



# Article Horizon Areas and Logarithmic Correction to the Charged Accelerating Black Hole Entropy

## Parthapratim Pradhan

Department of Physics, Hiralal Mazumdar Memorial College For Women, Dakshineswar, Kolkata 700035, India; pppradhan77@gmail.com

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**Abstract:** It has been shown by explicit and exact calculation that the geometric product formula i.e., area (or entropy) product formula of outer horizon ( $\mathcal{H}^+$ ) and inner horizon ( $\mathcal{H}^-$ ) for charged accelerating black hole (BH) should neither be mass-independent nor quantized. This implies that the area (or entropy) product is mass-independent conjecture has been broken down for charged accelerating BH. This also further implies that the mass-independent feature of the area product of  $\mathcal{H}^\pm$  is not a generic feature at all. We also compute the Cosmic-Censorship-Inequality for this BH. Moreover, we compute the specific heat for this BH to determine the local thermodynamic stability. Under certain criterion, the BH shows the second order phase transition. Furthermore, we compute logarithmic corrections to the entropy for the said BH due to small statistical fluctuations around the thermal equilibrium.

Keywords: black hole physics; gravitation; area product; entropy product; accelerating BH

## 1. Introduction

Perhaps, BHs are the most facinating objects in the universe. They are the direct consequences of Einstein's general relativity. They could be used as a tool for testing strong gravity and beyond. The most general class of BHs are characterized by three parameters namely the mass, charge and spin parameter. They are described by Kerr-Newman family of BH in 3 + 1 dimensions. It is now well established by fact that BH is a thermal object because it has characterized by thermodynamic variables like temperature, entropy etc. [1,2]. New thermodynamic product relations (particularly the horizon area (or entropy) product relations) of event horizon (EH) area and Cauchy horizon (CH) area for several class of BHs have been found universal [3–10].

This relation is said to be universal because of the product of area of multihorizons particularly two physical horizons namely  $\mathcal{H}^{\pm}$  is mass-independent. When a thermodynamic product relation is satisfied this criterion then it is said to be a universal quantity in BH thermodynamics. For example, in case of Kerr-Newman (KN) BH [3] which is an electrovacuum solution of Einstein's equations, it has been shown that the product of inner horizon (IH) area and outer horizon (OH) area should read

$$\mathcal{A}_{-}\mathcal{A}_{+} = 64\pi^{2}J^{2} + 16\pi^{2}Q^{4} , \qquad (1)$$

where  $A_{-}$  and  $A_{+}$  are area of CH and EH's respectively. This relation indicates that the universal product depend only on quantized angular momentum and quantized charges respectively [3–10]. When the charge parameter Q = 0, one obtains the area product for Kerr BH

$$\mathcal{A}_{-}\mathcal{A}_{+} = 64\pi^{2}J^{2}, \qquad (2)$$

and it indicates that the universal product depends only quantized angular momentum parameter. When the angular momentum parameter J = 0, one obtains the area product formula for spherically symmetric charged BH

$$\mathcal{A}_{-}\mathcal{A}_{+} = 16\pi^{2}Q^{4}, \qquad (3)$$

and it implies that the universal product depends only on quantized charge parameter. In the above three cases, it is indeed true that the horizon area (or entropy) product formula is mass-independent, thus it is universal in this sense. This is the only motivation behind this work and this is in fact an interesting topic in recent years in the scientific community particularly in the general relativity (GR) community [3] and in the string theory community [4,5] (see also references [6–10]).

It may be noted that for spherically-symmetric extremal charged BH (M = Q) the above Equation (3) coalesces to the following equation

$$\mathcal{A}_{+}^{2} = \mathcal{A}_{-}^{2} = 16\pi^{2}Q^{4} = 16\pi^{2}M^{4} , \qquad (4)$$

Hence in this case the area's square of outer horizon or the area's square of inner horizon depends on the mass parameter.

