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Metric Gravity in the Hamiltonian Form—Canonical Transformations—Dirac's Modifications of the Hamilton Method and Integral Invariants of the Metric Gravity

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Abstract: Two different Hamiltonian formulations of the metric gravity are discussed and applied to describe a free gravitational field in the d dimensional Riemann space-time. Theory of canonical transformations, which relates equivalent Hamiltonian formulations of the metric gravity, is investigated in detail. In particular, we have formulated the conditions of canonicity for transformation between the two sets of dynamical variables used in our Hamiltonian formulations of the metric gravity. Such conditions include the ordinary condition of canonicity known in classical Hamilton mechanics, i.e., the exact coincidence of the Poisson (or Laplace) brackets which are determined for both the new and old dynamical Hamiltonian variables. However, in addition to this, any true canonical transformations defined in the metric gravity, which is a constrained dynamical system, must also guarantee the exact conservation of the total Hamiltonians H_t (in both formulations) and preservation of the algebra of first-class constraints. We show that Dirac's modifications of the classical Hamilton method contain a number of crucial advantages, which provide an obvious superiority of this method in order to develop various non-contradictory Hamiltonian theories of many physical fields, when a number of gauge conditions are also important. Theory of integral invariants and its applications to the Hamiltonian metric gravity are also discussed. For Hamiltonian dynamical systems with first-class constraints this theory leads to a number of peculiarities some of which have been investigated.

1. Introduction

The main goal of this study is a deep analysis of canonical transformations of the Hamiltonian dynamical variables which are applied in the metric gravity. We investigate general properties of such transformations and formulate some criteria of canonicity. Another aim is to discuss modifications made by Dirac [1–3] in the classical Hamilton method. We want to show that Dirac's approach has many crucial advantages for the development and following improvement of various Hamiltonian formulations for the free gravitational field in metric gravity, where $2d$ additional gauge conditions exist. We also introduce the integral invariants of the metric gravity and discuss applications of these invariants to the current and new Hamiltonian formulations of the metric gravity.

In metric gravity the classical gravitational field is described as a symmetric tensor field which is defined in the d dimensional Riemann space-time. As is well known, the general theory of metric gravitational field(s) have been created more than 100 years ago by A. Einstein and it is based on his fundamental idea that the actual gravitational field has a tensor nature and it is defined in the four-dimensional (Riemann) space-time. Since then the gravitational field is designated as covariant components of the fundamental metric tensor $g_{\alpha\beta}$. In general, the metric gravity can be developed in the d dimensional space-time (or d-space), where $d \geq 3$, while the time is always one-dimensional. In other words, there is no need to restrict ourselves to the four-dimensional case only, where $d = 4$. In this

study we also deal with the d-dimensional Riemann space-time. Everywhere below the notation $\bar{x}$, designates the d-dimensional vector which has the contravariant components x^α, while its covariant components are designated as x_α, where $\alpha = 0, 1, \ldots, d-1$. The temporal component (or time component) of the coordinate vector $\bar{x}$ is x^0, while its spatial components are x^k ($k = 1, 2, \ldots, d-1$). The same rule is applied everywhere in this study: components of d-vectors are labelled by Greek indices, while spatial components of these vectors are denoted by Latin indices. In respect to this agreement all components of the covariant fundamental tensor (or metric tensor) are designated as $g_{\alpha\beta}$ (see above), while the notation $g^{\alpha\beta}$ stands for the components of contravariant fundamental (or metric) tensor [4,5]. The determinant of the $g_{\alpha\beta}$ is called the fundamental determinant and it is denoted by letter g. In the metric gravity the numerical value of g is always real and negative, but the $-g$ value is positive, which allows one to operate with the expression such as $\sqrt{-g}$ and consider functions of $\sqrt{-g}$. Any suffix with a comma before it denotes differentiation according to the general scheme $F_{,\mu} = \frac{dF}{dx^\mu}$. In particular, the temporal derivative are always designated as $F_{,0} \left(= \frac{dF}{dx^0} \right)$. For an arbitrary metric-dependent functional $F(g_{\alpha\beta})$ the notation $F_{,\gamma}$ means $F_{,\gamma} = \left(\frac{\partial F}{\partial g_{\mu\nu}} \right) g_{\mu\nu,\gamma}$, etc.

