

Article

Regular Black Holes with Asymptotically Minkowski Cores

Alex Simpson  and Matt Visser 

School of Mathematics and Statistics, Victoria University of Wellington, P.O. Box 600, Wellington 6140, New Zealand; alex.simpson@sms.vuw.ac.nz

* Correspondence: matt.visser@sms.vuw.ac.nz

Received: 21 November 2019; Accepted: 25 December 2019; Published: 31 December 2019



Abstract: Standard models of “regular black holes” typically have asymptotically de Sitter regions at their cores. Herein, we shall consider novel “hollow” regular black holes, those with asymptotically Minkowski cores. The reason for doing so is twofold: First, these models greatly simplify the physics in the deep core, and second, one can trade off rather messy cubic and quartic polynomial equations for somewhat more elegant special functions such as exponentials and the increasingly important Lambert W function. While these “hollow” regular black holes share many features with the Bardeen/Hayward/Frolov regular black holes, there are also significant differences.

Keywords: regular black holes; de Sitter core; Minkowski core; Lambert W function; exponential mass suppression

PACS: 03.65.-w; 03.65.Aa; 03.65.Ta

1. Introduction

It is well established that all static spherically symmetric spacetimes have a line element which can, without loss of generality, be represented in the following form [1,2]:

$$ds^2 = -e^{-2\Phi(r)} \left(1 - \frac{2m(r)}{r}\right) dt^2 + \frac{dr^2}{1 - \frac{2m(r)}{r}} + r^2 d\Omega_2^2. \quad (1)$$

Here, $\Phi(r)$ and $m(r)$ are a priori arbitrary functions of r . Historically, most of the prominent models for regular black holes are of this form, and exhibit an asymptotically de Sitter core with finite central energy density and an equal-but-opposite central pressure. (See, for instance, the common Bardeen/Hayward/Frolov, etc. [3–9] regular black holes, but also the well-known Mazur–Mottola gravastars [10–18], and Dymnikova’s models [19–25].)

Presented herein is a new and rather different form of the metric tensor, which is still a regular black hole in the sense of exhibiting everywhere finite curvature (as originally pioneered by Bardeen [3], see discussion below) but now with an asymptotically Minkowski core (the energy density and associated pressures asymptote to zero). We may make this physical difference mathematically explicit by examining the required conditions on the two functions $\Phi(r)$ and $m(r)$ appearing in the metric (1):

- *de Sitter core:* We would require that $\lim_{r \rightarrow 0} \rho(r) = \rho_0 \neq 0$; that is, $\rho(r) = O(1)$. For static spherically symmetric spacetimes, we have $\rho(r) = m'(r)/4\pi r^2$ [1,26–28], so this implies $m(r) = O(r^3)$. For $\Phi(r)$, it is sufficient to demand $\Phi(r) = O(1)$. All of the standard Bardeen/Hayward/Frolov/gravastar/Dymnikova models fall into this class.

- *Minkowski core*: We would require that $\lim_{r \rightarrow 0} \rho(r) = 0$; that is, $\rho(r) = o(r)$. This in turn implies $m(r) = o(r^3)$. For $\Phi(r)$, it is once again sufficient to demand $\Phi(r) = O(1)$. It is these class of objects that we focus on in this article.

For simplicity/tractability, we therefore might as well immediately set $\Phi(r) = 0$, which will preserve all of the key features of our desired physics. ($\Phi(r) = 0$ spacetimes have a long and respected history. Regarding this specific point, see specifically references [1] and [29–31], and more generally references [19–25].)

The choice for $m(r)$ is more subtle. Presented herein is a specific choice of mass function $m(r) = m e^{-a/r}$ which has the effect of “exponentially suppressing” the Misner–Sharp quasi-local mass of the centralised object as one nears the coordinate location $r = 0$, guaranteeing that the object has an asymptotically Minkowski core. The exponential function $e^{-a/r}$ guarantees that everything is everywhere infinitely smooth (C^∞), though not always analytic (C^ω).¹ That is to say, we analyse the specific line element:

$$ds^2 = - \left(1 - \frac{2m e^{-a/r}}{r} \right) dt^2 + \frac{dr^2}{\left(1 - \frac{2m e^{-a/r}}{r} \right)} + r^2 d\Omega_2^2. \quad (2)$$

Conducting a standard general relativistic analysis of the resulting spacetime, we shall demonstrate that this metric does indeed correspond to a regular black hole in the sense pioneered by Bardeen [3]. We find that in this toy model the resulting curvature tensors and curvature invariants are significantly simpler than for Bardeen/Hayward/Frolov, at the cost of having many physically interesting features (such as horizon radius, horizon area, and surface gravity) specified in terms of the Lambert W function, defined by $W(x)e^{W(x)} = x$, this being one of the special functions of mathematics [32–44], which is finding increasingly common applications in physics.

A rather different (extremal) version of this model spacetime, based on nonlinear electrodynamics, has been previously discussed by Culetu [45], with follow-up on some aspects of the non-extremal case in references [46–48]. See also references [49–51]. Part of the GRF essay [52] is based on a mass function of the form $m(r) = m e^{-a^2/r^2}$. Another very different type of exponential modification of Schwarzschild and Kerr geometries, obtained by inserting factors of the form $1 - a \exp(-b^3/r^3)$ into the spacetime metric, has been discussed by Takeuchi [53]. Further afield, accretion onto regular black holes has been studied in reference [54], and modified Smarr formulae have been discussed in reference [55].

The overall framework these authors are working with differs significantly from our own.

2. Metric Analysis

First, let us note that the metric is static and spherically symmetric. The areas of surfaces of spherical symmetry of constant r -coordinate are trivial, given by the area function $A(r) = 4\pi r^2$, which we can clearly see is minimised at $r = 0$.