Another motivation comes from Visser's work [11] (see also References [11–16]). Using this concept here we have tried to extend our analysis for charged accelerating anti-de Sitter (AdS) BH. By explicit and exact calculation, we show that the area (or entropy) product formula of OH and IH for charged accelerating BH should not be mass-independent and also it should not to be quantized. Thus, we conclude that the theorem of Ansorg-Hennig [3] "The area (or entropy) product formula is independent of mass" is not universal for charged accelerating BH. Moreover, we study other thermodynamic properties, particularly the local thermodynamic stability, by computing the specific heat. Under appropriate condition, the BH possesses second order phase transition.

One aspect is that the leading-order logarithmic corrections to BH entropy due to quantum fluctuations around the thermal equilibrium BH temperature for charged AdS BH have not been studied previously; we compute here the logarithmic corrections to Bekenstein-Hawking entropy for the said BH and it appears to be a generic feature of the BH. It should be noted that we have assumed that the BH is a thermodynamic system which is in equilibrium at Bekenstein-Hawking temperature.

Another strong motivation came from the work of Cvetič et al. [4], where the authors suggested that if the cosmological parameter is quantized then the area (or entropy) product relations for rotating BH in D = 4 and D > 4 should provide a "looking glass for probing the microscopics of general BHs". Thus it is quite interesting to investigate the area (or entropy) product formula after incorporating the cosmological constant.

The interesting property of the charged AdS BH is that it has an accelerating horizon and the OH possesses conical singularity [17]. In spite of the non-asymptotic structure, the first law of BH thermodynamics and Smarr formula are satisfied for this accelerating spacetime [17] (See also [18]). However, the cosmological horizon does not have accelerating horizon. This type of BH is said to be slowly accelerating BH. The other novel property of this accelerating BH is that it is described by the *C* metric [19–22]. Again the *C* metric has some peculiar features in the sense that it accelerates by pulling with a 'cosmic string', which is described by a 'conical deficit in the spacetime'; it connects the OH of the BH to infinity [17].

Actually, the idea of vacuum *C* metric was first given by Levi-Civita in 1918 [23]. Then it was rediscovered by Newman and Tamburino in 1961 [24], also by Robinson and Trautman [25] in the same years, and by Ehlers and Kundt [26] in 1963 but there was no explanation given. First Robinson and

Trautman discovered the interesting feature of this metric, that is, it emits gravitational radiation. Later, Podolsky et al. [27] studied the gravitational and electromagnetic radiation emitted by the uniformly accelerated charged BH in AdS spacetime.

In the next section, we have given the basic characteristics of the charged accelerating BH and we have also derived the area functional relation in terms of BH mass, charge, acceleration, cosmological constant. We have also derived the cosmic censorship inequality for this BH and finally we have computed the specific heat which determines the local thermodynamic stability. In the third section, we have discussed the logarithmic correction to BH entropy for this class of BHs. Finally, in the last section, we have given the conclusion.

#### 2. Thermodynamic Properties of Charged Accelerating BH

The metric of the charged accelerating BH [17,19,20,22] is described by

$$ds^{2} = \frac{1}{\Omega^{2}} \left[ -\mathcal{F}(r)dt^{2} + \frac{dr^{2}}{\mathcal{F}(r)} + r^{2} \left( \frac{d\theta^{2}}{\mathcal{G}(\theta)} + \mathcal{G}(\theta) \sin^{2}\theta \frac{d\phi^{2}}{K^{2}} \right) \right].$$
(5)

with the gauge potential F = dB and  $B = -\frac{q}{r}dt$ , and where

$$\mathcal{F}(r) = (1 - \chi^2 r^2) \left( 1 - \frac{2m}{r} + \frac{q^2}{r^2} \right) + \frac{r^2}{\ell^2}, \tag{6}$$