This paper has the following structure. The next two Sections play the role of 'introductory part' for our current analysis. In particular, in the next Section we introduce the regular $\Gamma - \Gamma$ Lagrangian of the metric gravity. Then, by using this $\Gamma - \Gamma$ Lagrangian we define the momenta $\pi^{\mu\nu}$ which are also the dynamical (Hamiltonian) variables. These momenta are considered as variables which are dynamically conjugate to the corresponding generalized coordinates $g_{\alpha\beta}$. In general, the momenta $\pi^{\mu\nu}$ are defined as the partial derivatives of the $\Gamma - \Gamma$ Lagrangian in respect to the 'velocities' $\frac{\partial g_{\mu\nu}}{\partial t} = g_{\mu\nu,0}$. The arising set of $2d$ dynamical variables $\{ g_{\alpha\beta}, \pi^{\mu\nu} \}$ includes all variables which are needed to develop the Hamiltonian formulation of the metric gravity. In particular, both canonical H_C and total H_t Hamiltonians of the metric gravity are written as the quadratic functions of all space-like momenta π^{pq} and as a linear combination of temporal momenta $\pi^{0\mu}(= \pi^{\mu 0})$. This essentially means that the free gravitational field is a constrained dynamical (Hamiltonian) system [6,7] and all d temporal momenta $\pi^{0\mu}(= \pi^{\mu 0})$ of this field can rigorously be defined only as the primary constraints. The commutators (or Poisson brackets) of these primary constraints with the canonical and total Hamiltonians produce d secondary constraints. In general, the properly defined Poisson brackets always play a central role in Hamiltonian formulations of many physical theories. Briefly, we can say that the Poisson brackets is the most important working tool of any consistent Hamiltonian theory.

In Section 5 by using our Poisson brackets we derive the Hamiltonian equations of motion for all dynamical variables of the metric gravity. Analogous Hamilton equations for the primary and secondary first-class constraints are also derived and discussed. The Dirac closure of this Hamiltonian formulation is explicitly demonstrated. Then we consider another Hamiltonian formulation of the metric gravity developed by Dirac [1]. For this Hamiltonian formulation we also derive the corresponding Hamilton equations of motion and determine the Poisson brackets between all essential first-class constraints (see, also our Appendix A). Then we define the canonical transformations of the dynamical Hamiltonian variables which relate both these Hamiltonian formulations, i.e., [1,6]. The criteria of canonicity for arbitrary transformations of the dynamical variables are formulated in the Section 6. Then we discuss modifications made by Dirac in the classical Hamiltonian method [1–3]. We have shown explicitly that these Dirac's modifications allowed him to create a logically closed and transparent Hamiltonian approach which has many advantages to study actual motions of various physical fields when numerous gauge conditions must also be taken into account. Here we also formulate the new principle of the "complete reverse recovery" which must be applied to any Hamiltonian formulation of the metric gravity to check its validity and correct relations with the original Einstein's field equations. This simple and physically clear principle can be used to "separate the wheat from the

chaff" (Matthew, 3:13) in Hamiltonian gravity. It allows us to operate only with the true Hamiltonian formulations of metric gravity and discard the fake ones.

In Section 8 we introduce integral invariants of the metric gravity. In some sense this is a central part of this study, since the method of integral invariants allows one to create a real foundation for the Hamiltonian formulations of the metric gravity. In particular, by applying the integral invariants of metric gravity one can easy perform all steps of the rigorous Hamiltonian procedure and formulate various criteria of canonicity which can be applied actual transformations of dynamical variables. Concluding remarks can be found in the last Section. This paper also includes three Appendixes. Appendix A is a 'pure technical' part, which contains derivations of some formulas. These formulas and expressions are of paramount importance for this study, but they could not be included in the main text, since this would damage the logic and harmony of our presentation. In Appendix B we discuss a number of tricky moments which traditionally complicate the correct definition of canonicity in classical and quantum mechanics. In Appendix C we explicitly derive some important formulas for the integral invariants. Along with the discussions of the latest achievements in Hamiltonian metric gravity, we also wanted to write a simple and transparent article which can be understood by any theoretical physicist who is familiar with the modern Hamiltonian methods developed for constrained dynamical systems.

