frac{x^a}{\sqrt{x^2}} (\bar{\sigma}_a \lambda)_{\dot{\alpha}}, \quad \bar{v}_{\dot{\alpha}} = \frac{x^a}{\sqrt{x^2}} (\bar{\sigma}_a \mu)_{\dot{\alpha}}. \tag{3.59}$$

In stereographic coordinates, the Kleinian space is covered in two charts with $0 \leq h^2 \leq 2$, and hence the Weyl tensors blow up in the limit $h^2 \rightarrow 0$ preventing the solution from approaching $H_{3,2}$ in this limit. In coordinate system introduced in (3.21), the metric reads $ds^2 = -(dr^2 + r^2 dt^2) + (1 + r^2) d\Omega_2^2$, and the pre-factor in this solution reads

$$\left(\frac{h^2 - 1}{h^2} \right)^s = 2^{-s} (1 - r \cos t)^s. \tag{3.60}$$

which, indeed, diverges at the boundary $r \rightarrow \infty$ as noted above. In [21], it was also found that

$$e_\mu^a = \frac{-2}{(h^2 + 2h^{-2})} \left[(1 + 2h^{-2}) \delta_\mu^a + 2\lambda^2 h^{-4} x_\mu x^a + \frac{6\lambda^2}{h^4 - 4} (Jx)_\mu (Jx)^a \right], \tag{3.61}$$

where

$$J_{ab} = (\sigma_{ab})^{\alpha\beta} b_{\alpha\beta}, \quad J_a^c J_c^b = -\delta_a^b. \tag{3.62}$$

For the spin connection, the result is

$$\begin{aligned} \omega^{\alpha\beta} &= \frac{1}{1 - 4h^{-4}} \left[\omega_0^{\alpha\beta} - 8h^{-4} (b \omega_0 b)^{\alpha\beta} \right] + \frac{4h^{-4}}{(1 - 2h^{-4})(1 - 4h^{-4})} b^{\alpha\beta} b_{\gamma\delta} \omega_0^{\gamma\delta}, \\ \bar{\omega}^{\dot{\alpha}\dot{\beta}} &= \bar{\omega}_0^{\dot{\alpha}\dot{\beta}} + 4(1 + h)^2 h^{-4} \left[-(\bar{a}b a)^{\dot{\alpha}\dot{\beta}} b_{\gamma\delta} + 2(\bar{a}b)^{\dot{\alpha}\gamma} (\bar{a}b)^{\dot{\beta}\delta} \right] \omega_{\gamma\delta}. \end{aligned} \tag{3.63}$$

Note that in the last term of the second equation the full spin connection $\omega_{\gamma\delta}$ arises. The metric $g_{\mu\nu} = e_\mu^a e_\nu^b \eta_{ab}$ takes the form

$$\begin{aligned} g_{\mu\nu} &= \frac{4}{(h^2 + 2h^{-2})^2} \left[(1 + 2h^{-2})^2 \eta_{\mu\nu} + 4h^{-4} \left((1 - h^2)h^{-4} + (1 + 2h^{-2}) \right) x_\mu x_\nu \right. \\ &\quad \left. + \frac{12}{1 - 4h^{-4}} \left(\frac{3(1 - h^2)}{1 - 4h^{-4}} + (1 + 2h^{-2}) \right) (Jx)_\mu (Jx)_\nu \right], \end{aligned} \tag{3.64}$$

The vierbein has potential singularities at $h^2 = 0$ and $h^2 = 2$. The limit $h^2 \rightarrow 0$ is a boundary at which $e_\mu^a \sim h^{-2} x_\mu x^a$, i.e., a scale factor times a degenerate vierbein. In the limit $h^2 \rightarrow 2$ one approaches the boundary of a coordinate chart. Also in this limit, the vierbein becomes degenerate, viz., $e_\mu^a \sim h^{-2} (Jx)_\mu (Jx)^a$.

3.5. Perturbative Construction of Domain-Wall Solution

As mentioned earlier, if we wish to construct a solution of the HS equations that has a symmetry group that includes any translation generators P^a , given that there is no known realization of these generators that would form a closed algebra with the full Lorentz generators $M_{\alpha\beta}$, the symmetry

conditions (3.22) need to be imposed in terms of the undeformed generators that are bilinear in the oscillators. The deformation of this symmetry to accommodate nonlinear corrections can then be computed perturbatively in a weak field expansion scheme. This is the framework which was pursued in considerable detail in [20], where the perturbative construction of solutions with 3, 4 and 6 parameter isometry subgroups of the AdS group were considered. Here we shall outline the key aspects of this constructions by describing the example of a domain-wall solution having $ISO(2, 1)$ symmetry and its appropriate HS extension [20].

The $ISO(2, 1)$ algebra is generated by

$$ISO(2, 1) : \quad M_{ij}, \quad P_i = (\alpha M_{ab} L^b + \beta P_a) L_i^a, \quad \alpha^2 - \beta^2 = 0, \quad (3.65)$$

where α and β are real parameters and (L_i^a, L^a) is a representative of the coset $SO(3, 1)/SO(2, 1)$, obeying

$$L^a L_a = \epsilon = \pm 1, \quad L_i^a L_a = 0, \quad L_i^a L_{ja} = \eta_{ij} = \text{diag}(+, +, -), \quad (3.66)$$

and the generators are taken from the oscillator realization of $SO(3, 2)$ algebra given by

$$M_{ab} = -\frac{1}{8} \left[(\sigma_{ab})^{\alpha\beta} y_\alpha y_\beta + (\bar{\sigma}_{ab})^{\dot{\alpha}\dot{\beta}} \bar{y}_{\dot{\alpha}} \bar{y}_{\dot{\beta}} \right], \quad P_a = \frac{1}{4} (\sigma_a)^{\alpha\dot{\beta}} y_\alpha \bar{y}_{\dot{\beta}}, \quad (3.67)$$

In particular $[P_i, P_j] = i(\beta^2 - \alpha^2) M_{ij}$ vanishes for $\alpha^2 = \beta^2$, as required for $ISO(2, 1)$. Thus, the symmetry conditions to be imposed are

$$[M_{ij}, C'] = 0, \quad [P_i, C']_\pi = 0. \quad (3.68)$$

As shown in [20], these conditions are solved by

$$C'(P) = (\mu_1 + \mu_2 P) e^{4iP}, \quad P := \frac{1}{4} L^a (\sigma_a)_{\alpha\dot{\alpha}} y^\alpha \bar{y}^{\dot{\alpha}}. \quad (3.69)$$

Denoting the $ISO(2, 1)$ transformations discussed above by $\epsilon'_{(0)}$, we can seek its nonlinear deformation by expanding

$$\epsilon' = \epsilon'_{(0)} + \epsilon'_{(1)} + \epsilon'_{(2)} + \dots, \quad (3.70)$$

where $\epsilon'_{(n)}$ is constant in spacetime but may depend on Y and Z . The symmetry condition at first order is satisfied by C' given in (3.69). To establish the symmetry at second order, we need to satisfy

$$\left([\epsilon'_{(1)}, C']_\pi + [\epsilon'_{(0)}, B'_{(2)}]_\pi \right)_{Z=0} = 0, \quad (3.71)$$

where $B'_{(2)}$ is obtained from the normal ordered perturbative scheme (see Section 6 below) to be [20]

$$\begin{aligned} B'_{(2)} &= f + \tau \bar{\pi} f + \pi (f + \tau \bar{f})^\dagger, \\ f &:= -z^\alpha \int_0^1 dt (V_\alpha'^{(1)} \star C')_{Z \rightarrow tZ}, \\ V_\alpha'^{(1)} &= -\frac{i}{2} z_\alpha \int_0^1 dt B'(-tz, \bar{y}) \kappa(tz, y). \end{aligned} \quad (3.72)$$

This condition (3.71) is solved in [20] where it is found that

$$\epsilon'_{(1)} = -\int_0^1 dt \int_0^1 t' dt' \left(\frac{itt'}{2} \lambda^{\alpha\beta} z_\alpha z_\beta + \lambda^{\alpha\dot{\beta}} z_\alpha \bar{\delta}_{\dot{\beta}} \right) C'(-tt'z, \bar{y}) e^{itt'yz} - h.c., \quad (3.73)$$

where $\lambda^{\alpha\beta}, \lambda^{\alpha\dot{\beta}}$ are arbitrary constant parameters. For a more detailed discussion of the procedure outlined above, see [20].

3.6. Other Known Solutions

The solutions described in Section 3.3 were generalized in [21] to find new Lorentz-invariant vacuum solutions, in which, in addition to the continuous parameter ν , an infinite set of independent and discrete parameters $\theta_k = \{0, 1\}$, each turning on a Fock-space projector $P_n(u)$, were activated. Should they be proved to be gauge-inequivalent to AdS_4 , as they seem to be, they would represent monodromies of flat but non-trivial connections $(V_\alpha, \bar{V}_{\dot{\alpha}})$ on \mathcal{Z} . An interesting limit of this solution arises upon setting $\nu = 0$ and $(\theta_k - \theta_{k+1})^2 = 1$, leading to the degenerate metric

$$ds^2 = \frac{4(x^a dx_a)^2}{\lambda^2 x^2 (1 - \lambda^2 x^2)}. \tag{3.74}$$

The methods above can also be extended to the Prokushkin-Vasiliev theory in $D = 3$, giving rise to Lorentz-invariant instanton solutions (with additional twisted sectors of the theory excited, and the characteristic extra deformation parameter λ that allows to vary the mass of the scalar), as well as to the above projector vacua [31].