$$\mathcal{G}(\theta) = 1 + 2m\chi\cos\theta + q^2\chi^2\cos^2\theta. \tag{7}$$

and the conformal factor is  $\Omega = 1 + \chi r \cos \theta$ . It determines the conformal infinity of the AdS BH. The quantities *m* and *q* are BH mass and BH charge respectively,  $\chi > 0$  determines the acceleration of the BH and  $-\frac{\Lambda}{3} = \frac{1}{\ell^2}$  where  $\ell$  is the radius of the AdS BH. It should be noted that when  $\chi < \frac{1}{\ell}$ , a single BH is present with single horizon [28] whereas when  $\chi > \frac{1}{\ell}$ , two BHs of opposite charge are separated by accelerating horizon [21,29] and when  $\chi = \frac{1}{\ell}$  is a special case and it has been explicitly described in Reference [30]. To obtain the angular coordinates as usual form on  $S^2$  we have set the restriction  $m\chi < \frac{1}{2}$ .

Now we discuss the angular part of the metric and the properties of  $\mathcal{G}(\theta)$  at the north pole  $(\theta = 0)$ and south pole  $(\theta = \pi)$ . In general,  $K = \mathcal{G}(\theta)$  but at north pole fixed with  $K = K_N = 1 + 2m\chi + q^2\chi^2$ and at south pole  $K_S = 1 - 2m\chi + q^2\chi^2$  that indicates the choices at the north pole and at the south pole are mutually incompatible. Thus the metric cannot be made regular at both the poles at the same time. Therefore, the convention is to make a choice that makes it regular at one of the poles—this leads to a 'conical deficit' at the other pole which is  $\delta = \frac{8\pi m\chi}{1+2m\chi+q^2\chi^2}$  and which corresponds to a 'cosmic string' [17] with tension  $\mu = \frac{\delta}{8\pi} = \frac{m\chi}{1+2m\chi+q^2\chi^2}$ .

Thus the *C* metric is described by the five physical parameters: the mass *m*, the charge *q*, the negative cosmological constant  $-\Lambda = \frac{3}{\ell^2}$ , the acceleration  $\chi$  and the 'tension of the cosmic string' on each axis which is represented by the periodicity of the angular coordinate. More discussion regarding the thermodynamic properties (particularly first law of BH thermodynamics [18], Smarr mass formula, thermodynamic volume, Gibbs free energy and Reverse isoperimetric inequality) of the *C* metric could be found in Reference [17].

Now we evaluate the radii of BH horizons by imposing the condition  $\mathcal{F}(r_i) = 0$ , i.e.,

$$\left(1-\chi^{2}\ell^{2}\right)r_{i}^{4}+2m\ell^{2}\chi^{2}r_{i}^{3}+\ell^{2}\left(1-\chi^{2}q^{2}\right)r_{i}^{2}-2m\ell^{2}r_{i}+q^{2}\ell^{2} = 0.$$
(8)

Apply Vieta's theorem, we find

$$\sum_{i=1}^{4} r_i = -\frac{2m\ell^2\chi^2}{(1-\chi^2\ell^2)} \,. \tag{9}$$

$$\sum_{1 \le i < j \le 4} r_i r_j = \ell^2 \frac{\left(1 - \chi^2 q^2\right)}{\left(1 - \chi^2 \ell^2\right)} \,. \tag{10}$$

$$\sum_{1 \le i < j < k \le 4} r_i r_j r_k = \frac{2m\ell^2}{(1 - \chi^2 \ell^2)}.$$
(11)

$$\sum_{1 \le i < j < k < l \le 4} r_i r_j r_k r_l = \frac{q^2 \ell^2}{(1 - \chi^2 \ell^2)} .$$
(12)

Eliminating the mass parameter, one obtains a single mass-independent relation as

$$r_1r_2 + rac{q^2\ell^2}{\left(1 - \chi^2\ell^2\right)r_1r_2} + \left(r_1 + r_2\right)^2 \times$$

$$\left[r_1 r_2 \frac{\chi^2}{(1+\chi^2 r_1 r_2)} - \frac{q^2 \ell^2 \chi^2}{r_1 r_2 (1-\chi^2 \ell^2) (1+\chi^2 r_1 r_2)} - 1\right] = \ell^2 \frac{(1-\chi^2 q^2)}{(1-\chi^2 \ell^2)}.$$
(13)

