

Symmetry Constrained Decoherence of Conditional Expectation Values [†]

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Abstract: Conditional expectation values of quantum mechanical observables reflect unique non-classical correlations, and are generally sensitive to decoherence. We consider the circumstances under which such sensitivity to decoherence is removed, namely, when the measurement process is subjected to conservation laws. Specifically, we address systems with additive conserved quantities and identify sufficient conditions for the system state such that its coherence plays no role in the conditional expectation values of observables that commute with the conserved quantity. We discuss our findings for a specific model where the system-detector coupling is given by the Jaynes-Cummings interaction, which is relevant to experiments tracking trajectories of qubits in cavities. Our results clarify, among others, the role of coherence in thermal measurements in current architectures for quantum thermodynamics experiments.

Keywords: quantum measurements; conservation laws; quantum coherence

1. Introduction

Quantum measurements stochastically disturb the state of the measured system. The paradigmatic example is ideal projective measurements, wherein the system “collapses” to the eigenstates of the measured observable [1]. More generally, the measurement induced back action is characterised by completely positive, trace non-increasing maps, or quantum instruments, which can be given a Kraus decomposition [2]. Such a state change will therefore allow us to define what the expectation value of some other observable will be, conditional on observing a given measurement outcome. In order to define the conditional *change* in the expectation value of an observable, however, we must have a method of defining what the conditional expectation value of an observable would be prior to observing a measurement outcome. As recently suggested in [3], this quantity can be defined by the weak value [4,5]. The weak value can be arbitrarily large, and even imaginary, which is incompatible with a probability distribution over the eigenvalues of the observable in question.

Conditional expectation values of observables have been associated to physically significant quantities like tunnelling times [6–9] or the spectroscopy of a wave function [10]. The possibility of tracking systems along single quantum trajectories conditioned to measurements outcomes in optical [11,12] and more recently in solid state [13,14] systems, opens also the possibility to observe individual quantum trajectories [15], implement feedback control protocols [13,16,17], determine weak values [18,19], produce deterministic entanglement [20,21], and realize Maxwell demons [22]. It is then possible to detect physical quantities along such quantum trajectories. Most prominently this idea has been used to define thermodynamic quantities along quantum trajectories based on the definition of conditional energy change along trajectories [23–25].

These conditional values are extremely sensitive to the coherence of the initial state of the system. This also means that they are extremely sensitive to decoherence mechanisms [26,27]. In the case of the weak value, for example, the effect of decoherence has been studied by analysing the quantum operation acting on the W -operator [28]. However, the effect of decoherence can be constrained in some situations by the presence of symmetries, i.e., conserved quantities, in the measurement process. In this work we address this issue by considering the measurement process that conserves an additive quantity across the object system that is measured, $\mathcal{S}$, and the measuring apparatus, $\mathcal{A}$, and derive sufficient conditions of the measurement process so that decoherence does not affect the result. First, we show that if the observable O_S commutes with the object system component of the conserved quantity L_S , then the conditional expectation value of O_S will not be sensitive to the coherence in the object system with respect to L_S so long as the apparatus state also commutes with the apparatus conserved quantity L_A . Second, we show that if the measurement process is generated by a Jaynes-Cummings Hamiltonian (which conserves the total excitation number) then the conditional expectation value of an observable that commutes with the number operator will be insensitive to the initial coherence in the system, even if the apparatus state does not commute with the number operator, so long as the compound system is symmetric in the number representation. This situation is relevant for experiments tracking quantum trajectories for qubit in resonant cavities and we comment on the consequence of our result for thermodynamic quantities investigated in these setups.

2. Measurement Models and Conservation Laws

2.1. Quantum Measurements

An observable on a quantum system $\mathcal{S}$, with Hilbert space $\mathcal{H}_S$, is described by a positive operator valued measure (POVM) $M := \{M(x)\}_{x \in \mathcal{X}}$, where $\mathcal{X}$ denotes the set of measurement outcomes and $M(x)$ are positive *effect* operators acting on $\mathcal{H}_S$ that sum to the identity [29]. Given an initial preparation of the system in state ρ , the probability of observing outcome x of the POVM M is given by the Born rule as

$$p_\rho^M(x) := \text{tr}[M(x)\rho] \quad (1)$$

The physical implementation of a POVM, by means of a suitable interaction with an apparatus, is referred to as a measurement model [30,31]. Every POVM M admits infinitely many measurement models, described by the tuple $\mathcal{M} := (\mathcal{H}_A, \varrho, U, Z_A)$. Here $\mathcal{H}_A$ is the Hilbert space of apparatus $\mathcal{A}$, and ϱ is the state in which the apparatus is initially prepared; U is a unitary operator acting on $\mathcal{H}_S \otimes \mathcal{H}_A$; and $Z_A = \sum_{x \in \mathcal{X}} x P_A^x$ is a self-adjoint operator defining a projective valued measure (PVM) on $\mathcal{A}$, where P_A^x denote projection operators on $\mathcal{H}_A$. Every outcome x of the PVM Z_A on $\mathcal{A}$ is associated with outcome x of the POVM M on $\mathcal{S}$. Consequently, a measurement model can be considered as a method of transferring information from the object system $\mathcal{S}$ to the measurement apparatus $\mathcal{A}$ by the unitary operator U , such that a measurement of the apparatus by Z_A after this process will replicate the statistics of directly measuring the object system by the POVM M . Because of this, the unitary interaction stage of measurement is referred to as “premeasurement”, while the projective measurement at the end, responsible for leaving a permanent record of measurement outcomes, is referred to as “objectification” [32].