2. Regular $\Gamma - \Gamma$ Lagrangian Density of the Metric Gravity

In this Section we introduce the regularized (or regular) Lagrangian density of metric gravity. As is well known (see, e.g., [8,9]) the original Lagrangian density of the metric gravity coincides with the integrand in the Einstein–Hilbert integral-action L_{EH} which equals to the product of scalar (or Gauss) curvature of the d-dimensional space $R = g^{\alpha\beta}R_{\alpha\beta}$ and factor $\sqrt{-g}$, which is Jacobian of the transformation from the flat space to the curved Riemann space (see, e.g., [8,9]). The invariant integral $\int R\sqrt{-g}d\Omega$ is called the gravitational action. The explicit form of Lagrangian density is $L_{EH} = \sqrt{-g}R = \sqrt{-g}g^{\alpha\beta}R_{\alpha\beta} = \sqrt{-g}g^{\alpha\beta}g^{\gamma\sigma}R_{\gamma\alpha\sigma\beta}$, where $R = g^{\alpha\beta}R_{\alpha\beta}$ is the scalar (or Gauss) curvature of d-dimensional space-time, while $R_{\alpha\beta}$ is the Ricci tensor

$$R_{\alpha\beta} = \frac{\partial \Gamma^{\gamma}_{\alpha\beta}}{\partial x^{\gamma}} - \frac{\partial \Gamma^{\gamma}_{\alpha\gamma}}{\partial x^{\beta}} + \Gamma^{\gamma}_{\alpha\beta}\Gamma^{\lambda}_{\gamma\lambda} - \Gamma^{\lambda}_{\alpha\gamma}\Gamma^{\gamma}_{\beta\lambda} , \text{ or } R_{\alpha\beta} = g^{\mu\nu}R_{\mu\alpha\nu\beta} = g^{\nu\mu}R_{\nu\beta\mu\alpha} = R_{\beta\alpha} \quad (1)$$

In this equation and everywhere below in this study the notation $\Gamma^{\gamma}_{\alpha\beta} = \frac{1}{2}g^{\gamma\nu}\left(\frac{\partial g_{\nu\alpha}}{\partial x^{\beta}} + \frac{\partial g_{\nu\beta}}{\partial x^{\alpha}} - \frac{\partial g_{\alpha\beta}}{\partial x^{\nu}}\right)$ are the Cristoffel symbols of the second kind (see, e.g., [4]). The Ricci tensor $R_{\alpha\beta} = g^{\gamma\sigma}R_{\gamma\alpha\sigma\beta}$ is simply related to the Einstein tensor $G_{\alpha\beta} = g^{\gamma\sigma}R_{\alpha\gamma\sigma\beta}$, since $R_{\alpha\beta} = -G_{\alpha\beta}$ [4]. In this notation the governing equations of the free gravitational field (famous Einstein's equations) are written in one of the following forms: $R_{\alpha\beta} = 0 = G_{\alpha\beta}$. Any Hamiltonian formulation of the metric gravity must reproduce these original field equations exactly and unambiguously. This is the new fundamental principle of the "complete reverse recovery" and its applications to various new and old Hamiltonian formulations of the metric gravity allows one quickly "to separate the wheat from the chaff" (Mathew, 13, 24–30) (see below).

An alternative (but equivalent!) form of the same Lagrangian density is written as: $L_{EH} = \sqrt{-g}g^{\alpha\beta}g^{\gamma\sigma}R_{\gamma\alpha\sigma\beta}$, where the notation $R_{\alpha\beta\gamma\sigma}$ designates the Riemann curvature tensor (or Riemann-Cristoffel tensor) which is

$$R_{\alpha\beta\gamma\sigma} = \frac{1}{2}\left[\frac{\partial^2 g_{\alpha\sigma}}{\partial x^{\beta}\partial x^{\gamma}} + \frac{\partial^2 g_{\beta\gamma}}{\partial x^{\alpha}\partial x^{\sigma}} - \frac{\partial^2 g_{\alpha\gamma}}{\partial x^{\beta}\partial x^{\sigma}} - \frac{\partial^2 g_{\beta\sigma}}{\partial x^{\alpha}\partial x^{\gamma}}\right] + \Gamma_{\rho,\alpha\sigma}\Gamma^{\rho}_{\beta\gamma} - \Gamma_{\rho,\beta\sigma}\Gamma^{\rho}_{\alpha\gamma} , \quad (2)$$

where $\Gamma_{\lambda,\mu\nu} = \frac{1}{2}\left(\frac{\partial g_{\nu\alpha}}{\partial x^{\beta}} + \frac{\partial g_{\nu\beta}}{\partial x^{\alpha}} - \frac{\partial g_{\alpha\beta}}{\partial x^{\nu}}\right)$ are the Cristoffel symbols of the first kind. The Riemann–Cristoffel tensor defined in Equation (2) is a covariant tensor of the fourth rank. As follows from the last equation the Einstein–Hilbert Lagrangian density $L_{EH} = $