In terms of two BH physical horizons area  $A_i = \frac{4\pi r_i^2}{K(1-\chi^2 r_i^2)}$  (where i = 1 for EH and i = 2 for CH), the mass independent functional relationship is given by

$$\left( \frac{KA_{1}}{4\pi + K\chi^{2}A_{1}} \sqrt{\frac{KA_{2}}{4\pi + K\chi^{2}A_{2}}} + \frac{q^{2}\ell^{2}}{(1-\chi^{2}\ell^{2})} \sqrt{\frac{4\pi + K\chi^{2}A_{1}}{KA_{1}}} \sqrt{\frac{4\pi + K\chi^{2}A_{2}}{KA_{2}}} + \left( \sqrt{\frac{KA_{1}}{4\pi + K\chi^{2}A_{1}}} + \sqrt{\frac{KA_{2}}{4\pi + K\chi^{2}A_{2}}} \right)^{2} \times \left[ \sqrt{\frac{KA_{1}}{4\pi + K\chi^{2}A_{1}}} \sqrt{\frac{KA_{2}}{4\pi + K\chi^{2}A_{2}}} \frac{\chi^{2}}{\left(1 + \chi^{2}\sqrt{\frac{4\pi + K\chi^{2}A_{1}}{KA_{1}}} \sqrt{\frac{4\pi + K\chi^{2}A_{2}}{KA_{2}}} \right)} - 1 \right] - \left( \sqrt{\frac{KA_{1}}{4\pi + K\chi^{2}A_{1}}} + \sqrt{\frac{KA_{2}}{4\pi + K\chi^{2}A_{2}}} \right)^{2} \times \sqrt{\frac{4\pi + K\chi^{2}A_{1}}{KA_{1}}} \sqrt{\frac{4\pi + K\chi^{2}A_{2}}{KA_{2}}} \right)^{2} \times \sqrt{\frac{4\pi + K\chi^{2}A_{1}}{KA_{1}}} \sqrt{\frac{4\pi + K\chi^{2}A_{2}}{KA_{2}}} \times \left[ \frac{q^{2}\ell^{2}\chi^{2}}{\left(1 - \chi^{2}\ell^{2}\right)\left(1 + \chi^{2}\sqrt{\frac{4\pi + K\chi^{2}A_{1}}{KA_{1}}} \sqrt{\frac{4\pi + K\chi^{2}A_{2}}{KA_{2}}} \right)} \right] = \ell^{2} \frac{\ell^{2}(1 - \chi^{2}q^{2})}{\left(1 - \chi^{2}\ell^{2}\right)} \tag{14}$$

where,

$$K = 1 + 2m\chi + q^2\chi^2.$$
(15)

Equation (13) is indeed mass independent but the difficulties arise when we write the expression in terms of area of the BH physical horizons (Equation (14)) because there is a factor K where a mass term m is present. Therefore, it is indeed true that the mass-independent relation has been violated for charged accelerating BH. Thus the "Ansorg-Hennig [3] area theorem" conjecture breaks down for charged accelerating BH. This is another example we have provided in the literature in which the area product of OH and IH is not always universal.

The BH entropy [17] is given by

$$S_{i} = \frac{\mathcal{A}_{i}}{4} = \frac{\pi r_{i}^{2}}{K \left(1 - \chi^{2} r_{i}^{2}\right)}.$$
(16)

and the electric potential on the horizon should read

$$\Phi_i = \frac{q}{r_i}.$$
(17)

Now the BH temperature is given by

$$T_i = \frac{\mathcal{F}'(r_i)}{4\pi} = \frac{1}{4\pi} \left[ \frac{2m}{r_i^2} - \frac{2q^2}{r_i^3} + 2m\chi^2 - 2\chi^2 r_i + \frac{2r_i}{\ell^2} \right].$$
(18)