Note that this representation of the metric corresponds to the central mass being r -dependent in the following manner: $m(r) = m e^{-a/r}$. We immediately have $a \in \mathbb{R}^+$ in order to ensure the mass is being exponentially “suppressed” as $|r| \rightarrow 0$; in contrast, if $a = 0$, we simply have the Schwarzschild solution, and, if $a < 0$, we have an altogether different scenario where asymptotic behaviour for small r indicates massive exponential “growth”. For $a > 0$, the exponential expression factor has the properties:

¹ Alternatively, one could try, for instance, to represent $m(r)$ by some rational polynomial in the r coordinate, $m(r) = m p(r)/q(r)$. This is closer in spirit to the Bardeen/Hayward/Frolov regular black holes — but then one has to worry about potential poles and zeros in the mass function $m(r)$, and locating the horizons, and evaluating the surface gravities, typically involves intractable polynomials. The exponential suppression $e^{-a/r}$ side-steps these messy issues.

$$\lim_{r \rightarrow 0^+} e^{-a/r} = 0, \quad \lim_{r \rightarrow 0^-} e^{-a/r} = +\infty. \quad (3)$$

The metric is C^∞ smooth but not C^ω analytic at coordinate location $r = 0$. Looking at the behaviour as $r \rightarrow 0^-$, we see that the region $r < 0$ can safely be omitted from the analysis; the severe discontinuity at $r = 0$ implies that behaviour in the negative r domain is grossly unphysical. This does not affect our coordinate patch, as $r = 0$ also marks a coordinate singularity. (Demonstration that this is a coordinate singularity and not a curvature singularity is a corollary of analysis in Section 3.) We may trivially avoid these “issues” by enforcing $r \in \mathbb{R}^+$, i.e., strictly remaining within “our” universe, which is the primary region of interest. In view of the diagonal metric environment, we may now examine horizon locations for the spacetime by setting $g_{tt} = 0$:

$$g_{tt} = 0 \quad \implies \quad r = -\frac{a}{W(-\frac{a}{2m})} \quad \implies \quad r = 2m e^{W(-\frac{a}{2m})}. \quad (4)$$

We have a coordinate location for the horizon defined *explicitly* in terms of the Lambert W function, defined by $W(x)e^{W(x)} = x$ [32–44]. Given that we have taken $a > 0$ and $r > 0$, we must enforce the output of the Lambert W function to be negative. This presents two possibilities:

- Taking the $W_0(x)$ branch of the real-valued Lambert W function:

$$W_0\left(-\frac{a}{2m}\right) < 0 \quad \implies \quad a \in \left(0, \frac{2m}{e}\right]. \quad (5)$$

Provided a lies in this interval, we have a well-defined coordinate location for a horizon when taking the $W_0(x)$ branch of the Lambert W function. Keeping in mind that fixing a in this interval causes $W_0(-\frac{a}{2m}) \in [-1, 0)$, the possible coordinate locations of *this* horizon are given by $r_H \in [a, +\infty)$.

- Taking the $W_{-1}(x)$ branch of the real-valued Lambert W function:

The $W_{-1}(x)$ branch only returns outputs for $x \in [-\frac{1}{e}, 0)$, hence we have the same restriction on a as before to ensure a defined coordinate location for the horizon—that $a \in (0, \frac{2m}{e}]$. The range of the $W_{-1}(x)$ branch is entirely negative so all possible solutions will correspond to $r_H > 0$. However, the difference is that fixing a in this interval causes $W_{-1}(-\frac{a}{2m}) \in [-1, -\infty)$, hence the possible coordinate locations for *this* horizon are given by $r_H \in (0, a]$.

In view of the possible horizon locations corresponding to the two possible branches, we see that there is an *outer horizon* at coordinate location $r = 2m e^{W_0(-\frac{a}{2m})}$, and an *inner horizon* at $r = 2m e^{W_{-1}(-\frac{a}{2m})}$. When $a = \frac{2m}{e}$, we see that $W_{-1}(-\frac{1}{e}) = W_0(-\frac{1}{e}) = -1$; the two horizons merge and we in fact have an extremal black hole.

It follows that, in order to have $r_H > 0$, we require $a \in (0, \frac{2m}{e}]$. Then, subject to the choice of branch for the Lambert W function, we find an inner horizon located in the region $r_H \in (0, a)$, and an outer horizon in the region $r_H \in (a, +\infty)$. In both cases, the geometry is certainly modeling a black hole region of some description; it remains to demonstrate that the spacetime is gravitationally nonsingular in order to show this is a regular black hole in the sense pioneered by Bardeen [3].

As an aside, let us take a look at what happens to the geometry when $a > \frac{2m}{e}$:

$$a > \frac{2m}{e} \quad \implies \quad -\frac{a}{2m} < -\frac{1}{e} \quad \implies \quad W\left(-\frac{a}{2m}\right) \text{ is undefined.} \quad (6)$$

Therefore, there are no horizons in this geometry. It follows that, when $a > \frac{2m}{e}$, there is no black hole of any kind; this horizonless geometry is modeling something qualitatively different, closer in spirit to a gravastar or an exotic compact object.

In any of these situations, the geometry admits an almost-global coordinate patch with the following domains: $t \in (-\infty, +\infty)$, $r \in \mathbb{R}^+$, $\theta \in (0, \pi)$, and $\phi \in (-\pi, \pi)$. We now examine the non-zero components of the curvature tensors, as well as the curvature invariants, to show that, for $a \in (0, \frac{2m}{e}]$, this metric does indeed model a regular black hole geometry.