For each measurement outcome x , the measurement model defines an instrument [33] on $\mathcal{S}$, given as

$$\mathcal{I}_x^{\mathcal{M}}(\rho) := \text{tr}_A[(\mathbb{1}_S \otimes P_A^x)U(\rho \otimes \varrho)U^\dagger] \quad (2)$$

where $\text{tr}_A[\cdot]$ denotes the partial trace over $\mathcal{H}_A$. The probability reproducibility criterion, which ensures that $\mathcal{M}$ replicates the measurement statistics of M , is defined as

$$p_\rho^M(x) = \text{tr}[\mathcal{I}_x^M(\rho)] \text{ for all } \rho \text{ on } \mathcal{H}_S \quad (3)$$

2.2. Expected Value of a Self-Adjoint Operator Conditioned on the Outcome of a POVM

The state of the system, after observing outcome x of the POVM M , implemented by the measurement model $\mathcal{M}$, is denoted as $\rho(x) := \mathcal{I}_x^M(\rho)/p_\rho^M(x)$. The conditional expectation value of a self adjoint operator O_S , evaluated *after* observing outcome x of the POVM M , given an initial preparation of the system in state ρ , is thus given as

$$\begin{aligned} \langle O_S \rangle_{\text{after}}^{\rho, \mathcal{M}, x} &:= \text{tr}[O_S \rho(x)] \\ &= \frac{1}{p_\rho^M(x)} \text{tr}[(O_S \otimes P_A^x) U(\rho \otimes \varrho) U^\dagger] \end{aligned} \quad (4)$$

The average value of $\langle O_S \rangle_{\text{after}}^{\rho, \mathcal{M}, x}$, over all measurement outcomes, is simply

$$\sum_{x \in \mathcal{X}} p_\rho^M(x) \langle O_S \rangle_{\text{after}}^{\rho, \mathcal{M}, x} = \text{tr}[(O_S \otimes \mathbb{1}_A) U(\rho \otimes \varrho) U^\dagger] \quad (5)$$

Similarly, we may define the conditional expectation value of O_S , evaluated *before* observing outcome x of the POVM M , given an initial preparation of the system in state ρ , as the real part of the generalised weak value of O_S [34]

$$\begin{aligned} \langle O_S \rangle_{\text{before}}^{\rho, \mathcal{M}, x} &:= \frac{\text{Re}(\text{tr}[M(x) O_S \rho])}{p_\rho^M(x)} \equiv \frac{\text{Re}(\text{tr}[\mathcal{I}_x^M(O_S \rho)])}{p_\rho^M(x)} \\ &= \frac{1}{2p_\rho^M(x)} \text{tr}[(\mathbb{1}_S \otimes P_A^x) U((O_S \rho + \rho O_S) \otimes \varrho) U^\dagger] \end{aligned} \quad (6)$$

while the average value of $\langle O_S \rangle_{\text{before}}^{\rho, \mathcal{M}, x}$ over all measurement outcomes is

$$\sum_{x \in \mathcal{X}} p_\rho^M(x) \langle O_S \rangle_{\text{before}}^{\rho, \mathcal{M}, x} = \text{tr}[O_S \rho] \quad (7)$$

Note that while $\langle O_S \rangle_{\text{after}}^{\rho, \mathcal{M}, x}$ depends on the specific measurement model $\mathcal{M}$ for the POVM M , the same is not true for $\langle O_S \rangle_{\text{before}}^{\rho, \mathcal{M}, x}$, which is uniquely determined by the POVM M . Finally, we may define the conditional change in the quantity O_S , given outcome x of the POVM M , as the difference between Equations (4) and (6), which is

$$\Delta O_S^{\rho, \mathcal{M}, x} := \langle O_S \rangle_{\text{after}}^{\rho, \mathcal{M}, x} - \langle O_S \rangle_{\text{before}}^{\rho, \mathcal{M}, x} \quad (8)$$

The average change in this quantity is thus simply $\langle \Delta O_S^{\rho, \mathcal{M}, x} \rangle = \text{tr}[(O_S \otimes \mathbb{1}_A) U(\rho \otimes \varrho) U^\dagger] - \text{tr}[O_S \rho]$. While Equations (4) and (6) can only be interpreted as statistical properties of the system in general, they can be definite properties under specific conditions. The most trivial case is for Equation (4), which is a definite property if $\rho(x)$ only has support on a single (possibly degenerate) subspace of O_S . The more interesting case where the conditional expectation value can be a definite property is for Equation (6). In the special case where ρ is a pure state $|\psi\rangle$, and $M(x)$ is a projection on the pure state $|\phi\rangle$, Equation (6) simplifies to the real component of the familiar weak value $\langle \phi | O_S | \psi \rangle / \langle \phi | \psi \rangle$. As discussed in [35], in such a case the weak value can be interpreted as the “eigenvalue” obtained by measuring the observable O_S on the system described by the two-state vector [36] $\langle \phi | | \psi \rangle$, where $|\psi\rangle$ describes the evolution of the system forwards in time, while $|\phi\rangle$ describes the evolution of the system