The other useful thermodynamic relations are

$$T_i S_i = \frac{M\left(1 + \chi^2 r_i^2\right)}{2\left(1 - \chi^2 r_i^2\right)} - \frac{\Phi Q}{2\left(1 - \chi^2 r_i^2\right)} - \frac{\chi^2 r_i^3}{2K\left(1 - \chi^2 r_i^2\right)} + P \frac{4\pi}{3K} \frac{r_i^3}{\left(1 - \chi^2 r_i^2\right)}.$$
 (19)

where we have set the parameter m = MK, q = QK and  $P = -\frac{\Lambda}{8\pi} = \frac{3}{8\pi\ell^2}$ .

The Smarr-Gibbs-Duhem relation becomes

$$M = 2\left(\frac{1-\chi^2 r_i^2}{1+\chi^2 r_i^2}\right) (T_i S_i - PV) + \frac{\Phi Q}{(1+\chi^2 r_i^2)} + \frac{\chi^2 r_i^3}{K\left(1+\chi^2 r_i^2\right)}.$$
 (20)

The thermodynamic volume is derived to be

$$V = \left(\frac{\partial M}{\partial P}\right)_{S,Q} = \frac{4\pi}{3K} \frac{r_i^3}{\left(1 - \chi^2 r_i^2\right)} \,. \tag{21}$$

beacuse the mass parameter becomes

$$M = \frac{1}{2K} \left[ r_i + \frac{K^2 Q^2}{r_i} + \frac{8\pi P}{3} \frac{r_i^3}{\left(1 - \chi^2 r_i^2\right)} \right].$$
(22)

When the acceleration  $\chi$  vanishes, one obtains indeed the result of charged AdS BH. When we have taken into the concept of extended phase space then the first law has taken to be the form as

$$dM = T_i dS_i + V dP + \Phi_i dQ.$$
<sup>(23)</sup>

where the thermodynamic volume is defined in Equation (21).

The reverse isoperimetric inequality is indeed violated for this BH as

$$\mathcal{R} = \left(1 - \chi^2 r_+^2\right)^{\frac{1}{6}} \le 1.$$
 (24)

Finally the Gibb's free energy is defined to be

$$G_i = M - T_i S_i . (25)$$

where  $T_i S_i$  and *M* are defined in Equations (19) and (22).

We know the famous Cosmic-Censorship-Inequality [which requires cosmic-censorship hypothesis [31] (See References [32–36])] for Schwarzschild BH is given by

$$m \geq \sqrt{\frac{A}{16\pi}}$$
 (26)

It is a very challenging topic in mathematical relativity since 1973. The above inequality<sup>1</sup> for accelerating BH becomes

$$m \geq \frac{1}{2}\sqrt{\frac{K\mathcal{A}_i}{4\pi + K\chi^2\mathcal{A}_i}} \left[1 + q^2\left(\frac{4\pi + K\chi^2\mathcal{A}_i}{K\mathcal{A}_i}\right) + \frac{K\mathcal{A}_i}{4\pi\ell^2}\right].$$
(27)

This idea was first given in 1973 by Penrose [31], which is an important topic in GR which relates ADM mass (i.e., total mass of the spacetime) and the area of the even horizon. It is called Cosmic-Censorship-Inequality or Cosmic-Censorship-Bound [37].

This inequality implies that it provides the lower bound on the mass (or energy) for any time-symmetric initial data which satisfied the famous Einstein equations with negative cosmological constant, and which also satisfied the dominant energy condition which possesses no naked singularity.