3. Curvature Tensors and Curvature Invariants

Before proceeding, it is prudent to introduce a relevant piece of mathematical detail: for any polynomial function $p(r)$, we have $e^{-a/r}/p(r) \rightarrow 0$ as $|r| \rightarrow 0$. That is, the exponential expression dominates the asymptotic behaviour for small r . Keeping this in mind, let us examine the mixed non-zero curvature tensor components and the curvature invariants.

Ricci scalar:

$$R = \frac{2m a^2 e^{-a/r}}{r^5}. \quad (7)$$

Ricci tensor non-zero components:

$$R^t_t = R^r_r = \frac{m a (a - 2r) e^{-a/r}}{r^5}, \quad R^\theta_\theta = R^\phi_\phi = \frac{2m a e^{-a/r}}{r^4}. \quad (8)$$

Einstein tensor non-zero components:

$$G^t_t = G^r_r = -\frac{2m a e^{-a/r}}{r^4}, \quad G^\theta_\theta = G^\phi_\phi = -\frac{m a (a - 2r) e^{-a/r}}{r^5}. \quad (9)$$

Riemann tensor non-zero components:

$$\begin{aligned} R^{tr}_{tr} &= \frac{m (a^2 - 4ar + 2r^2) e^{-a/r}}{r^5}, \\ R^{t\theta}_{t\theta} &= R^{t\phi}_{t\phi} = R^{r\theta}_{r\theta} = R^{r\phi}_{r\phi} = \frac{m (a - r) e^{-a/r}}{r^4}, \\ R^{\theta\phi}_{\theta\phi} &= \frac{2m e^{-a/r}}{r^3}. \end{aligned} \quad (10)$$

Weyl tensor non-zero components:

$$\begin{aligned} -\frac{1}{2} C^{tr}_{tr} &= -\frac{1}{2} C^{\theta\phi}_{\theta\phi} = C^{t\theta}_{t\theta} = C^{t\phi}_{t\phi} = C^{r\theta}_{r\theta} = C^{r\phi}_{r\phi} \\ &= -\frac{m (a^2 - 6ar + 6r^2) e^{-a/r}}{6r^5}. \end{aligned} \quad (11)$$

The Ricci contraction $R_{ab} R^{ab}$:

$$R_{ab} R^{ab} = \frac{2m^2 a^2 (a^2 - 4ar + 8r^2) e^{-2a/r}}{r^{10}}. \quad (12)$$

The Kretschmann scalar $R_{abcd} R^{abcd}$:

$$R_{abcd} R^{abcd} = \frac{4m^2 (a^4 - 8a^3r + 24a^2r^2 - 24ar^3 + 12r^4) e^{-2a/r}}{r^{10}}. \quad (13)$$

The Weyl contraction $C_{abcd} C^{abcd}$:

$$C_{abcd} C^{abcd} = \frac{4m^2 (a^2 - 6ar + 6r^2)^2 e^{-2a/r}}{3r^{10}}. \quad (14)$$

As $|r| \rightarrow +\infty$, we have $e^{-a/r} \rightarrow 1$, and all non-zero tensor components and invariants become proportional to some Laurent polynomial function of r (i.e., all asymptote to $O(r^{-n})$). For large r , all non-zero components and invariants tend to zero, consistent with the fact that asymptotic infinity models Minkowski space. As $r \rightarrow 0^+$, all non-zero tensor components and invariants also asymptotically head to zero (they become $o(r)$). This of course represents the fact that the core of the black hole region is asymptotically Minkowski. Thus, we have a geometry that approaches Minkowski both near the centre and at asymptotic infinity, with some maximised region of curvature located in between. We may conclude that all non-zero tensor components and invariants are most certainly globally finite, and, as such, the geometry possesses no curvature singularities as predicted—we are indeed dealing with a regular black hole spacetime in the sense pioneered by Bardeen.

4. Surface Gravity, Hawking Temperature, and Horizon Area

Let us calculate the surface gravity at the two horizons when $a \in (0, \frac{2m}{e})$. The Killing vector which is null at the event horizon is $\xi = \partial_t$; that is, $\xi^\mu = (1, 0, 0, 0)^\mu$. This yields the norm:

$$\xi^\mu \xi_\mu = g_{\mu\nu} \xi^\mu \xi^\nu = g_{tt} = - \left(1 - \frac{2m e^{-\frac{a}{r}}}{r} \right). \quad (15)$$

Then, we have the following relation for the surface gravity κ (see for instance [26–28]):

$$\nabla_\nu (-\xi^\mu \xi_\mu) = 2\kappa \xi_\nu. \quad (16)$$

That is:

$$\nabla_\nu \left(1 - \frac{2m e^{-\frac{a}{r}}}{r} \right) = 2\kappa \xi_\nu. \quad (17)$$

Alternatively, we can apply the formalism of reference [2].

Now, perform a case-by-case analysis for each horizon:

- *Outer horizon:* The outer horizon is located at radial coordinate $r = 2m e^{W_0(-\frac{a}{2m})}$; we therefore have:

$$\begin{aligned} \kappa_{outer} &= \frac{1}{2} \partial_r \left(1 - \frac{2m e^{-\frac{a}{r}}}{r} \right) \Big|_{r=2m e^{W_0(-\frac{a}{2m})}} \\ &= \kappa_{Schwarzschild} \left\{ -\frac{2m W_0(-\frac{a}{2m})}{a} \left[1 + W_0\left(-\frac{a}{2m}\right) \right] \right\}. \end{aligned} \quad (18)$$

The Hawking temperature for our regular black hole is [26–28]:

$$T_{outer} = \frac{\hbar \kappa_{outer}}{2\pi k_B} = T_{Schwarzschild} \left\{ -\frac{2m W_0(-\frac{a}{2m})}{a} \left[1 + W_0\left(-\frac{a}{2m}\right) \right] \right\}. \quad (19)$$

For the area of the outer horizon, we have:

$$A_{outer} = 4\pi \left(2m e^{W_0(-\frac{a}{2m})} \right)^2 = \frac{4\pi a^2}{[W_0(-\frac{a}{2m})]^2}. \quad (20)$$

As a sanity check, let us check the behaviour of κ_{outer} when $a = 0$ (corresponding to the Schwarzschild solution).