backwards in time. To see this, consider the case where the measurement model for the observable O_S is given by an apparatus that is a particle on a line, given by a pure state described by the position operator Q , and the premeasurement unitary interaction with the system is $U(g) = e^{-igO_S \otimes P}$, with P the conjugate momentum to Q , and g a strength parameter. If $|\psi\rangle$ is an eigenstate of O_S , with eigenvalue o , the apparatus will remain in a pure state after the measurement interaction, and its position will shift by go . As such, the system in state $|\psi\rangle$ has a definite property of the observable O_S . If, however, $|\psi\rangle$ is not an eigenstate of O_S , then the apparatus will be in a statistical mixture of pure states, each shifted by a different amount; here, $|\psi\rangle$ will not have a definite property of O_S . But if we also post-select the system onto the pure state $|\phi\rangle$ after its interaction with the apparatus, then if g is sufficiently small the apparatus will remain pure, with its position shifted by $\approx g\langle\phi|O_S|\psi\rangle/\langle\phi|\psi\rangle$; analogously, we may say that the two-state vector $\langle\phi||\psi\rangle$ has a definite property of the observable O_S , even though this property may lie outside the range of the eigenvalues of O_S . However, such an interpretation does not hold in the general case, where either the initial state ρ is mixed, or the POVM element $M(x)$ is not a projection on a pure state. This is because in such a case, the apparatus will again be in a statistical mixture of pure states, each shifted by a different amount.

2.3. Measurements Restricted by Additive Conservation Laws

Measurements, as other physical processes, may be subject to conservation laws. These are characterised by the commutation of the measurement unitary operator U with some quantity L . A class of interest are additive conservation laws, meaning that $L = L_S + L_A$, with L_S and L_A being self adjoint operators on $\mathcal{H}_S$ and $\mathcal{H}_A$, respectively.

In the presence of additive conservation laws, measurements will be restricted by the Wigner-Araki-Yanase (WAY) theorem [37–39]. Consider a measurement model $\mathcal{M} := (\mathcal{H}_A, \varrho, U, Z_A)$ that defines a PVM M on the system S , such that U commutes with $L_S + L_A$. The WAY theorem states that either if M is repeatable, or Z_A commutes with L_A (the Yanase condition), it will follow that M must commute with L_S .

A PVM M is said to be repeatable if, conditional on observing outcome x , a subsequent measurement of M will result in outcome x with certainty. While it is not necessary for a measurement of M on S to be repeatable, it is necessary for the measurement of Z_A on the apparatus to be repeatable; this is because the record of a measurement outcome stored in the apparatus must be a permanent fixture of the world.

Measurement models can be extended ad infinitum, by means of introducing a measurement model for the apparatus observable Z_A with the aid of an additional apparatus $\mathcal{B}$. Therefore, if the measurement of Z_A is to be implemented under an additive conservation law $L_A + L_B$, then by the WAY theorem the requirement of repeatability will necessitate that Z_A commutes with L_A .

3. Results

3.1. Conserved Quantities and Decoherence Maps

Let us now define the decoherence map with respect to a self-adjoint operator L as

$$\Phi_L(A) := \sum_l Q^l A Q^l \quad (9)$$

where Q^l are the spectral projections of L . If any physical phenomenon, such as measurement, does not distinguish between A and $\Phi_L(A)$, i.e., the two states give the same result, we may say that the coherence of A with respect to L is a symmetry of that phenomenon.

We now explore the situations in which the coherence of the system and apparatus states ρ and ϱ , with respect to the conserved quantities L_S and L_A , will be symmetries of $\Delta O_S^{\rho, \mathcal{M}, x}$, defined in Equation (8), if O_S commutes with L_S . First, we shall prove that if a measurement model conserves an additive quantity $L = L_S + L_A$, then the coherence of the object system state ρ (apparatus system state

ϱ) with respect to L_S (L_A) will be a symmetry of $\Delta O_S^{\rho, \mathcal{M}, x}$, if O_S commutes with L_S and ϱ commutes with L_A (ρ commutes with L_S).

Theorem 1. Consider the POVMs M_1 and M_2 , induced by the measurement models $\mathcal{M}_1 := (\mathcal{H}_A, \varrho_1, U, Z_A)$ and $\mathcal{M}_2 := (\mathcal{H}_A, \varrho_2, U, Z_A)$. Let U commute with $L = L_S + L_A$ and Z_A commute with L_A . Finally, let $\varrho_2 = \Phi_{L_A}(\varrho_1)$, with Φ_{L_A} defined by Equation (9). It follows that if both O_S and ρ commute with L_S , then

$$\begin{aligned} \langle O_S \rangle_{\text{before}}^{\rho, M_1, x} &= \langle O_S \rangle_{\text{before}}^{\rho, M_2, x} \\ \langle O_S \rangle_{\text{after}}^{\rho, \mathcal{M}_1, x} &= \langle O_S \rangle_{\text{after}}^{\rho, \mathcal{M}_2, x} \end{aligned} \quad (10)$$