Local thermodynamic stability and phase transitions [38,39], particularly second order phase transition, are important phenomena in BH thermodynamics and it can be determined by computing the specific heat which is calculated to be for accelerating BH

$$C_{i} = 2\pi r_{i}^{2} \frac{\left[1 - \frac{q^{2}}{r_{i}^{2}} + \frac{r_{i}^{2}}{\ell^{2}} \frac{(3 - \chi^{2} r_{i}^{2})}{(1 - \chi^{2} r_{i}^{2})^{2}}\right]}{\left[\frac{4\chi^{2} r_{i}^{4}}{\ell^{2} (1 - \chi^{2} r_{i}^{2})^{2}} + \frac{r_{i}^{2}}{\ell^{2}} \left(\frac{3 - \chi^{2} r_{i}^{2}}{1 - \chi^{2} r_{i}^{2}}\right) + \frac{3q^{2}}{r_{i}^{2}} - (1 + q^{2}\chi^{2} + \chi^{2} r_{i}^{2})\right]}.$$
(28)

Now we analyze the above expression of specific heat for different parameter space. *Case I* When

$$1 + \frac{r_i^2}{\ell^2} \frac{(3 - \chi^2 r_i^2)}{(1 - \chi^2 r_i^2)^2} > \frac{q^2}{r_i^2} \text{ and}$$

$$\frac{4\chi^2 r_i^4}{\ell^2 (1 - \chi^2 r_i^2)^2} + \frac{r_i^2}{\ell^2} \left(\frac{3 - \chi^2 r_i^2}{1 - \chi^2 r_i^2}\right) + \frac{3q^2}{r_i^2} > \left(1 + q^2 \chi^2 + \chi^2 r_i^2\right)$$
(29)

<sup>&</sup>lt;sup>1</sup> The Penrose inequality is indeed violated for charged accelerating BH because in the right side of the Equation (14) the factor *K* depends on the mass parameter.

the specific heat is positive i.e.,  $C_i > 0$ , which indicates that the BH is thermodynamically stable. This inequality is plotted in Figure 1a. Along the abcissa we have taken the value of horizon radius  $r_+$  and along the ordinate we have taken the value of q.

Case II When

$$1 + \frac{r_i^2}{\ell^2} \frac{(3 - \chi^2 r_i^2)}{(1 - \chi^2 r_i^2)^2} > \frac{q^2}{r_i^2} \text{ and}$$

$$\frac{4\chi^2 r_i^4}{\ell^2 (1 - \chi^2 r_i^2)^2} + \frac{r_i^2}{\ell^2} \left(\frac{3 - \chi^2 r_i^2}{1 - \chi^2 r_i^2}\right) + \frac{3q^2}{r_i^2} < \left(1 + q^2 \chi^2 + \chi^2 r_i^2\right) \text{ or }$$

$$1 + \frac{r_i^2}{\ell^2} \frac{(3 - \chi^2 r_i^2)}{(1 - \chi^2 r_i^2)^2} < \frac{q^2}{r_i^2} \text{ and}$$

$$\frac{4\chi^2 r_i^4}{\ell^2 (1 - \chi^2 r_i^2)^2} + \frac{r_i^2}{\ell^2} \left(\frac{3 - \chi^2 r_i^2}{1 - \chi^2 r_i^2}\right) + \frac{3q^2}{r_i^2} > \left(1 + q^2 \chi^2 + \chi^2 r_i^2\right)$$
(30)

the specific heat is negative i.e.,  $C_i < 0$ , which implies that the BH is thermodynamically unstable. These two inequalities are shown in Figure 1b,c.

Case III When

$$\frac{\left[\frac{4\chi^2 r_i^4}{\ell^2 (1-\chi^2 r_i^2)^2} + \frac{r_i^2}{\ell^2} \left(\frac{3-\chi^2 r_i^2}{1-\chi^2 r_i^2}\right) + \frac{3q^2}{r_i^2}\right]}{(1+q^2\chi^2 + \chi^2 r_i^2)} = 1.$$
(31)

the specific heat  $C_i$  diverges. It signals a second order phase transition for such BHs. It could be observed from the Figure 2.



Figure 1. Cont.



**Figure 1.** The inequality for specific heat of the Case-I is plotted in (**a**), Case-II is plotted in (**b**,**c**). Along the abcissa (*X*) we have chosen the value of horizon radius  $r_+$  and along the ordinate (*Y*) we have chosen the value of *q*. We have set  $\ell = \chi = 1$ .