We have:

$$\begin{aligned}
 \lim_{a \rightarrow 0} \kappa_{outer} &= \kappa_{Schwarzschild} \lim_{a \rightarrow 0} \left\{ -\frac{2m W_0 \left(-\frac{a}{2m} \right)}{a} \left[1 + W_0 \left(-\frac{a}{2m} \right) \right] \right\}, \\
 &= \kappa_{Schwarzschild} \lim_{a \rightarrow 0} \left\{ -\frac{2m W_0 \left(-\frac{a}{2m} \right)}{a} \right\}, \\
 &= \kappa_{Schwarzschild} \lim_{a \rightarrow 0} e^{-W_0 \left(-\frac{a}{2m} \right)} \\
 &= \kappa_{Schwarzschild}.
 \end{aligned} \tag{21}$$

This is the expected result.

- *Inner horizon:* The inner horizon is located at $r = 2m e^{W_{-1} \left(-\frac{a}{2m} \right)}$; and through similar analysis we have the following:

$$\kappa_{inner} = \kappa_{Schwarzschild} \left\{ -\frac{2m W_{-1} \left(-\frac{a}{2m} \right)}{a} \left[1 + W_{-1} \left(-\frac{a}{2m} \right) \right] \right\}, \tag{22}$$

$$T_{inner} = T_{Schwarzschild} \left\{ -\frac{2m W_{-1} \left(-\frac{a}{2m} \right)}{a} \left[1 + W_{-1} \left(-\frac{a}{2m} \right) \right] \right\}, \tag{23}$$

$$A_{inner} = \frac{4\pi a^2}{[W_{-1} \left(-\frac{a}{2m} \right)]^2}. \tag{24}$$

- *Extremal horizon:* Extremality arises when the two horizons merge; this occurs when $a = 2m/e$, at coordinate location $r = 2m/e = a$, where $W_0 \left(-\frac{1}{e} \right) = -1 = W_{-1} \left(-\frac{1}{e} \right)$. Consequently, $\kappa_{extremal} = 0$, while $T_{extremal} = 0$, and $A_{extremal} = 4\pi a^2$.

5. Stress-Energy Tensor and Energy Conditions

Consider the Einstein field equations for this spacetime, defining the stress-energy-momentum tensor by $G^\mu{}_\nu = 8\pi T^\mu{}_\nu$. From Equation (9), we have:

$$T^\mu{}_\nu = \begin{bmatrix} -\rho & 0 & 0 & 0 \\ 0 & p_{\parallel} & 0 & 0 \\ 0 & 0 & p_{\perp} & 0 \\ 0 & 0 & 0 & p_{\perp} \end{bmatrix}. \tag{25}$$

Specifically, for the principal pressures:

$$\begin{aligned}
 \rho &= -p_{\parallel} = \frac{2ma e^{-a/r}}{8\pi r^4}, \\
 p_{\perp} &= -\frac{ma(a-2r) e^{-a/r}}{8\pi r^5}.
 \end{aligned} \tag{26}$$

Let us examine where the energy density is maximised for this spacetime (this is of specific interest due to the exponential suppression of the mass—we wish to see how the suppression affects the distribution of energy densities in through the geometry):

$$\frac{\partial \rho}{\partial r} = \frac{2ma e^{-a/r}}{8\pi r^6} (a - 4r). \tag{27}$$

We see that ρ is maximised at coordinate location $r = \frac{a}{4}$. Let us now analyse the various energy conditions [56–67] and see whether they are violated in our spacetime.

5.1. Null Energy Condition

In order to satisfy the null energy condition, we require that both $\rho + p_{\parallel} \geq 0$ and $\rho + p_{\perp} \geq 0$ globally in our spacetime. Let us first consider $\rho + p_{\parallel}$ and note that it is identically zero. (This result is common to all $\Phi(r) = 0$ spacetimes, see [1,29–31].) Now, noting that for all $\Phi(r) = 0$ spacetimes $\rho + p_{\perp} = \frac{r}{2} \rho'$ (see reference [31]), we have:

$$\rho + p_{\perp} = \frac{r}{2} \rho' = \frac{m a e^{-a/r}}{8\pi r^5} (a - 4r). \quad (28)$$

This changes sign when $r = \frac{a}{4}$. We therefore have the somewhat non-typical situation where the radial NEC is satisfied by the geometry whilst the transverse NEC is violated in the deep core. Violation of the NEC implies that all of the classical energy conditions must be violated in the deep core [56].

5.2. Strong Energy Condition

In order to satisfy the strong energy condition (SEC), one of the conditions we require is that $\rho + p_{\parallel} + 2p_{\perp} \geq 0$ globally in our spacetime. Evaluating:

$$\rho + p_{\parallel} + 2p_{\perp} = 2p_{\perp} = -\frac{m a (a - 2r) e^{-a/r}}{4\pi r^5}. \quad (29)$$

This is negative for the region $r < \frac{a}{2}$, and we may conclude that the SEC is violated in the deep core—this is consistent with expectations [56].