$\sqrt{-g}g^{\alpha\gamma}g^{\beta\sigma}R_{\alpha\beta\gamma\sigma}$ contains a number of second-order derivatives $\frac{\partial^2 g_{\alpha\beta}}{\partial x^\gamma \partial x^\lambda}$ and cannot be used directly in the principle of least action. However, as it follows from Equations (1) and (2) all these derivatives of the second order are included in the Lagrangian density L_{EH} only as a linear combination with the constant coefficients which do not contain any derivative of the metric tensor. Such a linearity of the invariant integral $\int R\sqrt{-g}d\Omega$ upon the second-order derivatives of the metric tensor can be used to transform this integral (by means of Gauss's theorem) to the integral which does not include any second-order derivative. After a few simple transformations the invariant integral S_g is reduced to the form

$$\int R\sqrt{-g}d\Omega = \int g^{\alpha\beta}\left(\Gamma^\lambda_{\alpha\gamma}\Gamma^\gamma_{\beta\lambda} - \Gamma^\gamma_{\alpha\beta}\Gamma^\lambda_{\gamma\lambda}\right)\sqrt{-g}d\Omega + \int \frac{\partial\left[\sqrt{-g}\left(g^{\alpha\beta}\Gamma^\gamma_{\alpha\beta} - g^{\alpha\gamma}\Gamma^\beta_{\alpha\beta}\right)\right]}{\partial x^\gamma}d\Omega , \tag{3}$$

where the integrand of the first integral in the right-hand side of this equation contains only products of different powers of components of the metric tensor and their first-order derivatives, while the second integral has the form of a divergence of the vector-like quantity $\sqrt{-g}\left(g^{\alpha\beta}\Gamma^\gamma_{\alpha\beta} - g^{\alpha\gamma}\Gamma^\beta_{\alpha\beta}\right)$. It is clear that the second integral can be transformed (with the help of Gauss's theorem) into an integral over a hyper-surface surrounding the *d*-dimensional volume over which the integration is carried out in other two integrals. When we vary the gravitational action S_g, the variation of this (second) term in the right-hand side of Equation (3) vanishes, since in respect to the principle of least action, the variation of the gravitational field at the limits of the region of integration must be equal zero.

Now, from Equation (3), one finds

$$\delta S_g = \delta\int L_{EH}d\Omega = \delta\int R\sqrt{-g}d\Omega = \delta\int L_{\Gamma-\Gamma}d\Omega , \text{ or } \frac{\delta L_{EH}}{\delta g_{\mu\nu}} = \frac{\delta\left(R\sqrt{-g}\right)}{\delta g_{\mu\nu}} = \frac{\delta L_{\Gamma-\Gamma}}{\delta g_{\mu\nu}} \tag{4}$$

where the notation δ means variation, while the notation $\frac{\delta F}{\delta g_{\mu\nu}}$ means the variational derivative (or Lagrange derivative) of the functional F. Furthermore, in this equation the symbol $L_{\Gamma-\Gamma} = \sqrt{-g}g^{\alpha\beta}\left(\Gamma^\lambda_{\alpha\gamma}\Gamma^\gamma_{\beta\lambda} - \Gamma^\gamma_{\alpha\beta}\Gamma^\lambda_{\gamma\lambda}\right)$ stands for the regularized (or regular) $\Gamma - \Gamma$ Lagrangian density of the metric gravity which plays a central role in numerous Hamiltonian approaches developed for the metric gravity. As follows from Equation (4) the variational derivative of the $L_{\Gamma-\Gamma}$ Lagrangian density is a true tensor, while the original $L_{\Gamma-\Gamma}$ Lagrangian density is not a true scalar. The equality, Equation (4), expresses the fact that we can replace the 'singular' Einstein–Hilbert Lagrangian density $L_{EH} = \sqrt{-g}R$ by the regular $\Gamma - \Gamma$ Lagrangian density $L_{\Gamma-\Gamma} = \sqrt{-g}g^{\alpha\beta}\left(\Gamma^\lambda_{\alpha\gamma}\Gamma^\gamma_{\beta\lambda} - \Gamma^\gamma_{\alpha\beta}\Gamma^\lambda_{\gamma\lambda}\right)$ which is variationally equivalent to the original Einstein–Hilbert Lagrangian density and contains no second-order derivative. This $\Gamma - \Gamma$ Lagrangian density is also written in the following form