Finally, in [42] a different class of exact solutions was constructed by means of the gauge function method coupled with a different choice of gauge in twistor space, there referred to as axial gauge. As for the perturbative construction of solutions, it is worth mentioning the plane wave solution of [4,19], whose elevation to an exact solution remains to be investigated, to our best knowledge.

4. Factorization Method and Solutions

4.1. The Method

The method developed in [6,7,22] for finding exact solutions of Vasiliev equations also exploits the gauge function method to solve for (B, V, W) in terms of (B', V') from (3.5) and (2.22), which, in turn, are to be determined by solving (3.4). It is here that the method differs from the method described above, by making the following factorized ansatz for (B', V') :

$$B'(Z, Y) = \nu \Psi(y, \bar{y}) \star \kappa_y, \tag{4.1}$$

$$V'_\alpha(Z, Y) = V'_\alpha(z; \Psi) = \sum_{n \geq 1} V_\alpha^{(n)}(z) \star \Psi^{\star n}. \tag{4.2}$$

Note that this ansatz for V'_α is holomorphic in z , and in the following we shall refer to the solution in this form as given in *holomorphic gauge* [7]. In this section we shall consider the nonminimal model in which we recall that conditions (2.35) apply. It is shown in [7] that this ansatz solves the fully non-linear Equation (3.4) provided that

$$\pi_z(V_\alpha^{(n)}(z)) = -V_\alpha^{(n)}(z), \tag{4.3}$$

and that

$$s_\alpha := z_\alpha - 2i \sum_{n \geq 1} V_\alpha^{(n)}(z) \nu^n, \tag{4.4}$$

obeys the deformed oscillator algebra

$$[s_\alpha, s_\beta]_\star = -2i \epsilon_{\alpha\beta} (1 - \nu \kappa_z), \quad \kappa_z \star s_\alpha = -s_\alpha \star \kappa_z. \tag{4.5}$$

Note that only the first power of v in (4.4) survives in the commutator (4.5). One class of solutions is given by [6]

$$\sum_{n \geq 1} V_{\alpha}^{(n)}(z) v^n = -\frac{iv}{2} z_{\alpha} \int_{-1}^{+1} \frac{dt}{(t+1)^2} \exp\left(i \frac{t-1}{t+1} z_{(0)}^{+} z_{(0)}^{-}\right) {}_1F_1\left(\frac{1}{2}; 1; bv \log t^2\right), \tag{4.6}$$

where the constant spinors $v_{(0)\alpha}^{\pm}$ are used to defined the projected oscillators

$$v_{(0)}^{+\alpha} v_{(0)\alpha}^{-} = 1, \quad z_{(0)}^{\pm} = v_{(0)}^{\pm\alpha} z_{\alpha}. \tag{4.7}$$

The presence of $z^{+}z^{-}$ term breaks manifest Lorentz covariance, which will be restored when we consider the field dependent gauge transformation in Section 7 that is needed to cast the results into a form that can be interpreted in terms of Fronsdal fields that obey the standard boundary conditions.

At this point, B' and V'_{α} are determined, with $\Psi(Y)$ representing an arbitrary initial datum. One can proceed to compute (B, V, W') from (3.4), (3.5) and (2.22). However, one needs to ensure that the star products involving Ψ are well defined. The analyticity properties of the resulting (B, V, W') also require special care. The strategy adopted in [6,7,22] is to employ projection operators with well defined group theoretical origin, and easily deducible symmetry properties. We shall illustrate aspects of this procedure below with a relatively simple example, namely the black hole-like solution [7] closely related to that of [24], which we shall also describe in a subsequent section below.

4.2. Black Hole Solution

We are seeking an exact solution of Vasiliev’s equations which has the symmetries of 4D static black hole solution that is asymptotically AdS_4 , namely spatial rotations and time translations generated by

$$SO(3) \times SO(2) : \quad M_{rs} (r, s = 1, 2, 3), \text{ and } M_{05} = E = \frac{1}{4} (\sigma_0)_{\alpha\dot{\alpha}} y^{\alpha} \bar{y}^{\dot{\alpha}}. \tag{4.8}$$

The first instance of such a solution was found in [24] in a different approach that will be summarized later. In both approaches the following projector plays an important role:

$$\Psi = v P', \quad P' \star P' = P', \tag{4.9}$$

where P' is given by

$$P' = 4e^{-4E}, \tag{4.10}$$

and the reality conditions dictate that

$$v = iM, \tag{4.11}$$

with $M \in \mathbb{R}$. This projector clearly has the desired symmetry property since $[M_{rs}, P']_{\star} = 0$ and $[E, P']_{\star} = 0$. In performing the L -dressing, we need the result

$$P = L^{-1} \star 4e^{-4E} \star L = 4 \exp\left(-\frac{1}{2} K_{AB} Y^A Y^B\right), \tag{4.12}$$

where K_{AB} are the Killing parameters taking the form

$$K_{AB} = \begin{pmatrix} u_{\alpha\beta} & v_{\alpha\dot{\beta}} \\ \bar{v}_{\dot{\alpha}\beta} & \bar{u}_{\dot{\alpha}\dot{\beta}} \end{pmatrix}. \tag{4.13}$$

In terms of the x -dependent eigenspinors of $u_{\alpha\beta}$, these parameters can be expressed as

$$\begin{aligned} u_{\alpha\beta} &= 2r u_{(\alpha}^+ u_{\beta)}^-, & v_{\alpha\beta} &= \sqrt{1+r^2} \left(u_{\alpha}^+ \bar{u}_{\beta}^+ + u_{\alpha}^- \bar{u}_{\beta}^- \right), \\ u_{\alpha\beta}^{-1} &= \frac{2}{r} u_{(\alpha}^+ u_{\beta)}^-, & 2u_{[\alpha}^- u_{\beta]}^+ &= \epsilon_{\alpha\beta}. \end{aligned} \tag{4.14}$$

The Kerr-Schild vector $k_{\alpha\dot{\alpha}}$ associated with E is obtained via a projection of $v_{\alpha\dot{\alpha}}$ with the eigenspinors of $u_{\alpha\beta}$, $\bar{u}_{\dot{\alpha}\dot{\beta}}$, and can be written as⁷

$$k_{\alpha\dot{\alpha}} = \frac{1}{\sqrt{1+r^2}} u_{\alpha}^+ \bar{u}_{\dot{\alpha}}^+. \tag{4.15}$$

In obtaining the above results, the computation is first done in stereographic coordinates, and then a coordinate change is made to go over to the spherical (global) coordinate system in which the AdS metric reads $ds^2 = -(1+r^2)dt^2 + (1+r^2)^{-1}dr^2 + r^2 d\Omega_2^2$.

From the point of view of the factorized Ansatz (4.1)–(4.2), the fact that Ψ is proportional to a projector can be seen as a way of enforcing the Kerr-Schild property of a black-hole solution, as it effectively causes a collapse of all non-linear correction in $V'_{\alpha}, \bar{V}'_{\dot{\alpha}}$ down to the linear order, at least from the point of view of the oscillator dependence. Coupled with the gauge freedom on S_{α} this allows to effectively reach a gauge in which the full solution only contains first order deformations in v [6]. Another noteworthy fact is that, due to the factorized dependence on Y and Z , one can effectively separately rotate the two oscillators by means of a factorized gauge function

$$g = L(x, Y) \star \tilde{L}(x, Z). \tag{4.16}$$

As discussed in [6,7,22], turning on the second factor is useful since by choosing it appropriately one can achieve collinearity between the spin-frame $(v^+, v^-)(x)$ on \mathcal{Z} (which is obtained by pointwise rotation of $v_{(0)}$ given in (4.7) by \tilde{L}) and the eigenspinors $(u_{\alpha}^+, u_{\beta}^-)$ of $u_{\alpha\beta}$ in order to remove singularities that appear in the solution for the master one-form, that are gauge artifacts. We note that the factor $\tilde{L}(x, Z)$, being purely Z -dependent, does not affect B and only acts non-trivially on $V_{\alpha}, \bar{V}_{\dot{\alpha}}$.