Conversely, if O_S commutes with L_S , then for any ρ

$$\begin{aligned} \langle O_S \rangle_{\text{before}}^{\rho, M_2, x} &= \langle O_S \rangle_{\text{before}}^{\Phi_{L_S}(\rho), M_2, x} \\ \langle O_S \rangle_{\text{after}}^{\rho, \mathcal{M}_2, x} &= \langle O_S \rangle_{\text{after}}^{\Phi_{L_S}(\rho), \mathcal{M}_2, x} \end{aligned} \quad (11)$$

The proof is provided in Appendix A. Equation (10) shows that the coherence in the apparatus state with respect to the conserved quantity L_A is a symmetry of $\Delta O_S^{\rho, \mathcal{M}, x}$ if both O_S and ρ commute with the conserved quantity L_S . Meanwhile, Equation (11) shows that the coherence of the system state with respect to the conserved quantity L_S is a symmetry of $\Delta O_S^{\rho, \mathcal{M}, x}$ if O_S commutes with L_S and the apparatus state ϱ commutes with L_A .

It is, however, possible for the coherence in both ρ and ϱ to be a symmetry of $\Delta O_S^{\rho, \mathcal{M}, x}$ even if neither of them commutes with the conserved quantity.

Theorem 2. Consider the POVMs M_1 and M_2 , induced by the measurement models $\mathcal{M}_1 := (\mathcal{H}_A, \varrho_1, U, Z_A)$ and $\mathcal{M}_2 := (\mathcal{H}_A, \varrho_2, U, Z_A)$. Let U commute with $L = L_S + L_A$ and Z_A commute with L_A . Finally, let $\varrho_2 = \Phi_{L_A}(\varrho_1)$, with Φ_{L_A} defined by Equation (9). Denote the eigenvectors of L_S and L_A as $|\phi_m^\alpha\rangle$ and $|\phi_\mu^\beta\rangle$ respectively, where m and μ are eigenvalues, while α and β label the degeneracy. It follows that if O_S commutes with L_S , $\rho \otimes \varrho_i$ is symmetric in the eigenbasis representation $|\phi_m^\alpha \otimes \phi_\mu^\beta\rangle$, and the real component of $\langle \phi_n^{\alpha'} \otimes \phi_v^{\beta'} | U^\dagger (O_S \otimes P_A^x) U | \phi_m^\alpha \otimes \phi_\mu^\beta \rangle$ is zero when $m \neq n$ and $\mu \neq v$, then

$$\begin{aligned} \langle O_S \rangle_{\text{before}}^{\rho, M_1, x} &= \langle O_S \rangle_{\text{before}}^{\rho, M_2, x} = \langle O_S \rangle_{\text{before}}^{\Phi_{L_S}(\rho), M_1, x} = \langle O_S \rangle_{\text{before}}^{\Phi_{L_S}(\rho), M_2, x} \\ \langle O_S \rangle_{\text{after}}^{\rho, \mathcal{M}_1, x} &= \langle O_S \rangle_{\text{after}}^{\rho, \mathcal{M}_2, x} = \langle O_S \rangle_{\text{after}}^{\Phi_{L_S}(\rho), \mathcal{M}_1, x} = \langle O_S \rangle_{\text{after}}^{\Phi_{L_S}(\rho), \mathcal{M}_2, x} \end{aligned} \quad (12)$$

The proof is provided in Appendix A. It will be instructive to consider physical situations where Theorem 2 applies.

3.2. Qubits Measured by a Jaynes-Cummings Interaction

A simple example of a measurement process with an additive conserved quantity is obtained by a Jaynes-Cummings system-apparatus interaction. Consider the case where the measurement unitary operator is $U = e^{-i\theta H_I}$, with the Jaynes-Cummings interaction Hamiltonian

$$H_I := \sigma_S^+ \otimes \sigma_A^- + \sigma_S^- \otimes \sigma_A^+ \quad (13)$$

where for $X \in \{S, A\}$,

$$\sigma_X^+ := \sum_{k=0}^{d_X-1} \sqrt{k+1} |k+1\rangle \langle k| = (\sigma_X^-)^\dagger \quad (14)$$

Here, d_x is the dimension of Hilbert space $\mathcal{H}_x$, and $|k\rangle$ denotes the excitation number k of the system. The unitary U conserves the total excitation number $\mathcal{N} = \mathcal{N}_S + \mathcal{N}_A$, where for $X \in \{S, A\}$,

$$\mathcal{N}_X = \sum_{k=0}^{d_X-1} k|k\rangle\langle k| \quad (15)$$

Given the observables O_S and Z_A that commute with $\mathcal{N}_S$ and $\mathcal{N}_A$ respectively, it follows that the real component of $\langle n \otimes v | U^\dagger (O_S \otimes P_A^x) U | m \otimes \mu \rangle$ will always be zero when $n \neq m$ and $v \neq \mu$ so long as either $d_S = 2$ or $d_A = 2$. Let us consider the case where $d_S = 2$, i.e., when the object system being measured is a qubit, while the measuring apparatus can have an arbitrarily large dimension. It follows that $U = \sum_{l=0}^{d_A+1} U_l$, where for $1 \leq l \leq d_A$, U_l is a 2-dimensional matrix acting on the subspace spanned by $\{|0 \otimes l\rangle, |1 \otimes l-1\rangle\}$, given as $U_l = e^{-i\theta\sqrt{l}\sigma_x} = \cos(\theta\sqrt{l})\mathbb{1} - i\sin(\theta\sqrt{l})\sigma_x$, with σ_x the Pauli-X matrix defined as $\sigma_x|0 \otimes l\rangle = |1 \otimes l-1\rangle$. Consequently, for $n \neq m$ and $v \neq \mu$, the real part of $\langle n \otimes v | U | m \otimes \mu \rangle$ is zero, while the imaginary part of $\langle m \otimes \mu | U | m \otimes \mu \rangle$ is zero. Consequently, by choosing $O_S = \lambda_0|0\rangle\langle 0| + \lambda_1|1\rangle\langle 1|$ we may write