$$
\begin{aligned}
L_{\Gamma-\Gamma} &= \frac{1}{4}\sqrt{-g}B^{\alpha\beta\gamma\mu\nu\rho}\left(\frac{\partial g_{\alpha\beta}}{\partial x^\gamma}\right)\left(\frac{\partial g_{\mu\nu}}{\partial x^\rho}\right) = \frac{1}{4}\sqrt{-g}B^{\alpha\beta\gamma\mu\nu\rho}g_{\alpha\beta,\gamma}g_{\mu\nu,\rho} \\
&= \frac{1}{4}\sqrt{-g}\left(g^{\alpha\beta}g^{\gamma\rho}g^{\mu\nu} - g^{\alpha\mu}g^{\beta\nu}g^{\gamma\rho} + 2g^{\alpha\rho}g^{\beta\nu}g^{\gamma\mu} - 2g^{\alpha\beta}g^{\gamma\mu}g^{\nu\rho}\right)g_{\alpha\beta,\gamma}g_{\mu\nu,\rho} , \tag{5}
\end{aligned}
$$

where $B^{\alpha\beta\gamma\mu\nu\rho}$ is a homogeneous cubic polynomial of the contravariant components of the metric tensor $g^{\alpha\beta}$. The explicit definition of the $B^{\alpha\beta\gamma\mu\nu\rho}$ quantities follows directly from Equation (5). Below, we shall deal with the $\Gamma - \Gamma$ Lagrangian density only. In order to simplify the following formulas we shall designate this Lagrangian density by the letter L, i.e., $L = L_{\Gamma-\Gamma}$. Now, by using the $\Gamma - \Gamma$ Lagrangian density we can derive the explicit expressions for all contravariant components of momenta $\pi^{\mu\nu}$ and obtain the closed expression for the Hamiltonian(s) of the metric gravity. These important steps are made in the next Section.

3. Momenta—Canonical and Total Hamiltonians of Metric Gravity

In the previous Section we have introduced the $\Gamma - \Gamma$ Lagrangian density, Equation (5), of the metric gravity. At the second step of any standard Hamiltonian procedure, by using the known Lagrangian density we have to define the corresponding momenta. Our current derivation of momenta in this study is based on the approaches developed in the two earlier papers [1,6] which still play a central role in all modern Hamiltonian formulations of the metric gravity. First, we need to re-write the formula, Equation (5), for the $\Gamma - \Gamma$ Lagrangian density to a slightly different form where all temporal derivatives of the covariant components of metric tensor, i.e., $g_{\alpha\beta,0}$, are explicitly separated from other similar derivatives (see, e.g., [6,10])

$$L = \frac{1}{4}\sqrt{-g}B^{\alpha\beta0\mu\nu0}g_{\alpha\beta,0}g_{\mu\nu,0} + \frac{1}{2}\sqrt{-g}B^{(\alpha\beta0|\mu\nu k)}g_{\alpha\beta,0}g_{\mu\nu,k} + \frac{1}{4}\sqrt{-g}B^{\alpha\beta k\mu\nu l}g_{\alpha\beta,k}g_{\mu\nu,l} \, , \quad (6)$$

where the notation $B^{(\alpha\beta\gamma|\mu\nu\rho)}$ means a 'symmetrical' $B^{\alpha\beta\gamma\mu\nu\rho}$ quantity which is symmetrized in respect to the permutation of two groups of indexes, i.e.,

$$\begin{aligned} B^{(\alpha\beta\gamma|\mu\nu\rho)} &= \frac{1}{2}\left(B^{\alpha\beta\gamma\mu\nu\rho} + B^{\mu\nu\rho\alpha\beta\gamma}\right) = g^{\alpha\beta}g^{\gamma\rho}g^{\mu\nu} - g^{\alpha\mu}g^{\beta\nu}g^{\gamma\rho} \\ &+ 2g^{\alpha\rho}g^{\beta\nu}g^{\gamma\mu} - g^{\alpha\beta}g^{\nu\rho}g^{\gamma\mu} - g^{\alpha\rho}g^{\beta\gamma}g^{\mu\nu} \, . \end{aligned} \quad (7)$$