In Figure 2, we have plotted specific heat with the horizon radius. From the figure one can observed that the phase transition occurs at the positive value of the EH radius.



**Figure 2.** In this figure, we have plotted the variation of specific heat  $C_i$  with horizon radius  $(r_+)$  for the values  $\chi = 1$  (**a**) and  $\ell = 1$  (**b**).

#### 3. Logarithmic Corrections to Entropy for Charged Accelerating BH

In this section, we will derive the logarithmic corrections to BH entropy for charged accelerating BH by assuming that the BH is considered as a thermodynamic system which is in equilibrium at Bekenstein-Hawking temperature, and due to the effects of statistical thermal fluctuations around the equilibrium.

To derive this correction, we will follow the classical work of Das et al. [40]. Where the authors first studied the general logarithmic corrections to BH entropy for higher dimensional AdS spacetime and BTZ BH. Since the BHs have been considered as a macroscopic object compared to the Planck scale length and this indicates that the logarithmic terms are much smaller compared to the Bekenstein-Hawking terms and they should be treated as corrections.

Now we can define the canonical partition function [41] as

$$Z_i(\beta_i) = \int_0^\infty \rho_i(E) e^{-\beta_i E} dE .$$
(32)

where  $T_i = \frac{1}{\beta_i}$  is the temperature of  $\mathcal{H}^i$ . We have chosen the value of Boltzman constant  $k_B = 1$ .

The density of states can be defined as an inverse Laplace transformation of the partition function

$$\rho_i(E) = \frac{1}{2\pi i} \int_{a-i\infty}^{a+i\infty} Z_i(\beta_i) e^{\beta_i E} d\beta_i$$
(33)

$$= \frac{1}{2\pi i} \int_{a-i\infty}^{a+i\infty} e^{S_i(\beta_i)} d\beta_i .$$
(34)

where *a* is a real positive constant and

$$S_i = \ln Z_i + \beta_i E . \tag{35}$$

is entropy of the system near its equilibrium.

Near equilibrium of the inverse Hawking temperature  $\beta_i = \beta_{0,i}$ , we can expand the entropy function as

$$S_i(\beta_i) = S_{0,i} + \frac{1}{2}(\beta_i - \beta_{0,i})^2 S_{0,i}'' + \dots$$
(36)

where,  $S_{0,i} = S_i(\beta_{0,i})$  and  $S_{0,i}'' = \frac{\partial^2 S_i}{\partial \beta_i^2}$  at  $\beta_i = \beta_{0,i}$ 

Putting the Equation (36) in Equation (32), one obtains

$$\rho_i(E) = \frac{e^{S_{0,i}}}{2\pi i} \int_{a-i\infty}^{a+i\infty} e^{\frac{(\beta_i - \beta_{0,i})^2 S_{0,i}''}{2}} d\beta_i$$
(37)

Let us choose  $\beta_i - \beta_{0,i} = iy_i$  and setting  $a = \beta_{0,i}$ ,  $y_i$  is a real variable and evaluating a contour integral we have

$$\rho_i(E) = \frac{e^{S_{0,i}}}{\sqrt{2\pi S_{0,i}''}}.$$
(38)

The logarithm of  $\rho_i(E)$  gives the corrected entropy of the thermodynamic system

$$S_i = \ln \rho_i = S_{0,i} - \frac{1}{2} \ln S_{0,i}'' + \dots$$
 (39)