$$\begin{aligned} \langle 0 \otimes l | U^\dagger (O_S \otimes P_A^x) U | 1 \otimes l-1 \rangle &= \lambda_0 \delta_{l,x} \langle 0 \otimes l | U^\dagger | 0 \otimes l \rangle \langle 0 \otimes l | U | 1 \otimes l-1 \rangle \\ &\quad + \lambda_1 \delta_{l-1,x} \langle 0 \otimes l | U^\dagger | 1 \otimes l-1 \rangle \langle 1 \otimes l-1 | U | 1 \otimes l-1 \rangle \end{aligned} \quad (16)$$

where $\delta_{k,x}$ is the Kronecker delta function determining if $|k\rangle$ lies in the range of the projector P_A^x . Each term of the above equation is a product of a purely real number, with a purely imaginary number. Therefore, the total value is purely imaginary.

Given such a model, the results of Theorem 2 will follow so long as the initial product state of system and apparatus, $\rho \otimes \varrho$, is symmetric in the excitation number representation. As such, let us consider the simplest example where both S and A are qubits, where $\rho = |\psi\rangle\langle\psi|$, with $|\psi\rangle := \cos(\theta_1/2)|1\rangle + e^{i\phi}\sin(\theta_1/2)|0\rangle$, while $\varrho = |\xi\rangle\langle\xi|$, with $|\xi\rangle := \cos(\theta_2/2)|1\rangle + \sin(\theta_2/2)|0\rangle$. Moreover, let us choose $Z_A = |1\rangle\langle 1| - |0\rangle\langle 0|$ (with outcomes $x = \pm$) and $O_S = |1\rangle\langle 1| - |0\rangle\langle 0|$. According to Theorem 2, therefore, $\Delta O_S^{\rho, \mathcal{M}, x} = \Delta O_S^{\Phi_{\mathcal{N}_S(\rho)}, \mathcal{M}, x}$, with these quantities defined in Equation (8), if $\phi \in \{0, \pi\}$. This is shown in Figure 1, where $\Delta O_S^{\rho, \mathcal{M}, x} - \Delta O_S^{\Phi_{\mathcal{N}_S(\rho)}, \mathcal{M}, x}$ is plotted as a function of ϕ , and deviates from zero as soon as $\phi \neq 0, \pi$.

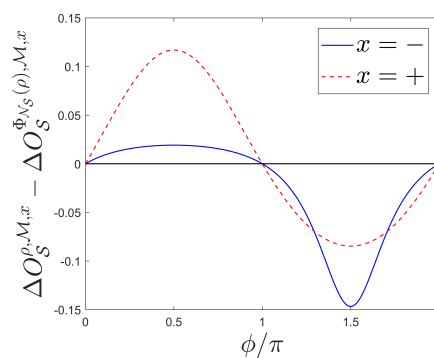


Figure 1. The measurement model for the POVM on $\mathcal{H}_S$ is $\mathcal{M} := (\mathcal{H}_A, |\xi\rangle, U, Z_A)$. Here, the measurement unitary is given as $U = e^{-i(\pi/3)H_I}$, with the Jaynes Communigs interaction Hamiltonian defined in Equation (13). This conserves the total excitation number $\mathcal{N}_S + \mathcal{N}_A$, defined in Equation (15). The apparatus observable is $Z_A = |1\rangle\langle 1| - |0\rangle\langle 0|$, with the corresponding outcomes $x = \pm$. The initial state of the apparatus is the coherent state $|\xi\rangle = \cos(\pi/6)|1\rangle + \sin(\pi/6)|0\rangle$, while the object system is initially prepared in the coherent state $|\psi\rangle = \cos(\pi/8)|1\rangle + e^{i\phi}\sin(\pi/8)|0\rangle$. Finally, the system observable is $O_S = |1\rangle\langle 1| - |0\rangle\langle 0|$. When $\phi \in \{0, \pi\}$, the state $|\psi\rangle\langle\psi| \otimes |\xi\rangle\langle\xi|$ is symmetric in the excitation number representation $|m, \mu\rangle$, and so by Theorem 2 $\Delta O_S^{\rho, \mathcal{M}, x} = \Delta O_S^{\Phi_{\mathcal{N}_S(\rho)}, \mathcal{M}, x}$, with these quantities defined by Equation (8).