6. Comparison with Existing Regular Black Hole Models

The primary mathematical benefit of this new model is the relative tractability of the curvature tensors and invariants analysis. Let us briefly compare some of the corresponding results for the Bardeen, Hayward, and Frolov regular black holes. The Bardeen regular black hole is defined by the line element [3]:

$$ds^2 = - \left(1 - \frac{2mr^2}{\left[r^2 + (2m\ell^2)^{\frac{2}{3}} \right]^{\frac{3}{2}}} \right) dt^2 + \frac{dr^2}{\left(1 - \frac{2mr^2}{\left[r^2 + (2m\ell^2)^{\frac{2}{3}} \right]^{\frac{3}{2}}} \right)} + r^2 d\Omega_2^2. \quad (30)$$

For the Hayward model, we have [4]:

$$ds^2 = - \left(1 - \frac{2mr^2}{r^3 + 2m\ell^2} \right) dt^2 + \frac{dr^2}{\left(1 - \frac{2mr^2}{r^3 + 2m\ell^2} \right)} + r^2 d\Omega_2^2, \quad (31)$$

and, for the static case of the Frolov regular black hole, we have [7,8]:

$$ds^2 = - \left(1 - \frac{2mr^2}{r^3 + 2m\ell^2 + \ell^3} \right) dt^2 + \frac{dr^2}{\left(1 - \frac{2mr^2}{r^3 + 2m\ell^2 + \ell^3} \right)} + r^2 d\Omega_2^2, \quad (32)$$

where, in all three cases, ℓ is some length scale, typically identified with the Planck length [9]. Now, consider a direct comparison of the resulting Einstein tensors:

- *Bardeen*: The Einstein tensor has non-zero components:

$$G^t_t = G^r_r = \frac{-6m (2ml^2)^{\frac{2}{3}}}{\left(r^2 + (2ml^2)^{\frac{2}{3}}\right)^{\frac{11}{2}}} \left\{ r^6 + 3r^4 (2ml^2)^{\frac{2}{3}} + 3r^2 (2ml^2)^{\frac{4}{3}} + (2ml^2)^2 \right\},$$

$$G^\theta_\theta = G^\phi_\phi = \frac{3m (2ml^2)^{\frac{2}{3}} \left(3r^2 - 2 (2ml^2)^{\frac{2}{3}}\right)}{\left(r^2 + (2ml^2)^{\frac{2}{3}}\right)^{\frac{7}{2}}}. \quad (33)$$

As $r \rightarrow 0$, we have $G^a_b \rightarrow -(3/\ell^2) \delta^a_b$.

- *Hayward*: The Einstein tensor has non-zero components:

$$G^t_t = G^r_r = -\frac{12m^2\ell^2}{(2m\ell^2 + r^3)^2}, \quad G^\theta_\theta = G^\phi_\phi = -\frac{24m^2\ell^2 (m\ell^2 - r^3)}{(2m\ell^2 + r^3)^3}. \quad (34)$$

As $r \rightarrow 0$, we have $G^a_b \rightarrow -(3/\ell^2) \delta^a_b$.

- *Frolov*: The Einstein tensor has non-zero components:

$$G^t_t = G^r_r = -\frac{6m\ell^2 (2m + \ell)}{(r^3 + 2m\ell^2 + \ell^3)^2},$$

$$G^\theta_\theta = G^\phi_\phi = \frac{6m\ell^2 (2r^3 [2m + \ell] - \ell^2(2m + \ell^2)^2)}{(r^3 + 2m\ell^2 + \ell^3)^3}. \quad (35)$$

As $r \rightarrow 0$, we have $G^a_b \rightarrow -(6m/[\ell^2(2m + \ell)]) \delta^a_b$.

- *Exponential suppression*: As displayed in Equation (9), we have:

$$G^t_t = G^r_r = -\frac{2mae^{-a/r}}{r^4}, \quad G^\theta_\theta = G^\phi_\phi = -\frac{ma(a - 2r)e^{-a/r}}{r^5}. \quad (36)$$

As $r \rightarrow 0$, we have $G^a_b \rightarrow 0$.

This relative simplicity comes at the cost of representing important physical features (such as horizon radius, horizon area, and surface gravity) in terms of the Lambert W function. For example, we may compare the horizon locations:

Bardeen: $r_H \in \left\{ \ell + \mathcal{O}(\ell^{5/3}/m^{2/3}), 2m + \mathcal{O}(\ell^{4/3}/m^{1/3}) \right\};$

Hayward/Frolov: $r_H \in \left\{ \ell + \mathcal{O}(\ell^2/m), 2m + \mathcal{O}(\ell^2/m) \right\};$

Exponential suppression: $r_H \in \left\{ 2m e^{W_{-1}(-\frac{a}{2m})}, 2m e^{W_0(-\frac{a}{2m})} \right\}.$

While this may seem somewhat unwieldy, the authors argue that the Lambert W function is now such a crucial tool in many quite standard GR analyses [36,37] that theoreticians would do well to become more comfortable in dealing with its subtleties.

Physically, the crucial comparison is that the canonical regular black hole solutions (Bardeen, Hayward, Frolov, etc.) have an asymptotically de Sitter core, whilst the model presented herein has an asymptotically Minkowski core. Whether there are scenarios where this has significant ramifications on black hole evolution or evaporation is a subject for further research.