Thus, dressing the primed fields given above by the gauge function g defined in (4.16), the following results have been obtained [6]

$$B = \frac{4M}{r} \exp \left[\frac{1}{r^2} \left(\frac{1}{2} y^{\alpha} u_{\alpha\beta} \tilde{y}^{\beta} + \frac{1}{2} \bar{y}^{\dot{\alpha}} \bar{u}_{\dot{\alpha}\dot{\beta}} \bar{\tilde{y}}^{\dot{\beta}} + i y^{\alpha} u_{\alpha\beta} v^{\beta}_{\dot{\alpha}} \bar{y}^{\dot{\alpha}} \right) \right], \tag{4.17}$$

$$S_{\alpha} = z_{\alpha} + 8P a_{\alpha} \int_{-1}^1 \frac{dt}{(t+1+ir(t-1))^2} j(t) \exp \left[\frac{i(t-1)}{t+1+ir(t-1)} a^+ a^- \right], \tag{4.18}$$

$$\begin{aligned} W' &= W_0 + \tilde{L}^{-1} \star d_x \tilde{L} - \left\{ \omega^{--} [y^+ y^+ + 8P(f_1 - f_2) a^+ a^+] \right. \\ &\quad \left. + \omega^{++} [y^- y^- + 8P(f_1 - f_3) a^- a^-] - 2\omega^{-+} [y^+ y^- - 8P(f_1 + f_4) a^+ a^- - r(f_5 + f_6)] \right\}, \end{aligned} \tag{4.19}$$

where $y^{\pm} = u^{\alpha\pm} y_{\alpha}$, and similarly $(a^{\pm}, \omega^{++}, \omega^{--}, \omega^{-+})$ are projections of $a_{\alpha}, \omega_{\alpha\beta}$ with $u^{\alpha\pm}$, and the function $j(t)$ is defined as

$$j(t) = -\frac{\nu}{4} {}_1F_1 \left[\frac{1}{2}; 2; \frac{\nu}{2} \log \frac{1}{t^2} \right]. \tag{4.20}$$

⁷ In stereographic coordinates these read [7] $u_{\alpha\beta} = 2x^i (\sigma_{i0})_{\alpha\beta} / (1-x^2)$ and $v_{\alpha\beta} = (\sigma_0)_{\alpha\beta} - 4x_{[0} x^i (\sigma_{i]}_{\alpha\beta}) / (1-x^2)$.

The modified oscillators $(a_\alpha, \bar{a}_{\dot{\alpha}})$ are defined as

$$\tilde{Z}_A := (a_\alpha, \bar{a}_{\dot{\alpha}}) = Z_A + iK_A^B Y_B, \quad [\tilde{Z}_A, \tilde{Z}_B] = 4i\epsilon_{AB}. \tag{4.21}$$

Furthermore, we have defined the functions

$$f_1 = \int_{-1}^1 dt q(t)(t+1)\tilde{\zeta}^3 \exp [i(t-1)\tilde{\zeta}a^+a^-], \tag{4.22a}$$

$$f_2 = \int_{-1}^1 \int_{-1}^1 dt dt' j(t)j(t') 2t \tilde{\zeta}^3 \exp [i(tt'-1)\tilde{\zeta}a^+a^-], \tag{4.22b}$$

$$f_3 = \int_{-1}^1 \int_{-1}^1 dt dt' j(t)j(t') 2t' \tilde{\zeta}^3 \exp [i(tt'-1)\tilde{\zeta}a^+a^-], \tag{4.22c}$$

$$f_4 = \int_{-1}^1 dt q(t)\tilde{\zeta}^2 \exp [i(t-1)\tilde{\zeta}a^+a^-], \tag{4.22d}$$

$$f_5 = \int_{-1}^1 \int_{-1}^1 dt dt' j(t)j(t') (tt'+1) \tilde{\zeta}^3 \exp [itt'\tilde{\zeta}a^+a^-], \tag{4.22e}$$

$$f_6 = \int_{-1}^1 \int_{-1}^1 dt dt' j(t)j(t') \tilde{\zeta}^2 \exp [i(tt'-1)\tilde{\zeta}a^+a^-], \tag{4.22f}$$

and

$$\tilde{\zeta} := \frac{1}{t+1+ir(t-1)}, \quad \tilde{\zeta}' := \frac{1}{tt'+1+ir(tt'-1)}. \tag{4.23}$$

The modified oscillators a_α appear naturally in V_α as a consequence of the factorized form of (4.2) with $\Psi^{*n} \propto P$, and of the fact that $z_\alpha \star P = a_\alpha P$ (analogously for $\bar{a}_{\dot{\alpha}}$ in $\bar{V}_{\dot{\alpha}}$). A consequence of the modified oscillator appearing in the solution is that V'_α does not obey the standard Vasiliev gauge $Z^A V_A = 0$.

As noted earlier, this solution is closely related to that of Didenko and Vasiliev [24]. Indeed, the solution for B is the same, while the relationship between the solutions for (S_A, W') is more subtle and it will be discussed in the next section.

Note that B does not depend on Z . Thus, $B|_{Z=0} = B = C(Y)$, and its holomorphic components give the Petrov type -D Weyl tensors

$$C_{\alpha_1 \dots \alpha_{2s}} = \frac{4M}{r^{s+1}} u_{\alpha_1}^+ \dots u_{\alpha_s}^+ u_{\alpha_{s+1}}^- \dots u_{\alpha_{2s}}^-, \tag{4.24}$$

and similarly for the anti-holomorphic components giving $\bar{C}_{\dot{\alpha}_1 \dots \dot{\alpha}_{2s}}$. The singularity of individual Weyl tensors does not necessarily imply a physical singularity in HS gravity for the following reasons. At the level of the master fields, $r = 0$ also appears as the only point at which V_α , as well as W' , acquire a pole on a plane in $\mathcal{Z} \times \mathcal{Y}$ defined by $a_\alpha|_{r=0} = z_\alpha + i(\sigma_0 \bar{y})_\alpha = 0$, due to the zero at $t = -1$ that the denominator of the integrand in (4.18) develops at $r = 0$. The master-field curvature is however given by $B \star \kappa$ and one can argue that at the master-field level, which is the only sensible way to look at such solution in the strong-field region, B remains, in fact, regular at $r = 0$. Qualitatively this can be understood as follows. r appears in (4.17) as the parameter of a delta sequence: away from the origin one has a smooth Gaussian function, approaching a Dirac delta function on \mathcal{Y} as r goes to zero [6]. However, unlike the delta function on a commutative space, the delta function in noncommutative twistor space, thought of as a symbol for an element of a star product algebra, is smooth. Indeed, it is possible to show [6] that by changing ordering prescription one can map the delta function to a regular element, and the smoothness of such change of basis manifests itself in the fact that the solution of the deformed oscillator problem obtained in the new ordering can be mapped back smoothly to the

solution above. In this sense, the singularity in $r = 0$ may be an artifact of the ordering choice for the infinite-dimensional symmetry algebra governing the Vasiliev system.

In order to extract the x -dependence of even just the spin-2 component of W' , one still needs to evaluate the complicated parametric integrals in (4.22a). However, as the Weyl-tensors take the simple form in x -space given in (4.24), we expect that a suitable gauge transformation exists that will give the metric in the standard Kerr-Schild form, namely $g_{\mu\nu} = g_{\mu\nu}^{AdS} + 2Mk_\mu k_\nu / r$.

Finally, let us note that the basic black hole-like solution reviewed above has a generalization in which infinite sets of projection operators P_n and twisted projection operators \tilde{P}_n are introduced⁸. The twisted projectors \tilde{P}_n are invariant under $SO(3) \times SO(2)$ discussed above while the projectors P_n are invariant under $SO(3)$ and the (B', S'_A) sector of the solution takes the form [6,7,22]

$$B' = \sum_{n=\pm 1, \pm 2, \dots} \left(v_n \tilde{P}'_n + \tilde{v}_n P'_n \right), \quad \tilde{P}'_n = P'_n \star \kappa_y, \tag{4.25}$$

$$S'_A = z_A - 2i \sum_{n=\pm 1, \pm 2, \dots} \left(V_{n,A} \star P'_n + \tilde{V}_{n,A} \star \tilde{P}'_n \right), \tag{4.26}$$

with

$$P'_n(E) = 2(-1)^{n-\frac{1+\varepsilon}{2}} \oint_{C(\varepsilon)} \frac{d\eta}{2\pi i} \left(\frac{\eta+1}{\eta-1} \right)^n e^{-4\eta E}, \quad \varepsilon := n/|n|, \tag{4.27}$$

$$\tilde{P}'_n(E) := P'_n(E) \star \kappa_y = 4\pi(-1)^{n-\frac{1+\varepsilon}{2}} \oint_{C(\varepsilon)} \frac{d\eta}{2\pi i} \left(\frac{\eta+1}{\eta-1} \right)^n \delta^2(y - i\eta\sigma_0\bar{y}), \tag{4.28}$$

where the contour integrals are performed around a small contour $C(\varepsilon)$ encircling ε . The expressions for $(V_{n,A}, \tilde{V}_{n,A})$, and their $g = L \star \tilde{L}$ dressing can be found in [7]. It turns out that the solutions with only v_n parameters switched on correspond to black hole-like solutions, of which the case summarized above arises for $n = 1$. Solutions with only nonvanishing \tilde{v}_n parameters correspond to massless particle modes, and surprisingly black hole modes as well entering from second order onwards in a perturbative treatment of the solution; for a detailed description of this phenomenon see [7,26]. Note that the $SO(3)$ invariant projectors are associated with spin-0 modes. To extend this construction to spin- s particle modes, one needs the spin- s generalization of the projectors P'_n discussed above. A particular presentation of such projectors can be found in [43].

4.3. Other Known Solutions

By means of the same factorized Ansatz (4.1)–(4.2) black-hole-like solutions with biaxial symmetry have also been found [6,22,44], some of which being candidate HS generalizations of the Kerr black hole [44]. The separation of variables in holomorphic gauge was also instrumental to finding solutions with \mathfrak{g}_6 isometries of cosmological interest, that we shall present in a forthcoming paper [23].