4. Discussion

Our findings have physical implications in all cases where conditional measurements are associated with physical quantities. A striking example is that of energy changes along quantum trajectories [3], with its thermodynamic implications. In particular, Theorem 1 can be seen as an extension of results pertaining to thermal operations to thermal measurements. Recall that a thermal operation is constituted of an energy conserving unitary interaction with an apparatus that is prepared in a Gibbs state. It is known that thermal operations are a subset of time-translation symmetric operations [40]. This implies that, since U commutes with the total Hamiltonian $H_S + H_A$, and the apparatus state ϱ commutes with H_A (which is clearly the case when ϱ is a Gibbs state $e^{-H_A/k_B T}/\text{tr}[e^{-H_A/k_B T}]$), then $\text{tr}[(H_S \otimes \mathbb{1}_A)U(\rho \otimes \varrho)U^\dagger] = \text{tr}[(H_S \otimes \mathbb{1}_A)U(\Phi_{H_S}(\rho) \otimes \varrho)U^\dagger]$. That is to say, the coherence in ρ with respect to H_S is a symmetry of the average change in energy for thermal operations. A thermal measurement can be seen as augmenting a thermal operation with a projective measurement of the apparatus by some observable Z_A . However, conditional energy changes given a thermal measurement are not necessarily invariant with respect to the coherence in the system; as shown in Theorem 1, for the conditional change in energies to be invariant with respect to the coherence in ρ , the Yanase condition must be satisfied, i.e., the apparatus observable Z_A must commute with H_A .

In Theorem 2 we show how, in some circumstances, the coherence in the system does not affect the conditional change in expectation values even if the apparatus state ϱ does not commute with its conserved quantity. A common example of such a situation is where the object system being measured is a qubit, and the measurement interaction is generated by a Jaynes-Cummings Hamiltonian, with the conserved quantity being the total excitation number. This setup describes, for example, the coupling of a superconducting qubit and a cavity mode used to track the energy changes of the system along its trajectories [Kater]. We show that so long as the state of the compound system prior to measurement is symmetric in the number representation, then the coherence in the object system w.r.t the number operator will not affect conditional expectation values of observables that commute with the number operator, such as the energy.

5. Conclusions

In this work we have studied the effect of quantum coherence on the expectation values of observables, conditioned on the outcome of generalized measurements, subject to symmetry constraints. To this end we have considered a general measurement model which preserves a quantity additive in the system and apparatus degrees of freedom (additive conservation law). We then identified the sufficient conditions such that the conditional expectation values given some initial state ρ will be identical to that given when the coherence of the initial state is removed. We have finally illustrated our results for a simple measurement model of a qubit coupled to a harmonic oscillator (e.g., an atom in a resonant electromagnetic cavity). The conditional expectation value can be controlled by tuning a relative phase of the qubit state and it is generically different from its incoherent counterpart; the two being equal for the value of the phase that fulfil the theorem conditions. This shows that additive symmetries, present in systems used for quantum measurement experiments, can strongly constrain the role of system coherences, to the point that they might not affect the conditional expectation values of observables.

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Abbreviations

The following abbreviations are used in this manuscript:

POVM	Positive operator valued measure
PVM	Projective valued measure
WAY	Wigner-Araki-Yanase

Appendix A. Proofs of Theorems

Proof of Theorem 1. Let us denote the spectral projections of L as $Q^l = \sum_{(m,\mu)_l} Q_S^m \otimes Q_A^\mu$, where Q_S^m and Q_A^μ are the spectral projections of L_S and L_A respectively, and $(m,\mu)_l$ denotes the pair of eigenvalues which sum to $m + \mu = l$. The commutation relation $[U, L]_- = 0$ will therefore imply that $U = \sum_l Q^l U Q^l$, and so we have

$$\begin{aligned} \langle O_S \rangle_{\text{after}}^{\rho, \mathcal{M}_i, x} &:= \frac{1}{p_\rho^{M_i}(x)} \text{tr}[(O_S \otimes P_A^x) U (\rho \otimes q_i) U^\dagger] \\ &= \frac{1}{p_\rho^{M_i}(x)} \sum_{l, l'} \text{tr}[(O_S \otimes P_A^x) Q^l U Q^{l'} (\rho \otimes q_i) Q^{l'} U^\dagger Q^l] \\ &= \frac{1}{p_\rho^{M_i}(x)} \sum_{l, l'} \sum_{(m,\mu)_l} \sum_{(n,\nu)_{l'}} \text{tr}[(Q_S^m O_S Q_S^m \otimes Q_A^\nu P_A^x Q_A^\mu) U Q^l (\rho \otimes q_i) Q^{l'} U^\dagger] \end{aligned} \quad (\text{A1})$$

Since Z_A commutes with L_A , and O_S commutes with L_S , the last line of this equation may be rewritten as

$$\begin{aligned} \langle O_S \rangle_{\text{after}}^{\rho, \mathcal{M}_i, x} &= \frac{1}{p_\rho^{M_i}(x)} \sum_{l, l'} \sum_{(m,\mu)_l} \sum_{(n,\nu)_{l'}} \text{tr}[(Q_S^m O_S Q_S^m \otimes Q_A^\mu P_A^x Q_A^\mu) U Q^l (\rho \otimes q_i) Q^{l'} U^\dagger] \\ &= \frac{1}{p_\rho^{M_i}(x)} \sum_l \text{tr}[(O_S \otimes P_A^x) U Q^l (\rho \otimes q_i) Q^l U^\dagger] \end{aligned} \quad (\text{A2})$$