7. Generalised Models

Now, consider a generalized model of the form

$$ds^2 = - \left(1 - \frac{2m \exp(-a^n/r^n)}{r} \right) dt^2 + \frac{dr^2}{\left(1 - \frac{2m \exp(-a^n/r^n)}{r} \right)} + r^2 d\Omega_2^2; \quad a, n \in \mathbb{R}^+. \quad (37)$$

This corresponds to the central mass being r -dependent in the following manner: $m(r) = m \exp(-a^n/r^n)$. We may immediately enforce $a \in \mathbb{R}^+$ and $r \in \mathbb{R}^+$. We examine horizon locations by setting $g_{tt} = 0$. Thence:

$$r_H = 2m \exp\left(-\frac{a^n}{r_H^n}\right) \implies r_H^n = (2m)^n \exp\left(-\frac{na^n}{r_H^n}\right). \quad (38)$$

That is:

$$r_H = 2m \exp\left(\frac{W\left(-\frac{na^n}{(2m)^n}\right)}{n}\right). \quad (39)$$

For general values of n , this defines a coordinate location of the horizon explicitly in terms of the real-valued branches of the Lambert W function. While we could repeat the entire analysis for general values of n , we feel there is little extra insight to be gained—all such models will have asymptotically Minkowski cores.

8. Discussion

The model spacetime presented above represents a regular black hole geometry in the sense of finite curvature, as pioneered by Bardeen, when the parameter $a \in (0, \frac{2m}{e}]$, and accordingly violates the SEC. The model satisfies the radial NEC, (there is no wormhole throat), but violates the tangential NEC in the deep core when $r < \frac{a}{4}$. The energy density ρ is maximised at $r = \frac{a}{4}$, and the exponential expression present in the metric implies that the regular black hole has an asymptotically Minkowski core as $r \rightarrow 0$. The curvature of the geometry is asymptotically flat at infinity, at the core, and has some maximal peak in between. We can calculate surface gravity at each horizon, and consequently the Hawking temperature. The curvature tensors and invariants are significantly simpler than for canonical regular black hole solutions (Bardeen/Hayward/Frolov etc.), at the cost of having some important physical features (such as horizon radius, horizon area, and surface gravity) defined explicitly in terms of the Lambert W function. We may extend the analysis beyond $n = 1$ to arbitrary (real) values of n whilst still preserving all desired physics. The authors contend that this regular black hole model is mathematically interesting due to its tractability, and physically interesting due to the non-standard asymptotically Minkowski core.

Regarding stability of this Minkowski-core spacetime, we would point out that (partially) proving perturbative stability of Schwarzschild spacetime took at least 40 years [68], and proving non-perturbative stability of Minkowski spacetime took close to 80 years [69,70]; while rigorous results regarding the non-perturbative stability of Schwarzschild spacetime are still at best very partial and incomplete—so a full stability analysis of this Minkowski-core spacetime is not at this stage really feasible.

In contrast, some crude estimates for the parameter “ a ” are quite feasible. On the one hand, since LIGO provides good evidence for the existence of stellar mass black holes, we must demand $a < (\text{a few kilometers})$. On the other hand, to avoid having these Minkowski-core geometries descend unobservably into the quantum spacetime foam, we should have $a > (\text{a few Planck lengths})$. While we do not have direct observational evidence of the existence of primordial black holes (PBHs), if they were to exist, we would require $a \lesssim 2m_{\text{PBH}}$.

Author Contributions: Conceptualization, A.S. and M.V.; Formal Analysis, A.S. and M.V.; Funding Acquisition, M.V.; Investigation, A.S. and M.V.; Methodology, A.S. and M.V.; Project Administration, M.V.; Supervision, M.V. All authors have read and agreed to the published version of the manuscript.

Funding: Alex Simpson acknowledges financial support via the PhD Doctoral Scholarship provided by Victoria University of Wellington. Alex Simpson is indirectly supported by the Marsden fund, administered by the Royal Society of New Zealand. Matt Visser is directly supported by the Marsden fund, administered by the Royal Society of New Zealand.

Conflicts of Interest: The authors declare no conflict of interest.