Next our task is to compute  $S_{0,i}''$ , for this we must choose any specific form of function  $S_i(\beta_i)$  which is modular invariant partition function and which is also admitted an extremum at some specific value  $\beta_{i,0}$ of  $\beta_i$  followed by underlying conformal field theory (CFT) [40,42]. Therefore, the exact entropy function followed by CFT is of the form

$$\mathcal{S}_i(\beta_i) = c\beta_i + \frac{d}{\beta_i} \tag{40}$$

where c, d are constants. It can be rewritten as a more general form which admits saddle point as

$$S_i(\beta_i) = c\beta_i^m + \frac{d}{\beta_i^n} \tag{41}$$

where m, n, c, d > 0. The special case we have considered here when m = n = 1 is due to the CFT. After some algebraic computation (See for more details [40,43]) one can find the value of  $S''_{0,i} = T_i^2 S_{0,i}$ , then we get the leading order corrections to BH entropy as

$$S_i = \ln \rho_i = S_{0,i} - \frac{1}{2} \ln \left| T_i^2 S_{0,i} \right| + \dots$$
(42)

Putting the values of  $S_{0,i} = \frac{\pi r_i^2}{K(1-\chi^2 r_i^2)}$  and  $T_i$ , one can obtain the logarithmic correction to BH entropy for charged accelerating BH

$$S_{i} = \frac{\pi r_{i}^{2}}{K\left(1 - \chi^{2} r_{i}^{2}\right)} - \ln \left| \frac{2m}{r_{i}} - \frac{2q^{2}}{r_{i}^{2}} + 2m\chi^{2}r_{i} - 2\chi^{2}r_{i}^{2} + \frac{2r_{i}^{2}}{\ell^{2}} \right| + \frac{1}{2}\ln \left| K\left(1 - \chi^{2}r_{i}^{2}\right) \right| + \dots$$

$$(43)$$

This corrected entropy formula indicates that it is a function of horizon radius, mass parameter, charge parameter, acceleration parameter and cosmological constant.

It should be noted that the product

$$\prod_{i=1}^{2} S_i \tag{44}$$

is explicitly depends on the value of mass parameter, etc. Thus the logarithmic corrected entropy formula is not mass-independent and it is not quantized.

We have plotted the logarithmic corrected entropy and without the logarithmic corrected entropy in Figure 3. It follows from the graph (Figure 3a (left panel)) when there is no acceleration i.e.,  $\chi = 0$  and there is no logarithmic correction, the entropy is increasing when horizon radius increases. When there is an acceleration, the value of entropy diverges at a certain horizon radius; this indicates the phase transition occurs at  $r_+ = 1$ . On the other hand, when we have taken the logarithmic correction to entropy in the presence of acceleration then the divergence disappears (Figure 3b (right panel)) and all the phenomena occurs in the regime  $0.5 < r_+ < 1$ . Three situations are also qualitatively different. This is an another interesting feature of charged accelerating BH.



**Figure 3.** In this graph, we have drawn the variation of logarithmic corrected entropy ( $S_+$ ) (**a**) and without logarithm corrected entropy ( $S_{0,+}$ ) (**b**) with horizon radius ( $r_+$ ).

### 4. Conclusions

In this work, we studied the thermodynamic properties of slowly accelerating BH which consists of five parameters, namely the mass, the charge, the acceleration, the cosmological constant and the cosmic string tension. We derived the geometric product formula i.e., area product formula of OH and IH for charged accelerating BH. We showed that this area product formula should not be mass-independent or quantized. This suggests that the mass-independent conjecture of Ansorg-Hennig breaks down for charged accelerating BH. This is an another example we have added in the literature in which the area product of two physical horizons is not always mass-independent. We also derived the famous Cosmic-Censorship-Inequality for this slowly accelerating BH. The physical significance of this inequality is to determine the lower bound of mass or energy for a time-symmetric initial data which fulfilled the dominant energy condition.

Moreover, we evaluated the criterion under which the BH showed the second order phase transition. Finally, we computed the logarithmic correction to entropy due to statistical quantum thermal fluctuations near the BH equilibrium temperature for this charged accelerating BH. The implication of the logarithmic corrections to BH entropy might be useful to understanding the Suskind's holographic hypothesis [44] and the AdS/CFT correspondence (which is a prime example of holography). This principle is based on string theory and quantum gravity. It would be an interesting topic of research if one could compute the quasilocal energy for this BH following the work [45]. It may be helpful to testify the Seifert conjecture for BH naked singularity [45].

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