Expanding Q^l into the spectral projections of L_S and L_A will thus yield the expression

$$\langle O_S \rangle_{\text{after}}^{\rho, \mathcal{M}_i, x} = \frac{1}{p_\rho^{M_i}(x)} \sum_l \sum_{(m,\mu)_l} \sum_{(n,\nu)_l} \text{tr}[(O_S \otimes P_A^x) U (Q_S^m \rho Q_S^n \otimes Q_A^\mu q_i Q_A^\nu) U^\dagger] \quad (\text{A3})$$

Similarly, we may write

$$\langle O_S \rangle_{\text{before}}^{\rho, \mathcal{M}_i, x} = \frac{1}{2p_\rho^{M_i}(x)} \sum_l \sum_{(m,\mu)_l} \sum_{(n,\nu)_l} \text{tr}[(\mathbb{1}_S \otimes P_A^x) U (Q_S^m (O_S \rho + \rho O_S) Q_S^n \otimes Q_A^\mu q_i Q_A^\nu) U^\dagger] \quad (\text{A4})$$

If ρ commutes with L_S , then the only terms that remain in Equations (A3) and (A4) are those with $m = n$. The fact that $m + \mu = m + \nu = l$ thus implies that $\mu = \nu$, and so we may make the substitution $q_i = \Phi_{L_A}(q_i)$, which gives Equation (10). Conversely, for the POVM M_2 , $\mu = \nu$, and by the same argument $m = n$, which implies Equation (11). $\square$

Proof of Theorem 2. Recall from Theorem 1 that, given the commutation relations $[U, L_S + L_A]_- = 0$, $[O_S, L_S]_- = 0$, and $[Z_A, L_A]_- = 0$, we arrive at Equations (A3) and (A4). We may write Equation (A3) as

$$\begin{aligned} \langle O_S \rangle_{\text{after}}^{\rho, \mathcal{M}_i, x} &= \frac{1}{p_\rho^{M_i}(x)} \sum_l \sum_{(m,\mu)_l} \text{tr}[(O_S \otimes P_A^x) U (Q_S^m \rho Q_S^m \otimes Q_A^\mu q_i Q_A^\mu) U^\dagger] \\ &\quad + \frac{1}{p_\rho^{M_i}(x)} \sum_l \sum_{(m,\mu)_l} \sum_{(n \neq m, \nu \neq \mu)_l} \text{tr}[(O_S \otimes P_A^x) U (Q_S^m \rho Q_S^n \otimes Q_A^\mu q_i Q_A^\nu) U^\dagger] \end{aligned} \quad (\text{A5})$$

Similarly, we may write Equaiton (A4) as

$$\begin{aligned} \langle O_S \rangle_{\text{before}}^{\rho, M_{i,x}} &= \frac{1}{p_\rho^{M_i}(x)} \sum_l \sum_{(m,\mu)_l} \left(\text{tr}[(\mathbb{1}_S \otimes P_A^x) U(O_S Q_S^m \rho Q_S^m \otimes Q_A^\mu q_i Q_A^\mu) U^\dagger] \right) \\ &+ \frac{1}{2p_\rho^{M_i}(x)} \sum_l \sum_{(m,\mu)_l} \sum_{(n \neq m, \nu \neq \mu)_l} \left(\text{tr}[(\mathbb{1}_S \otimes P_A^x) U(Q_S^m (O_S \rho + \rho O_S) Q_S^n \otimes Q_A^\mu q_i Q_A^\nu) U^\dagger] \right) \end{aligned} \quad (\text{A6})$$

For a given l , $m \neq n$, and $\mu \neq \nu$ the terms in Equaiton (A5) can be expanded as

$$\begin{aligned} &\text{tr}[(O_S \otimes P_A^x) U(Q_S^m \rho Q_S^n \otimes Q_A^\mu q_i Q_A^\nu) U^\dagger] + \text{tr}[(O_S \otimes P_A^x) U(Q_S^n \rho Q_S^m \otimes Q_A^\nu q_i Q_A^\mu) U^\dagger] \\ &= \sum_{\alpha, \alpha', \beta, \beta'} \langle \phi_n^{\alpha'} \otimes \varphi_\nu^{\beta'} | U^\dagger (O_S \otimes P_A^x) U | \phi_m^\alpha \otimes \varphi_\mu^\beta \rangle \langle \phi_m^\alpha | \rho | \phi_n^{\alpha'} \rangle \langle \varphi_\mu^\beta | q_i | \varphi_\nu^{\beta'} \rangle \\ &+ \sum_{\alpha, \alpha', \beta, \beta'} \langle \phi_m^\alpha \otimes \varphi_\mu^\beta | U^\dagger (O_S \otimes P_A^x) U | \phi_n^{\alpha'} \otimes \varphi_\nu^{\beta'} \rangle \langle \phi_n^{\alpha'} | \rho | \phi_m^\alpha \rangle \langle \varphi_\nu^{\beta'} | q_i | \varphi_\mu^\beta \rangle \end{aligned} \quad (\text{A7})$$