References

1. Visser, M. *Lorentzian Wormholes: From Einstein to Hawking*; Springer: New York, NY, USA, 1995.
2. Visser, M. Dirty black holes: Thermodynamics and horizon structure. *Phys. Rev. D* **1992**, *46*, 2445. [[CrossRef](#)] [[PubMed](#)]
3. Bardeen, J.M. Non-singular general-relativistic gravitational collapse. In Proceedings of the International Conference GR5, Tbilisi, Georgia, 9–13 September 1968; p. 174.
4. Hayward, S.A. Formation and evaporation of regular black holes. *Phys. Rev. Lett.* **2006**, *96*, 31103. [[CrossRef](#)] [[PubMed](#)]
5. Frolov, V.P. Information loss problem and a black hole model with a closed apparent horizon. *J. High Energy Phys.* **2014**, *1405*, 49. [[CrossRef](#)]
6. Frolov, V.P. Notes on nonsingular models of black holes. *Phys. Rev. D* **2016**, *94*, 104056. [[CrossRef](#)]
7. Frolov, V.P. Remarks on non-singular black holes. *EPJ Web Conf.* **2018**, *168*, 01001. [[CrossRef](#)]
8. Frolov, V.P.; Zelnikov, A. Quantum radiation from an evaporating nonsingular black hole. *Phys. Rev. D* **2017**, *95*, 124028. [[CrossRef](#)]
9. Carballo-Rubio, R.; Filippo, F.D.; Liberati, S.; Pacilio, C.; Visser, M. On the viability of regular black holes. *J. High Energy Phys.* **2018**, *1807*, 23. [[CrossRef](#)]
10. Mazur, P.O.; Mottola, E. Gravitational condensate stars: An alternative to black holes. *arXiv* **2002**, arXiv:gr-qc/0109035.
11. Mazur, P.O.; Mottola, E. Gravitational vacuum condensate stars. *Proc. Nat. Acad. Sci. USA* **2004**, *101*, 9545. [[CrossRef](#)]
12. Visser, M.; Wiltshire, D.L. Stable gravastars: An alternative to black holes? *Class. Quant. Grav.* **2004**, *21*, 1135. [[CrossRef](#)]
13. Cattöen, C.; Faber, T.; Visser, M. Gravastars must have anisotropic pressures. *Class. Quant. Grav.* **2005**, *22*, 4189. [[CrossRef](#)]
14. Lobo, F.S.N. Stable dark energy stars. *Class. Quant. Grav.* **2006**, *23*, 1525. [[CrossRef](#)]
15. Chirenti, C.B.M.H.; Rezzolla, L. How to tell a gravastar from a black hole. *Class. Quant. Grav.* **2007**, *24*, 4191. [[CrossRef](#)]
16. Moruno, P.M.; García, N.M.; Lobo, F.S.N.; Visser, M. Generic thin-shell gravastars. *J. Cosmol. Astropart. Phys.* **2012**, *1203*, 034. [[CrossRef](#)]
17. Lobo, F.S.N.; Martín-Moruno, P.; Montelongo-García, N.; Visser, M. Novel stability approach of thin-shell gravastars. *arXiv* **2015**, arXiv:1512.07659.
18. Chirenti, C.; Rezzolla, L. Did GW150914 produce a rotating gravastar? *Phys. Rev. D* **2016**, *94*, 084016. [[CrossRef](#)]
19. Dymnikova, I. Vacuum nonsingular black hole. *Gen. Rel. Grav.* **1992**, *24*, 235. [[CrossRef](#)]
20. Dymnikova, I.G. The algebraic structure of a cosmological term in spherically symmetric solutions. *Phys. Lett. B* **2000**, *472*, 33. [[CrossRef](#)]
21. Dymnikova, I. Variable cosmological constant: Geometry and physics. *arXiv* **2000**, arXiv:gr-qc/0010016 v1.
22. Dymnikova, I. Cosmological term as a source of mass. *Class. Quant. Grav.* **2002**, *19*, 725. [[CrossRef](#)]
23. Dymnikova, I. Spherically symmetric space-time with the regular de Sitter center. *Int. J. Mod. Phys. D* **2003**, *12*, 1015. [[CrossRef](#)]
24. Dymnikova, I.; Galaktionov, E. Stability of a vacuum nonsingular black hole. *Class. Quant. Grav.* **2005**, *22*, 2331. [[CrossRef](#)]
25. Dymnikova, I.; Korpusik, M. Regular black hole remnants in de Sitter space. *Phys. Lett. B* **2010**, *685*, 12. [[CrossRef](#)]
26. Wald, R.M. *Gravitation*; University of Chicago Press: Chicago, IL, USA, 1984.

27. Misner, C.W.; Thorne, K.S.; Wheeler, J.A. *Gravitation*; Freeman: San Francisco, CA, USA, 1973.
28. Hawking, S.W.; Ellis, G.F.R. *The Large Scale Structure of Space-Time*; Cambridge University Press: Cambridge, UK, 1973.
29. Jacobson, T. When is $g_{tt} g_{rr} = -1$? *Class. Quant. Grav.* **2007**, *24*, 5717. [[CrossRef](#)]
30. Visser, M. The Kiselev black hole is neither perfect fluid, nor is it quintessence. *arXiv* **2019**, arXiv:1908.11058.
31. Boonserm, P.; Ngampitipan, T.; Simpson, A.; Visser, M. Decomposition of total stress-energy for the generalised Kiselev black hole. *arXiv* **2019**, arXiv:1910.08008.
32. Valluri, S.R.; Jeffrey, D.J.; Corless, R.M. Some applications of the Lambert W function to physics. *Can. J. Phys.* **2000**, *78*, 823. [[CrossRef](#)]
33. Valluri, S.R.; Gil, M.; Jeffrey, D.J.; Basu, S. The Lambert W function and quantum statistics. *J. Math. Phys.* **2009**, *50*, 102103. [[CrossRef](#)]
34. Boonserm, P.; Visser, M. Bounding the greybody factors for Schwarzschild black holes. *Phys. Rev. D* **2008**, *78*, 101502. [[CrossRef](#)]
35. Boonserm, P.; Visser, M. Quasi-normal frequencies: Key analytic results. *J. High Energy Phys.* **2011**, *1103*, 073. [[CrossRef](#)]
36. Boonserm, P.; Ngampitipan, T.; Visser, M. Regge-Wheeler equation, linear stability, and greybody factors for dirty black holes. *Phys. Rev. D* **2013**, *88*, 41502. [[CrossRef](#)]
37. Boonserm, P.; Ngampitipan, T.; Simpson, A.; Visser, M. Exponential metric represents a traversable wormhole. *Phys. Rev. D* **2018**, *98*, 84048. [[CrossRef](#)]
38. Sonoda, H. Solving renormalization group equations with the Lambert W function. *Phys. Rev. D* **2013**, *87*, 85023. [[CrossRef](#)]
39. Sonoda, H. Analytic form of the effective potential in the large N limit of a real scalar theory in four dimensions. *arXiv* **2013**, arXiv:1302.6059.
40. Corless, R.; Gonnet, G.; Hare, D.; Jeffrey, D.; Knuth, D. On the Lambert W function. *Adv. Comput. Math.* **1996**, 329–359. [[CrossRef](#)]
41. Vial, A. Fall with linear drag and Wien's displacement law: Approximate solution and Lambert function. *Eur. J. Phys.* **2012**, *33*, 751. [[CrossRef](#)]
42. Stewart, S. Wien peaks and the Lambert W function. *Revista Brasileira de Ensino de Fisica* **2011**, *33*, 1–6. [[CrossRef](#)]
43. Stewart, S. Spectral peaks and Wien's displacement law. *J. Thermophys. Heat Transf.* **2012**, *26*, 689–692. [[CrossRef](#)]
44. Visser, M. Primes and the Lambert W function. *Mathematics* **2018**, *6*, 56. [[CrossRef](#)]
45. Culetu, H. On a regular modified Schwarzschild spacetime. *arXiv* **2013**, arXiv:1305.5964.
46. Culetu, H. On a regular charged black hole with a nonlinear electric source. *Int. J. Theor. Phys.* **2015**, *54*, 2855. [[CrossRef](#)]
47. Culetu, H. Nonsingular black hole with a nonlinear electric source. *Int. J. Mod. Phys. D* **2015**, *24*, 1542001. [[CrossRef](#)]
48. Culetu, H. Screening an extremal black hole with a thin shell of exotic matter. *Phys. Dark Univ.* **2016**, *14*. [[CrossRef](#)]
49. Balart, L.; Vagenas, E.C. Regular black holes with a nonlinear electrodynamics source. *Phys. Rev. D* **2014**, *90*, 24045. [[CrossRef](#)]
50. Junior, E.L.B.; Rodrigues, M.E.; Houndjo, M.J.S. Regular black holes in $f(T)$ Gravity through a nonlinear electrodynamics source. *J. Cosmol. Astropart. Phys.* **2015**, *1510*, 60. [[CrossRef](#)]
51. Rodrigues, M.E.; Junior, E.L.B.; Marques, G.T.; Zanchin, V.T. Regular black holes in $f(R)$ gravity coupled to nonlinear electrodynamics. *Phys. Rev. D* **2016**, *94*, 24062. [[CrossRef](#)]
52. Xiang, L.; Ling, Y.; Shen, Y.G. Singularities and the finale of black hole evaporation. *Int. J. Mod. Phys. D* **2013**, *22*, 1342016. [[CrossRef](#)]
53. Takeuchi, S. Hawking fluxes and anomalies in rotating regular black holes with a time-delay. *Class. Quant. Grav.* **2016**, *33*, 225016. [[CrossRef](#)]
54. Jawad, A.; Shahzad, M.U. Accretion onto some well-known regular black holes. *Eur. Phys. J. C* **2016**, *76*, 123. [[CrossRef](#)]
55. Balart, L.; Fernando, S. A Smarr formula for charged black holes in nonlinear electrodynamics. *Mod. Phys. Lett. A* **2017**, *32*, 1750219. [[CrossRef](#)]