5. Direct Method and the Didenko-Vasiliev Solution

While, as we have seen so far, the gauge function method is in general of great help in constructing exact solutions, it is sometimes possible to attack the equations directly, by virtue of some other simplifying Ansatz or gauge condition. We shall generically refer here to any method which does not rely on the use of gauge function as direct method. One such solution has been found so far in this way by Didenko and Vasiliev [24], which has nonvanishing HS fields, and which contains the Schwarzschild black hole solution in the spin 2 sector.

⁸ Even though $\tilde{P}'_n \star \tilde{P}'_n = P'_n$, we refer to \tilde{P}' as twisted projector to emphasize the fact that it is related to the projector P'_n by the relation $\tilde{P}'_n = P'_n \star \kappa_y$.

Indeed, motivated by the phenomenon that solutions of Einstein equations that can be put in Kerr-Schild form solve the linearized as well as the nonlinear form of the equations, the Authors of [24] thought of an Ansatz that would generalize some distinctive features of black hole solutions in gravity. First, it is based on an AdS timelike Killing vector, in the sense that the Weyl tensor will be a function of some element K_{AB} as (4.13). More precisely, if $f(K)$ satisfies the Killing vector equation, a proper ansatz for a solution of the linearized twisted adjoint equation will be given by $f(K) \star \kappa_y$. Second, they chose $f(K)$ in such a way that the Ansatz linearize the Vasiliev equations. For the latter purpose it is important that the function $f(K)$ is a projector—in fact, a Fock-space vacuum projector that coincides with (4.12). Such choice in particular reduces Equations (2.31) and (2.32) to two copies of the 3D (anti)holomorphic deformed oscillator problem that arises in Prokushkin-Vasiliev HS theory in 3D [8] in terms of the oscillators in (4.21)⁹.

The ansatz [24]

$$B = MP \star \kappa_y, \quad S_\alpha = z_\alpha + Pf_\alpha(a|x), \quad \bar{S}_{\dot{\alpha}} = \bar{z}_{\dot{\beta}} + P\bar{f}_{\dot{\alpha}}(\bar{a}|x), \quad (5.1)$$

where M is a constant and $(f_\alpha, \bar{f}_{\dot{\alpha}})$ are functions to be determined, indeed reduces the Equations (2.31) and (2.32) to two deformed oscillator problems in terms of the latter functions. A specific gauge choice on the $(f_\alpha, \bar{f}_{\dot{\alpha}})$, while bringing about a further breaking of the manifest Lorentz covariance, effectively linearizes their equations, which are then solved by means of the standard perturbative methods of Section 6. A further ansatz [24]

$$W = W_0 + P[g(a|x) + \bar{g}(\bar{a}|x)], \quad (5.2)$$

where g is another function to be determined is then employed to deal with the remaining equations that involve W , namely (2.28), (2.29) and (2.30). The resulting exact solution is given by [24]

$$B = \frac{4M}{r} \exp \left[\frac{1}{r^2} \left(\frac{1}{2} y^\alpha u_{\alpha\beta} y^\beta + \frac{1}{2} \bar{y}^{\dot{\alpha}} \bar{u}_{\dot{\alpha}\dot{\beta}} \bar{y}^{\dot{\beta}} + y^\alpha u_{\alpha\beta} v_{\dot{\alpha}}^\beta \bar{y}^{\dot{\alpha}} \right) \right], \quad (5.3a)$$

$$S^+ = z^+ + MP \frac{a^+}{r} \int_0^1 dt \exp \left(\frac{t}{r} a^+ a^- \right), \quad \text{idem } \bar{S}^+, \quad (5.3b)$$

$$S^- = z^-, \quad \text{idem } \bar{S}^-, \quad (5.3c)$$

$$W = W_0 + \frac{1}{2r} MP \left[d\tau^{--} a_\alpha^+ a_\beta^+ \int_0^1 dt (1-t) \exp \left(\frac{t}{r} a^+ a^- \right) + h.c. \right] + \frac{1}{8r} MP \left[u_{\alpha\beta} \omega_0^{\alpha\beta} + h.c. + 2(v_{\alpha\dot{\alpha}} + k_{\alpha\dot{\alpha}}) e_0^{\alpha\dot{\alpha}} \right], \quad (5.3d)$$

where

$$\tau_{\alpha\beta} = \frac{u_{\alpha\beta}}{r}, \quad \tau^{--} = u^{\alpha-} u^{\alpha-} \tau_{\alpha\beta}, \quad z^\pm = u^{\alpha\pm} z_\alpha, \quad a^\pm = u^{\alpha\pm} a_\alpha, \quad (5.4)$$

$$d\tau_{\alpha\beta} = -\frac{1}{r} \omega_{0(\alpha}{}^\gamma u_{\beta)\gamma} + \frac{1}{r} e_0^{\gamma\dot{\gamma}} \left(\epsilon_{\gamma(\alpha} v_{\beta)\dot{\gamma}} + \frac{1}{2r^3} v_{\dot{\gamma}}{}^\delta u_{\alpha\beta} u_{\gamma\delta} \right).$$

Note also that S_A does not satisfy the Vasiliev gauge $S(Z, Y)|_{Z=0} = 0$, and that W above has not been redefined as in (2.22). Nonetheless, it has been noted in [24] that a HS transformation of the form

⁹ In this section we shall use the conventions of [24] which differ from ours.

$\delta W = D_0 \epsilon^{(1)}$ with $\epsilon^{(1)} = \left(-\frac{1}{2} \int_0^1 dt z^\alpha S_\alpha|_{z \rightarrow tz} + h.c. + f(Y|x) \right)$ and arbitrary $f(Y|x)$ maps W to W^{phys} given by

$$W^{\text{phys}} = \frac{4M}{r} e_0^{\alpha\dot{\alpha}} k_{\alpha\dot{\alpha}} \exp \left(-\frac{1}{2} k_{\beta\dot{\beta}} y^\beta \bar{y}^{\dot{\beta}} \right), \tag{5.5}$$

whose spin 2 component gives the frame field and the associated metric

$$e_\mu^a = e_\mu^{a(AdS)} + \frac{M}{r} k_\mu k^a, \quad g_{\mu\nu} = g_{\mu\nu}^{\text{AdS}} + \frac{2M}{r} k_\mu k_\nu, \tag{5.6}$$

where $k_\mu = e_0^{\alpha\dot{\alpha}} k_{\alpha\dot{\alpha}}$. This is the metric of a black hole of mass M in AdS_4 in Kerr-Schild form. The terminology of black hole in HS context requires caution as discussed in the introduction. In addition to the $SO(3) \times SO(2)$, the solution summarized above has been shown to also have 1/4 of the $\mathcal{N} = 2$ supersymmetric HS symmetry of the model, and their infinite dimensional extension thereof [24].

The solution (5.3) differs from (4.17)–(4.19) both in the form of the internal connection (V^\pm, \bar{V}^\pm) and in that of the gauge field generating functions. As for the internal connection, the difference can be ascribed to the two choices employed in [6] and [24] for solving the deformed oscillator problem (referred to as “symmetric” and “most asymmetric”, respectively, in [6,22]). One can show that the resulting internal connections can be connected via a gauge transformation (see [6]), although the small or large nature of this transformation is yet to be investigated. The comparison of W' in (4.19) and W in (5.3) is technically more complicated, as the two differ also by the shift of the Lorentz connection (2.22), and it will be postponed to a future work, but we note that having the same B identical in both solutions strongly suggests that the physical gauge fields should be equivalent, and in particular equivalent to (5.5).

It is worth mentioning that even by working without the gauge function method, with a specific choice of gauge the Didenko-Vasiliev solution can be simplified in such a way that the W connection is reduced to the vacuum one W_0 . This simplification was studied in [45], along with the embedding of the solution in the $\mathcal{N} = 2$ and $\mathcal{N} = 4$ supersymmetric extensions of the bosonic Vasiliev equations.

6. Perturbative Expansion of Vasiliev Equations

In this Section, we shall summarize the standard perturbative expansion of the Vasiliev equations, benefiting from [4,6,12,25,27,46] (for more recent treatments, see [47,48]).