while those of Equaiton (A6) are expanded as

$$\begin{aligned} &\text{tr}[(\mathbb{1}_S \otimes P_A^x) U(Q_S^m (O_S \rho + \rho O_S) Q_S^n \otimes Q_A^\mu q_i Q_A^\nu) U^\dagger] + \text{tr}[(\mathbb{1}_S \otimes P_A^x) U(Q_S^n (O_S \rho + \rho O_S) Q_S^m \otimes Q_A^\nu q_i Q_A^\mu) U^\dagger] \\ &= \sum_{\alpha, \alpha', \beta, \beta'} \langle \phi_n^{\alpha'} \otimes \varphi_\nu^{\beta'} | U^\dagger (\mathbb{1}_S \otimes P_A^x) U | \phi_m^\alpha \otimes \varphi_\mu^\beta \rangle \langle \phi_m^\alpha | (O_S \rho + \rho O_S) | \phi_n^{\alpha'} \rangle \langle \varphi_\mu^\beta | q_i | \varphi_\nu^{\beta'} \rangle \\ &+ \sum_{\alpha, \alpha', \beta, \beta'} \langle \phi_m^\alpha \otimes \varphi_\mu^\beta | U^\dagger (\mathbb{1}_S \otimes P_A^x) U | \phi_n^{\alpha'} \otimes \varphi_\nu^{\beta'} \rangle \langle \phi_n^{\alpha'} | (O_S \rho + \rho O_S) | \phi_m^\alpha \rangle \langle \varphi_\nu^{\beta'} | q_i | \varphi_\mu^\beta \rangle \end{aligned} \quad (\text{A8})$$

If $\rho \otimes q$ is symmetric in the representation of $|\phi_m^\alpha \otimes \varphi_\mu^\beta\rangle$, it follows that $\langle \phi_m^\alpha | \rho | \phi_n^{\alpha'} \rangle \langle \varphi_\mu^\beta | q_i | \varphi_\nu^{\beta'} \rangle = \langle \phi_n^{\alpha'} | \rho | \phi_m^\alpha \rangle \langle \varphi_\nu^{\beta'} | q_i | \varphi_\mu^\beta \rangle$, while $\langle \phi_m^\alpha | (O_S \rho + \rho O_S) | \phi_n^{\alpha'} \rangle \langle \varphi_\mu^\beta | q_i | \varphi_\nu^{\beta'} \rangle = \langle \phi_n^{\alpha'} | (O_S \rho + \rho O_S) | \phi_m^\alpha \rangle \langle \varphi_\nu^{\beta'} | q_i | \varphi_\mu^\beta \rangle$ and so we may write the right hand sides of Equaitons (A7) and (A8) as

$$\sum_{\alpha, \alpha', \beta, \beta'} \left(\langle \phi_n^{\alpha'} \otimes \varphi_\nu^{\beta'} | U^\dagger (O_S \otimes P_A^x) U | \phi_m^\alpha \otimes \varphi_\mu^\beta \rangle + \text{complex conjugate} \right) \langle \phi_m^\alpha | \rho | \phi_n^{\alpha'} \rangle \langle \varphi_\mu^\beta | q_i | \varphi_\nu^{\beta'} \rangle \quad (\text{A9})$$

and

$$\begin{aligned} &\sum_{\alpha, \alpha', \beta, \beta'} \left(\langle \phi_n^{\alpha'} \otimes \varphi_\nu^{\beta'} | U^\dagger (\mathbb{1}_S \otimes P_A^x) U | \phi_m^\alpha \otimes \varphi_\mu^\beta \rangle + \text{complex conjugate} \right) \\ &\langle \phi_m^\alpha | (O_S \rho + \rho O_S) | \phi_n^{\alpha'} \rangle \langle \varphi_\mu^\beta | q_i | \varphi_\nu^{\beta'} \rangle \end{aligned} \quad (\text{A10})$$

Finally, if $\langle \phi_n^{\alpha'} \otimes \varphi_\nu^{\beta'} | U^\dagger (O_S \otimes P_A^x) U | \phi_m^\alpha \otimes \varphi_\mu^\beta \rangle$ is purely imaginary (which implies that so is $\langle \phi_n^{\alpha'} \otimes \varphi_\nu^{\beta'} | U^\dagger (\mathbb{1}_S \otimes P_A^x) U | \phi_m^\alpha \otimes \varphi_\mu^\beta \rangle$), it follows that the terms inside the parantheses of Equaitons (A9) and (A10) vanish. Consequently, the second lines of Equaitons (A5) and (A6) will be zero, and we are left with

$$\begin{aligned} \langle O_S \rangle_{\text{after}}^{\rho, M_{i,x}} &= \frac{1}{p_\rho^M(x)} \sum_l \sum_{(m,\mu)_l} \text{tr}[(O_S \otimes P_A^x) U(Q_S^m \rho Q_S^m \otimes Q_A^\mu q_i Q_A^\mu) U^\dagger] \\ \langle O_S \rangle_{\text{before}}^{\rho, M_{i,x}} &= \frac{1}{p_\rho^M(x)} \sum_l \sum_{(m,\mu)_l} \text{tr}[(\mathbb{1}_S \otimes P_A^x) U(O_S Q_S^m \rho Q_S^m \otimes Q_A^\mu q_i Q_A^\mu) U^\dagger] \end{aligned} \quad (\text{A11})$$

which results in Equaiton (12). $\square$

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