56. Barceló, C.; Visser, M. Twilight for the energy conditions? *Int. J. Mod. Phys. D* **2002**, *11*, 1553. [[CrossRef](#)]
57. Friedman, J.L.; Schleich, K.; Witt, D.M. Topological censorship. *Phys. Rev. Lett.* **1993**, *71*, 1486. [[CrossRef](#)] [[PubMed](#)]
58. Flanagan, E.E.; Wald, R.M. Does back reaction enforce the averaged null energy condition in semiclassical gravity? *Phys. Rev. D* **1996**, *54*, 6233. [[CrossRef](#)] [[PubMed](#)]
59. Ford, L.H.; Roman, T.A. Averaged energy conditions and quantum inequalities. *Phys. Rev. D* **1995**, *51*, 4277. [[CrossRef](#)]
60. Hartman, T.; Kundu, S.; Tajdini, A. Averaged null energy condition from causality. *J. High Energy Phys.* **2017**, *1707*, 066. [[CrossRef](#)]
61. Roman, T.A. Quantum stress energy tensors and the weak energy condition. *Phys. Rev. D* **1986**, *33*, 3526. [[CrossRef](#)]
62. Buniy, R.V.; Hsu, S.D.H. Instabilities and the null energy condition. *Phys. Lett. B* **2006**, *632*, 543. [[CrossRef](#)]
63. Buniy, R.V.; Hsu, S.D.H.; Murray, B.M. The null energy condition and instability. *Phys. Rev. D* **2006**, *74*, 063518. [[CrossRef](#)]
64. Martín-Moruno, P.; Visser, M. Semiclassical energy conditions for quantum vacuum states. *J. High Energy Phys.* **2013**, *1309*, 50. [[CrossRef](#)]
65. Martín-Moruno, P.; Visser, M. Classical and quantum flux energy conditions for quantum vacuum states. *Phys. Rev. D* **2013**, *88*, 61701. [[CrossRef](#)]
66. Martín-Moruno, P.; Visser, M. Semi-classical and nonlinear energy conditions. *arXiv* **2015**, arXiv:1510.00158.
67. Martín-Moruno, P.; Visser, M. Classical and semi-classical energy conditions. *Fundam. Theor. Phys.* **2017**, *189*, 193. [[CrossRef](#)]
68. Regge, T.; Wheeler, J.A. Stability of a Schwarzschild singularity. *Phys. Rev.* **1957**, *108*, 1063. [[CrossRef](#)]
69. Christodoulou, D.; Klainerman, S. *The Global Nonlinear Stability of the Minkowski Space*; Princeton University Press: Princeton, NJ, USA, 1993.
70. Bieri, L. An Extension of the Stability Theorem of the Minkowski Space in General Relativity. *J. Diff. Geom.* **2010**, *86*, 17. [[CrossRef](#)]



© 2019 by the authors. Licensee MDPI, Basel, Switzerland. This article is an open access article distributed under the terms and conditions of the Creative Commons Attribution (CC BY) license (<http://creativecommons.org/licenses/by/4.0/>).