In the normal order, defined by the star product formula (2.4), the inner Klein operators become real analytic in Y and Z space, viz.,

$$\kappa = \kappa_y \star \kappa_z = \exp(iy^\alpha z_\alpha), \quad \bar{\kappa} = \kappa_{\bar{y}} \star \kappa_{\bar{z}} = \exp(i\bar{y}^{\dot{\alpha}} \bar{z}_{\dot{\alpha}}). \tag{6.1}$$

Assuming that the full field configurations are real-analytic on \mathcal{Z}_4 for generic points in \mathcal{X}_4 , one may thus impose initial conditions

$$B|_{Z=0} = C, \quad W_\mu|_{Z=0} = a_\mu, \tag{6.2}$$

where the x -dependence is understood. In order to compute the Z -dependence of the fields one may choose $V_A|_{C=0}$, which is a trivial flat connection, to vanish, and a homotopy contractor for the de Rham differential on \mathcal{Z}_4 , which entails imposing a gauge condition on V_A . One may then solve

the constraints on $D_A B$, F_{AB} and $F_{A\mu}$ on \mathcal{Z}_4 in a perturbative expansion in C . This procedure gets increasingly complex with increasing order in the expansion, which schematically can be written as

$$B = \sum_{n \geq 1} B^{(n)}, \quad B^{(1)}(C) \equiv C, \tag{6.3}$$

$$V = \sum_{n \geq 1} V^{(n)}, \tag{6.4}$$

$$W = \sum_{n \geq 0} W^{(n)}(a_\mu), \quad W^{(0)}(a_\mu) \equiv a_\mu, \tag{6.5}$$

where $B^{(n)}$, $V^{(n)}$ and $W^{(n)}(a_\mu)$ are n -linear functionals in C , and $W^{(n)}(a_\mu)$ is a linear functional in a_μ . These quantities, which are constructed using the homotopy contractor on \mathcal{Z}_4 , depend on Z , and are real-analytic in $\mathcal{Y}_4 \times \mathcal{Z}_4$ provided that C and a_μ are real analytic in Y -space and all star products arising along the perturbative expansion are well-defined. As for the remaining equations, that is, that $F_{\mu\nu} = 0$ and $D_\mu B = 0$, it follows from the Bianchi identities that they are perturbatively equivalent to $F_{\mu\nu}|_{Z=0} = 0$ and $D_\mu B|_{Z=0} = 0$, which form a perturbatively defined Cartan integrable system on \mathcal{X}_4 for C and a_μ .

To Lorentz covariantize, one imposes

$$W'|_{Z=0} = w, \tag{6.6}$$

with W' from (2.22), that is

$$a_\mu = w_\mu + \frac{1}{4i} (\omega_\mu^{\alpha\beta} M_{\alpha\beta} + \bar{\omega}_\mu^{\dot{\alpha}\dot{\beta}} \bar{M}_{\dot{\alpha}\dot{\beta}})|_{Z=0}, \tag{6.7}$$

where w does not contain any component field proportional to $y_\alpha y_\beta$ and $\bar{y}_{\dot{\alpha}} \bar{y}_{\dot{\beta}}$ in view of (2.23). Upon substituting the above relation into $W^{(n)}(a_\mu; C, \dots, C)$, it follows from the manifest Lorentz covariance that the dependence of $F_{\mu\nu}|_{Z=0}$ and $D_\mu B|_{Z=0}$ on the Lorentz connection arises only via the Lorentz covariant derivative ∇ and the Riemann two-form $(r^{\alpha\beta}, \bar{r}^{\dot{\alpha}\dot{\beta}})$. Thus, the resulting equations in spacetime take the form [6]

$$\begin{aligned} \nabla w + w \star w + \frac{1}{4i} (r^{\alpha\beta} y_\alpha y_\beta + h.c.) + i \sum_{\substack{n_1+n_2 \geq 1 \\ n_{1,2} \geq 0}} (r^{\alpha\beta} V_\alpha^{(n_1)} \star V_\beta^{(n_2)} + h.c.)|_{Z=0} \\ + \sum_{\substack{n_1+n_2 \geq 1 \\ n_{1,2} \geq 0}} W^{(n_1)}(w) \star W^{(n_2)}(w)|_{Z=0} = 0, \end{aligned} \tag{6.8}$$

$$\nabla C + \sum_{\substack{n_1+n_2 \geq 1 \\ n_1 \geq 0, n_2 \geq 1}} [W^{(n_1)}(w_\mu), B^{(n_2)}]_\pi|_{Z=0} = 0, \tag{6.9}$$

where

$$\nabla w := d_x w + [\omega^{(0)}, w]_\star, \quad \nabla C := d_x C + [\omega^{(0)}, C]_\star, \quad \omega^{(0)} = \frac{1}{4i} \omega^{\alpha\beta} M_{\alpha\beta}^{(0)} - h.c., \tag{6.10}$$

$$r^{\alpha\beta} := d\omega^{\alpha\beta} - \omega^{\alpha\gamma} \wedge \omega_\gamma^\beta, \quad \bar{r}^{\dot{\alpha}\dot{\beta}} := d\bar{\omega}^{\dot{\alpha}\dot{\beta}} - \bar{\omega}^{\dot{\alpha}\dot{\gamma}} \wedge \bar{\omega}_{\dot{\gamma}}^{\dot{\beta}}. \tag{6.11}$$

Alternatively, in order to stress the perturbation expansion around the maximally symmetric background, including the spin-two fluctuations, it is more convenient to work in terms of the original one-form a_μ . The perturbative expansion up to 3rd order in Weyl curvatures reads

$$da = a \star a + \mathcal{V}(a, a, C) + \mathcal{V}^2(a, a, C, C) + \mathcal{O}(C^3), \tag{6.12}$$

$$dC = a \star C - C \star \pi(a) - \mathcal{U}(a, C, C) + \mathcal{O}(C^3), \tag{6.13}$$

where $(\mathcal{V}, \mathcal{V}^2, \mathcal{U})$ are functionals that can be determined from (6.8) and (6.9). One next assumes that the homotopy contraction in Z-space is performed such that

$$z^A V_A = 0, \tag{6.14}$$

which we refer to as the Vasiliev gauge, and expands

$$a = W_0 + a_1 + a_2 + \dots, \quad C = C_1 + C_2 + \dots. \tag{6.15}$$

where W_0 is the maximally symmetric background; a_1 a and C_1 are linearized fields; a_n and C_n are n th order fluctuations. The resulting linearized field equations on X-space provide an unfolded description of a dynamical scalar field

$$\phi = C_1 |_{Y=0}, \tag{6.16}$$

and a tower of spin- s Fronsdal fields

$$\phi_{a(s)} = \left(e_0^{\mu\alpha} \left((\sigma_a)^{\alpha\dot{\alpha}} \frac{\partial^2}{\partial y^\alpha \partial \bar{y}^{\dot{\alpha}}} \right)^{s-1} a_{1,\mu} \right) \Big|_{Y=0=Z}, \tag{6.17}$$

where we use the convention that repeated indices are symmetrized. Computing the functional $\mathcal{V}(W_0, W_0, C)$, the linearized unfolded system is given by [4]

$$D_0 a_1 = -\frac{i}{4} H^{\alpha\beta} \partial_\alpha \partial_\beta C_1(y, 0) - \frac{i}{4} \bar{H}^{\dot{\alpha}\dot{\beta}} \bar{\partial}_{\dot{\alpha}} \bar{\partial}_{\dot{\beta}} C_1(0, \bar{y}) \tag{6.18}$$

$$D_0 C_1 = 0, \tag{6.19}$$

where

$$D_0 a_1 := d_x a_1 + \{W_0, a_1\}_\star, \quad D_0 C_1 := d_x C + [W_0, C_1]_\star, \tag{6.20}$$

$$h^{\alpha\dot{\alpha}} := e_0^{\alpha\dot{\alpha}}, \quad H^{\alpha\beta} := h^{\alpha\dot{\gamma}} \wedge h^{\beta}_{\dot{\gamma}}, \quad \bar{H}^{\dot{\alpha}\dot{\beta}} := \bar{h}^{\dot{\alpha}\gamma} \wedge \bar{h}^{\dot{\beta}}_{\gamma}. \tag{6.21}$$

The oscillator expansion of (6.18) furnishes the definition of the spin- s Weyl tensors, and gives the field equations for spin- s fields which remarkably do not contain higher than second order derivatives, and indeed they are the well known Fronsdal equations for massless spin s -fields in AdS . As for (6.19), its oscillator expansion gives the AdS massless scalar field equation in unfolded form.

Perturbative expansion around AdS at second order is rather complicated but still manageable. Schematically the equations take the form

$$D_0 a_2 - \mathcal{V}(W_0, W_0, C_2) = a_1 \star a_1 + 2\mathcal{V}(W_0, a_1, C_1) + \mathcal{V}(W_0, W_0, C_1, C_1), \tag{6.22a}$$

$$D_0 C_2 = [a_1, C_1]_\pi + \mathcal{U}(W_0, C_1, C_1). \tag{6.22b}$$

Even though these terms have been known in the form of parametric integrals for some time, their detailed structure and consequences for the three point functions were considered much later in [11,49], where the field equation for the scalar field was examined, and used for computing the three-point

amplitude for spins $0 - s_1 - s_2$. If $s_1 \neq s_2$, only the first source term in (6.22b) contributes and gives a finite result, in agreement with the boundary CFT prediction. However, if $s_1 = s_2$, only the second source term in (6.22b) contributes and gives a divergent result [11,49]. This divergence was confirmed later in [12,50] where the divergence in the three point amplitude for spins $s - 0 - 0$, resulting from the last term in (6.22a). Soon after, it was shown that a suitable redefinition of the master zero-form cures this problem [48,51], as has been also confirmed with the computation of relevant three-point amplitudes [52,53]. A similar redefinition in the one-form sector has also been determined so that the divergence problem arising in the last term in the first equation above has also been removed [54].

In determining the higher order terms in the perturbative expansions of Vasiliev equations, it remains to be established in general what field redefinitions are allowed in choosing the appropriate basis for the description of the physical fields. In wrestling with this problem, the remarkable simplicity of the holographic duals of this highly nonlinear and seemingly very complicated interactions may provide a handle by means of their holographic reconstruction. Such reconstruction has been achieved for the three and certain four-point interactions [13–15]. Putting aside the analysis and interpretation of the nonlocalities [14,55,56], which are present, and nonetheless in accordance with holography by construction, the issue of how to extract helpful hints from them with regard to the nature of the allowed field redefinitions in perturbative analysis of Vasiliev equation remains to be seen.

Of course, ultimately it would be desirable to have a direct formulation of the principles that govern the nonlocal interactions, based on the combined boundary conditions in twistor space as well as spacetime, as we shall comment further below.

7. A Proposal for an Alternative Perturbation Scheme

In what follows, we shall show that for physically relevant initial data $\Psi(Y)$ given by particle and black hole-like states, the solutions obtained using the factorization method can be mapped to the Vasiliev gauge used in the normal ordered perturbation scheme at the linearized level. Whether the Vasiliev gauge is compatible with an asymptotic description in terms of Fronsdal fields to all orders in perturbation theory, or if it has to be modified, possibly together with a redefinition of the zero-form initial data, is an open problem. In finding the proper boundary conditions in both spacetime and twistor space it may turn out to be necessary to require finiteness of a set of classical observables involving integration over these spaces.

One can define formally the aforementioned map to all orders of classical perturbation theory by applying a gauge function

$$G = L \star H, \tag{7.1}$$

to the holomorphic gauge solution space, where $H = 1 + \sum_{n \geq 1} H^{(n)}$ is a field dependent gauge function to be fixed as to impose the Vasiliev gauge in normal ordering. To begin with, let us consider fluctuations around AdS for which $\Psi(Y)$ consists of particle states, focusing, for concreteness, to the case of scalar particle states worked out in [7]. Upon switching on the gauge function L , the field $V_\alpha^{(L)} := L^{-1} \star V'_\alpha \star L$ develops non-analyticities in the form of poles in Y -space in the particle sector [7]. Applying H removes these poles and expresses the results in terms of Fronsdal fields (at the same time restoring the manifest Lorentz covariance, as we shall see), at least in the leading order. This can be seen as follows. We want to obtain $V_\alpha^{(G)}$ such that

$$z^\alpha V_\alpha^{(G)} + \text{h.c.} = z^+ V^{(G)-} - z^- V^{(G)+} + \text{h.c.} = 0. \tag{7.2}$$

The leading order gauge transformation reads

$$V_\alpha^{(G)(1)} = V_\alpha^{(L)(1)} + \partial_\alpha H^{(1)}. \tag{7.3}$$

Contracting by z^α and using the fact that by definition $Z^A V_A^{(G)} = 0$, one finds

$$H^{(1)} = - \left(\frac{1}{z^\beta \partial_\beta} z^A V_A^{(L)(1)} + \text{h.c.} \right). \tag{7.4}$$

In particular, activating only the scalar ground state (and its negative-energy counterpart) via the parameters $\tilde{v}_{\pm 1}$ in the projector ansatz for B (see (4.25)), this was computed in [7] with the result

$$H^{(1)} = -\frac{i}{4} \frac{1-x^2}{1-2ix_0+x^2} \frac{1}{\tilde{y}^+ \tilde{y}^-} \frac{\tilde{y}^+ z^- + \tilde{y}^- z^+}{\tilde{u}} \left(e^{i\tilde{u}} - 1 \right) \Big|_{\eta=+1} + \text{idem}|_{\eta=-1}, \tag{7.5}$$

where we have set $b = \tilde{v}_1 = \tilde{v}_{-1} = 1$, and

$$\tilde{u} := \tilde{y}^\alpha z_\alpha = \tilde{y}^+ z^- - \tilde{y}^- z^+, \quad \tilde{y}_\alpha = y_\alpha + M_\alpha^{\hat{\beta}}(x, \eta) \tilde{y}_{\hat{\beta}}. \tag{7.6}$$

The matrix $M_\alpha^{\hat{\beta}}(x, \eta)$ can be found in Section 5.2.1 of [7].

This result for $H^{(1)}$ is regular in \mathcal{Z} but has a pole in \mathcal{Y} .

It follows that

$$V_\alpha^{(G)(1)} = -\frac{1}{2} \frac{1-x^2}{1-2ix_0+x^2} \frac{z_\alpha}{\tilde{u}} \left[e^{i\tilde{u}} - \frac{e^{i\tilde{u}} - 1}{i\tilde{u}} \right] + \text{idem}|_{\eta=-1}, \tag{7.7}$$

$$B(Y|x) = \frac{4(1-x^2)}{1-2ix_0+x^2} e^{iy^\alpha M_\alpha^{\hat{\beta}} \tilde{y}_{\hat{\beta}}} \Big|_{\eta=+1} + \text{idem}|_{\eta=-1}. \tag{7.8}$$

We observe that $V_\alpha^{(G)(1)}$ is now real-analytic *everywhere* on \mathcal{C} .

Furthermore, it was shown in [7] that the above expressions for (B, V_α) lead to the relation

$$V_\alpha^{(G)(1)} = z_\alpha \int_0^1 dt t B(-tz, \tilde{y}) e^{ity^\alpha z_\alpha} \tag{7.9}$$

in agreement with the result obtained in the standard perturbative analysis of Vasiliev equations at leading order. The emergence of z_α in (7.7) shows that manifest Lorentz covariance is restored, in comparison with the expression for $V_\alpha^{(L)(1)}$. The prefactor in (7.7) is a consequence of the fact that we are considering the lowest mode alone in the solution for Fronsdal equation the scalar field, as opposed to summing all the full set of modes.

Expressing the exact solutions obtained in holomorphic gauge in terms of Fronsdal fields amounts to setting up an alternative perturbation scheme in which one constructs the higher orders of the gauge function $H^{(n)}$ subject to *dual boundary conditions*, that is, to conditions restricting the both the twistor-space dependence of the master fields and their spacetime asymptotic behaviour. Indeed, one requires that after having switched on H , the master fields have symbols in normal order that are real-analytic at $Y = Z = 0$, and symbols in Weyl order that belong to a (associative) star-product subalgebra with well-defined classical observables defined by traces over twistor space and integrations over cycles in spacetime. The proposal is that this problem admits a non-trivial solutions, and that it fixes $H^{(n)}$ up to residual small HS gauge transformations and the initial data for the master zero-form B to all orders in classical perturbation theory. It would be interesting to see whether these type of field redefinitions are related to those recently proposed by Vasiliev in order to obtain a quasi-local perturbation theory in terms of Fronsdal fields [51,54]. An important related issue is whether the gauge function G is large in the sense that it affects the values of the HS zero-form charges, which are special types of classical observables given by traces over twistor space defining zero-forms in spacetime that are de Rham closed [20,27,57].

A nontrivial test of the factorization approach is first to show that the solution is finite after performing the higher order $H^{(n)}$ gauge transformations, and second, to show that the resulting n -point correlators are in agreement with the result expected from holography. The corrections beyond the leading order remain to be determined, while the computation of correlators has been performed in which the second order solution in standard perturbative scheme has been used. It has been shown that a naive computation of $B^{(2)}$ in the standard perturbative scheme leads to divergences [11,12], and later it was shown that these divergences can be removed by a suitable redefinition of $C(Y|x)$ [48,51]. Whether there exists a principle based on any notion of quasi-locality in spacetime that governs the nature of such redefinitions to all orders in perturbation is not known, to our best knowledge.

An advantage of the factorization method is that here we start from a full solution to Vasiliev’s theory, defined as a classical field theory on the product of spacetime and twistor space (not referring a priori to the conventional perturbative approach). This provides a convenient framework for the description of the solutions with particles fluctuating around nontrivial backgrounds. The key principle here is that linear superposition principle holds for the zero-form initial data $\Psi(Y)$. For example, if we want to describe the solution to Vasiliev equations for particles propagating around BH solution of Section 4, one simply takes $\Psi = v_n P_n + \tilde{v}_n \tilde{P}_n$. The exact solution for the combined system is obtained in this way, but in small fluctuations of a particle propagating in a fixed and exact black hole background, one may treat the parameter \tilde{v}_n and v_n as small and large, respectively. A very interesting open problem is thus to combine this scheme with the aforementioned proposal for dual boundary conditions in order to work out new types of generating functions for HS amplitudes.

8. Aspects of Higher Spin Geometry

HS theory has been mostly studied at the level of the field equations and in terms of locally defined quantities in ordinary spacetimes or twistor space extensions thereof. It is clearly desirable to develop a globally defined framework for the classical theory, in order to provide geometric interpretations of the exact solutions, and shed further light on the important issue of the choice of boundary conditions on the master fields on the total space required for Vasiliev’s equations to produce physically meaningful anti-holographic duals of boundary conformal field theories. An attempt at a global description of HS theory was made in [27] where bundle structures based on different choices of structure groups, soldering forms and classical observables were considered¹⁰. Here, we shall highlight a particularly interesting choice of structure group, and the resulting infinite dimensional coset description involving tensorial coordinates, and associated generalized frame field and resulting metrics.

8.1. Structure Group

Noting that Vasiliev equations are Cartan integrable in arbitrary number of commuting dimensions, yet without changing the local degrees of freedom associated to the zero-forms (more on this below), we consider the formulation of Vasiliev’s theory in terms of the master fields $(W, B, S_\alpha, \bar{S}_{\dot{\alpha}})$ thought of as horizontal forms on a noncommutative fibered space \mathcal{C} with eight-dimensional fibers given by $\mathcal{Y}_4 \times \mathcal{Z}_4$ and base given by an infinite-dimensional commuting real manifold \mathcal{M} , consisting of charts coordinatized by X^M . In each chart, the one-form field $W = dX^M W_M$ thus takes its values in the HS Lie algebra

$$\widehat{\mathfrak{hs}}(4) := \left\{ \widehat{P}(Y, Z) : \tau(\widehat{P}) = (\widehat{P})^\dagger = -\widehat{P} \right\}, \tag{8.1}$$

and the deformed oscillators $(S_\alpha, \bar{S}_{\dot{\alpha}})$ and the deformation field B are thought of as zero-forms on \mathcal{M} , valued in representations of the HS Lie algebra as described in Section 2, where the τ -map is also defined. The extended equations of motion are given by Equations (2.28)–(2.32), with the only difference being that the manifold χ_4 is replaced by \mathcal{M} .

¹⁰ We refer the reader to [27] for considerable amount of details albeit using a a considerably different notation.

In order to define the theory globally on \mathcal{M} , we glue together the locally defined master fields using transition functions from a structure group, which by its definition is generated by a structure algebra \mathfrak{h} given by a subalgebra of $\mathfrak{hs}(4)$. One interesting choice is [27]

$$\mathfrak{h} = \widehat{\mathfrak{hs}}_+(4) \oplus \mathfrak{sl}(2, \mathbb{C}), \quad \widehat{\mathfrak{hs}}_+(4) := \frac{1}{2}(1 + \pi)\widehat{\mathfrak{hs}}(4), \tag{8.2}$$

where $\mathfrak{sl}(2, \mathbb{C})$ is the algebra of canonical Lorentz transformations. The corresponding connection, also referred to as the generalized Lorentz connection, is given by

$$\Omega := W^+ \oplus \omega, \quad W^\pm := \frac{1}{2}(1 \pm \pi)W', \tag{8.3}$$

where ω is the canonical Lorentz connection, while the projection

$$\mathcal{E} := W^-, \tag{8.4}$$

is assumed to form a section¹¹. The Vasiliev equations and gauge transformations in terms of these master fields are spelled out in [27].

8.2. Soldering Mechanism

The horizontal differential algebra on \mathcal{C} , which is quasi-free in the sense that the curvature constraints are cartan integrable modulo zero-form constraints, can be projected to a free horizontal differential algebra on the reduced total space $\mathcal{C}|_{Z=0}$. To this end, one first solves the constraints on \mathcal{Z} given initial data

$$w(Y|X) \equiv dX^M w_M = dX^M \widehat{W}'_M \Big|_{Z=0}, \quad C(Y|X) = B \Big|_{Z=0}, \tag{8.5}$$

where $w(Y|X)$ belongs to the reduced HS algebra

$$\mathfrak{hs}(4) := \widehat{\mathfrak{hs}}(4) \Big|_{Z=0}, \tag{8.6}$$

and $C(Y|X)$ belongs to its twisted adjoint representation. Next, defining

$$\Gamma \oplus \omega := \Omega \Big|_{Z=0} \in \mathfrak{hs}_+(4) \oplus \mathfrak{sl}(2, \mathbb{C}), \quad E := \mathcal{E} \Big|_{Z=0} \in \mathfrak{hs}(4) / \mathfrak{hs}_+(4), \tag{8.7}$$

where $\mathfrak{hs}_+(4) := \frac{1}{2}(1 + \pi)\mathfrak{hs}(4)$, we assume that \mathcal{M} is soldered by E , that is, the tangent space of \mathcal{M} is assumed to be identified with the coset $\mathfrak{hs}(4) / \mathfrak{hs}_+(4)$ via E . Thus, expanding

$$E = dX^M E_M^A P_A, \quad \pi(P_A) = -P_A, \tag{8.8}$$

where P_A is a basis for the π -odd elements of $\mathfrak{hs}(4)$, and denoting the inverse of the frame field E_M^A by E^M_A , the local translations with gauge parameters $\zeta = \zeta^A P_A$ can be identified by usual means [27] as infinitesimal diffeomorphisms generated by globally defined vector fields $\vec{\zeta} = \zeta^A \mathcal{E}^M_A \vec{\partial}_M$ combined with local generalized Lorentz transformations with parameters $(t_{\vec{\zeta}} \Gamma, t_{\vec{\zeta}} \omega)$.

¹¹ It is worth noting that in [27], the form \mathcal{E} was considered to be a soldering form on a manifold \mathcal{M} with tangent space isomorphic to the coset $\widehat{\mathfrak{hs}}(4) / \widehat{\mathfrak{hs}}_+(4)$ and containing \mathcal{M} as a submanifold. We have simplified the geometrical framework here by formulating the system directly on \mathcal{M} .

8.3. Elimination of Tensorial Coordinates

The framework described above can be used to write the zero-form constraint on $\mathcal{M} \times \{Z = 0\}$, up to leading order in C , as

$$D_A(\Gamma, \omega)C + \{P_A, C\}_\star = 0, \tag{8.9}$$

where $D(\Gamma, \omega) = \nabla + \text{ad}_\Gamma$ with ∇ representing the Lorentz covariant derivative. This constraint can be analyzed by decomposing $P_A = \{P_{A_\ell}\}_{\ell=0}^\infty$ into levels of increasing tensorial rank, viz.,

$$P_{A_\ell} = \left\{ P_{a(2\ell+1), b(2k)} \right\}_{k=0}^\ell = \left\{ M_{\{a_1 b_1\}} \star \cdots \star M_{\{a_{2k} b_{2k}\}} \star P_{a_{2k+1}} \star \cdots \star P_{a_{2\ell+1}} \right\}_{k=0}^\ell, \tag{8.10}$$

where (M_{ab}, P_a) are the generators of $SO(3, 2)$ and $P_{a(2\ell+1), b(2k)}$ is a Lorentz tensor of type $(2\ell + 1, 2k)$. The zeroth level of the zero-form constraint reads

$$D_a(\Gamma, \omega)C + \{P_a, C\}_\star = 0, \quad a = 1, \dots, 4, \tag{8.11}$$

where the translation generator is given by the twistor relation $P_a = (\sigma_a)^{\alpha\dot{\alpha}} y_\alpha \bar{y}_{\dot{\alpha}} / 4$, which implies that $C_{\alpha(n+2s), \dot{\alpha}(n)}$ is identifiable as the n th order symmetrized covariant vectorial derivative of the primary spin- s Weyl tensor $C_{\alpha(2s)}$ [25]. On the other hand, the ℓ th level of the zero-form constraint implies

$$D_{a(2\ell+1)}(\Gamma, \omega)C + \{P_{a(2\ell+1)}, C\}_\star = 0, \tag{8.12}$$

where the higher translation generator is now given by the enveloping formula

$$P_{a(2\ell+1)} = P_{\{a_1 \star \cdots \star a_{2\ell+1}\}}. \tag{8.13}$$

As a result, the tensorial derivatives $\nabla_{a(2\ell+1)}(\Gamma, \omega)$ factorize on-shell into multiple vectorial derivatives; for example, the tensorial derivative of the physical scalar $\phi = C|_{Y=0}$ factorizes into

$$D_{a(2\ell+1)}(\Gamma, \omega)\phi \propto D_{\{a_1}(\Gamma, \omega) \cdots D_{a_{2\ell+1}}(\Gamma, \omega)\phi. \tag{8.14}$$

It follows that no new strictly local degrees of freedom are introduced due to the presence of the tensorial coordinates of \mathcal{M} .

8.4. Generalized Metrics

Taking traces of \star -products of generalized vielbeins $\mathcal{E} = dX^M \mathcal{E}_M$ and adjoint operators on twistor space, one can construct structure group invariants that are tensor fields on \mathcal{M} [27]; in particular, we may consider symmetric rank- r tensor fields

$$G_{(r)} := dX^{M_1} \cdots dX^{M_r} \text{Tr} \left[\mathcal{K} \star \left(\mathcal{E}_{M_1} \star \mathcal{V}_{\lambda_1, \bar{\lambda}_1}^{k_1, \bar{k}_1} \right) \star \cdots \star \left(\mathcal{E}_{M_r} \star \mathcal{V}_{\lambda_r, \bar{\lambda}_r}^{k_r, \bar{k}_r} \right) \right], \tag{8.15}$$

where $\mathcal{K} \in \{1, \kappa \bar{\kappa}\}$ and

$$\mathcal{V}_{\lambda, \bar{\lambda}}^{k, \bar{k}} := \left\{ \exp_\star \left[i(\lambda^\alpha S_\alpha + \bar{\lambda}^{\dot{\alpha}} \bar{S}_{\dot{\alpha}}) \right] \right\} \star (B \star \kappa)^{\star k} \star (B \star \bar{\kappa})^{\star \bar{k}}, \quad k, \bar{k} \in \mathbb{N}, \tag{8.16}$$

where $(\lambda^\alpha, \bar{\lambda}^{\dot{\alpha}})$ are auxiliary twistor variables, and

$$\text{Tr} [f(Z, Y)] := \int_{\mathcal{Y} \times \mathcal{Z}} \frac{d^4 Y d^4 Z}{(2\pi)^4} f(Z, Y). \tag{8.17}$$

Given a (compact) p -cycle $[\Sigma]$ in the homology of \mathcal{M} , one can consider the formally defined minimum

$$\mathcal{A}_{\min}[\Sigma, G_{(r)}] := \min_{\Sigma' \in [\Sigma]} \mathcal{A}[\Sigma', G_{(r)}], \tag{8.18}$$

of the area functional

$$\mathcal{A}[\Sigma', G_{(r)}] := \int_{\Sigma'} d^p \sigma \left(\epsilon^{m_1[p]} \dots \epsilon^{m_r[p]} \underbrace{(f^* G)_{m_1 \dots m_r} \dots (f^* G)_{m_1 \dots m_r}}_{p \text{ times}} \right)^{1/r}, \tag{8.19}$$

defined using the totally anti-symmetric tensor density $\epsilon^{m_1 \dots m_p}$ and induced metric

$$(f^* G)_{m_1 \dots m_r} = \partial_{m_1} X^{M_1} \dots \partial_{m_r} X^{M_r} G_{M_1 \dots M_r}, \tag{8.20}$$

on Σ' equipped with local coordinates σ^m . The area functional is structure group invariant, and its minimum, if well-defined, is $\text{Diff}(\mathcal{M})$ invariant, hence serving as a classical observable for the HS theory. Clearly, the dressings by the vertex operator-like operators result in a large number of inequivalent metrics for each r , but this is not a novelty in HS theory, as it is possible to consider similar dressings of the Einstein frame metric in ordinary gravity. The tensorial calculus pertinent to examining the variational principles for $r = 2$ is well understood, whereas for $r > 2$ it remains to be investigated further. In particular, one may ask whether there exists any principle for singling out a preferred metric (possibly of rank two), using, for example, calibrations based on abelian p -forms of the type that will be discussed below. The application of these ideas to a geometrical characterization of the exact solutions of HS theory remains to be investigated.

8.5. Abelian p -Form Charges

Another type of intrinsically defined classical observables facilitated by the introduction of soldering one-forms are the charges of on shell closed abelian p -forms, viz.,

$$\mathcal{Q}[\Sigma, H] = \oint_{\Sigma} H(\mathcal{E}, B, S_{\alpha}, \bar{S}_{\dot{\alpha}}, r_{\alpha\beta}, \bar{r}_{\dot{\alpha}\dot{\beta}}), \tag{8.21}$$

where Σ are closed p -cycles in \mathcal{M} or open p -cycles with suitable boundary conditions, and H are globally defined differential forms that are cohomologically nontrivial on shell, namely $dH = 0$ and $\delta_{\Lambda} H = 0$ ($\Lambda \in \mathfrak{h}$), and H is not globally exact. We also recall that $r_{\alpha\beta}$ is defined in (6.11). The abelian p -forms were considered in [27]

$$H_{[p]} = \text{Tr} \left[(\mathcal{E} \star \mathcal{E} + r^{(S)+})^{\star(p/2)} \star \kappa \right], \quad p = 2, 4, 6, \dots \tag{8.22}$$

where

$$r^{(S)+} = \frac{1}{8i} (1 + \pi) r^{\alpha\beta} S_{\alpha} \star S_{\beta} - h.c. \tag{8.23}$$

The conservation of these charges can be checked directly by replacing the exterior derivative by the structure group covariant derivative $D(\Omega)$ inside the trace and using the fact that $D(\Omega)(\mathcal{E} \star \mathcal{E} + r^{(S)+}) = 0$ on shell.

8.6. On-Shell Actions

Actions have been proposed that imply the Vasiliev equations [16,17] upon applying the variational principle. Their on-shell evaluations involve subtleties stemming from their global formulation and the crucial role played by the boundary conditions. Thus, it remains unclear whether these actions vanish on-shell. Nonetheless, one can still employ the abelian p -form charges discussed

above as off-shell topological deformations. To this end, one set of candidates that have been considered are [27]

$$\begin{aligned}
 S_{\text{top}}[\Sigma_2] &= \text{Re} \left\{ \tau_2 \oint_{\Sigma_2} \text{Tr} [\kappa \star R] \right\} \\
 S_{\text{top}}[\Sigma_4] &= \text{Re} \left\{ \oint_{\Sigma_4} \text{Tr} \left[\kappa \star \left(\tau_4 R \star R + \tilde{\tau}_4 \left((\mathcal{E} \star \mathcal{E} + r^{(S)+}) \star R + \frac{1}{2} (\mathcal{E} \star \mathcal{E} + r^{(S)+}) \star^2 \right) \right) \right] \right\},
 \end{aligned}
 \tag{8.24}$$

where τ_2 , τ_4 and $\tilde{\tau}_4$ are complex constants, $\Sigma_{2,4}$ are submanifolds of \mathcal{M} (where Lagrange multipliers vanish), and

$$R := \nabla W^+ + W^+ \star W^+ + \frac{1}{4i} (\omega^{\alpha\beta} M_{\alpha\beta}^{(0)} + \bar{\omega}^{\dot{\alpha}\dot{\beta}} \bar{M}_{\dot{\alpha}\dot{\beta}}^{(0)}),
 \tag{8.25}$$

which is the curvature of Ω . Using the field equation [27]

$$R + \mathcal{E} \star \mathcal{E} + r^{(S)+} = 0,
 \tag{8.26}$$

we can express $S_{\text{top}}[\Sigma_4]$ on-shell as

$$S_{\text{top}}[\Sigma_4] \approx \text{Re} \left\{ \left(\tau_4 - \frac{1}{2} \tilde{\tau}_4 \right) \oint_{\Sigma_4} H_{[4]} \right\},
 \tag{8.27}$$

where $H_{[4]}$ given in (8.22). Infinities arise from the integration over Σ_4 as well as twistor space. Assuming that the divergence from the AdS vacuum has a definite reality property, such that it can be removed by choosing τ_4 appropriately, one is left with an integral over perturbations that may in principle be finite modulo a prescription for integration contours; for related discussions, see [49,58]. Provided that one considers perturbations corresponding to boundary sources, it would be natural to interpret $S_{\text{top}}[\Sigma_4]$ as the generating functional for the boundary correlation functions.

One may also construct topological two-forms given on-shell by

$$S_{\text{top}}[\Sigma_2] \approx - \text{Re} \left\{ \tau_2 \oint_{\Sigma_2} H_{[2]} \right\},
 \tag{8.28}$$

with $H_{[2]}$ given in (8.22). One application of this surface operator is to wrap Σ_2 around a point-like defect or singularity such as the center of the rotationally symmetric and static solution of [24]. In this case, the leading order contribution, which comes from the AdS vacuum, is a divergent integral over twistor space. If the divergence has a definite reality property, one can cancel it by choosing τ_2 appropriately. One would then be left with an integral over the perturbations. As the latter involve nontrivial functions in twistor space, the integral may be finite. It would be interesting to seek an interpretation of the resulting value of $S_{\text{top}}[\Sigma_2]$ as some form of entropy of the black hole solutions reviewed above.

An alternative framework in which certain 2- and 4-forms in x -space, referred to as the Lagrangian forms, are introduced was proposed in [59], where their possible application for the computation of HS invariant charges and generating functional for the boundary correlations functions were discussed; see also [60] for the computation of such HS charges on black-hole solutions at the first order in the deformation parameters. Asymptotic charges in HS theories have also been described in [61–63].

Their relation to the HS charges¹² and their evaluation on certain exact solutions will be discussed elsewhere [26].

Acknowledgments: We thank David De Filippi, Gary Gibbons, Dmitry Ponomarev, Eugene Skvortsov and Xi Yin for helpful discussions in the course the writing of this review. The work of CI is supported in part by the Russian Science Foundation grant 14-42-00047 in association with the Lebedev Physical Institute in Moscow. The work of ES is supported in part by NSF grant PHY-1214344. The work of P.S. is supported by Fondecyt Regular grants N° 1140296 and N° 1151107 and Conicyt grant DPI 2014-0115.

Conflicts of Interest: The authors declare no conflict of interest.

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¹² It is worth noting that the HS charges described here, as well as those based on the above mentioned Lagrangian 2-form and evaluated in [60], present significant differences with respect to the asymptotic charges. For instance, the first ones are closed and gauge invariant everywhere in spacetime, which makes them proper classical observables also in the strong-coupling regions. Crucial for this to happen is that they contain contributions from all spins. On the other hand, each spin- s asymptotic charge is conserved per se, but is crucially defined only via an integration over a two-cycle at spatial infinity under the hypothesis of *AdS* asymptotics, i.e., only in the weak-coupling region. Indeed, it was shown in [60] that the HS charge based on the Lagrangian 2-form can be written as a combination of contributions from each spin- s field integrated over any two-cycle. Each of the latter contributions gives rise to a separately conserved spin- s charge only when the latter two-cycle is pushed to spatial infinity. Moreover, the asymptotic charges depend for their definition on the existence of asymptotic symmetries, whereas the HS charges, being purely master field constructs, are conserved even without assuming any symmetry. As stressed in [60], this can be seen as a consequence of the non-local expansion in derivatives of the physical fields that is naturally contained in the master fields on-shell.

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