The contravariant components of momentum $\pi^{\gamma\sigma}$ are defined as partial derivatives of the Lagrangian density, Equation (6), in respect to the corresponding velocities $g_{\gamma\sigma,0}$ (see, e.g., [3]). The expressions for the contravariant components of gravitational momenta (or momenta, for short) are

$$\pi^{\gamma\sigma} = \frac{\partial L}{\partial g_{\gamma\sigma,0}} = \frac{1}{2}\sqrt{-g}B^{((\gamma\sigma)0|\mu\nu0)}g_{\mu\nu,0} + \frac{1}{2}\sqrt{-g}B^{((\gamma\sigma)0|\mu\nu k)}g_{\mu\nu,k} \, , \quad (8)$$

where $B^{((\gamma\sigma)0|\mu\nu0)} = \frac{1}{2}\left(B^{(\gamma\sigma0|\mu\nu0)} + B^{(\sigma\gamma0|\mu\nu0)}\right)$ is the symmetrized linear combination of the two $B^{(\gamma\sigma0|\mu\nu0)}$ quantities. The first term in the right-hand side of this equation is written in the form

$$\frac{1}{2}\sqrt{-g}B^{((\gamma\sigma)0|\mu\nu0)}g_{\mu\nu,0} = \frac{1}{2}\sqrt{-g}g^{00}\left(e^{\mu\nu}e^{\gamma\sigma} - e^{\mu\gamma}e^{\nu\sigma}\right)g_{\mu\nu,0} = \frac{1}{2}\sqrt{-g}g^{00}E^{\mu\nu\gamma\sigma}g_{\mu\nu,0} \, , \quad (9)$$

where the notations $e^{\mu\nu}$ and $E^{\mu\nu\gamma\sigma}$ stands for the Dirac contravariant tensors of the second and fourth ranks, respectively. The explicit expressions for these tensors are

$$e^{\mu\nu} = g^{\mu\nu} - \frac{g^{0\mu}g^{0\nu}}{g^{00}} \, , \quad \text{and} \quad E^{\mu\nu\gamma\rho} = e^{\mu\nu}e^{\gamma\rho} - e^{\mu\gamma}e^{\nu\rho} \, , \quad (10)$$

i.e., each component of these two tensors is a function of the contravariant components of the metric tensor only. For these tensors one finds the following symmetries in respect to permutations of their indexes: $e^{\mu\nu} = e^{\nu\mu}$ and $E^{\mu\nu\gamma\sigma} = E^{\gamma\sigma\mu\nu}$. Furthermore, as follows directly from the formulas, Equation (10), the tensor $e^{\mu\nu}$ equals zero, if either index μ, or index ν (or both) equals zero. Analogously, for the Dirac $E^{\mu\nu\gamma\sigma}$ tensor one finds that $E^{0\nu\gamma\sigma} = 0$, $E^{\mu0\gamma\sigma} = 0$, $E^{\mu\nu0\sigma} = 0$ and $E^{\mu\nu\gamma0} = 0$. Therefore, it is more productive to discuss the space-like quantities e^{mn} and E^{mnpq} only.

The space-like E^{pqkl} quantity is, in fact, the space-like Dirac tensor of the fourth rank. This space-like tensor E^{pqkl} do not have components which are equal zero identically. Furthermore, this space-like tensor E^{pqkl} is a positively defined, invertible tensor. Its inverse space-like tensor I_{mnpq} is also positively defined and invertible space-like tensor of the fourth rank which is written in the form [6]

$$I_{mnqp} = \frac{1}{d-2}g_{mn}g_{pq} - g_{mp}g_{nq} \, . \quad (11)$$

The relation between space-like tensors I_{mnpq} and E^{pqkl} is written in the form $I_{mnpq}E^{pqkl} = g_m^k g_n^l = \delta_m^k \delta_n^l$ (see, Appendix A), where $g_\beta^\alpha = \delta_\beta^\alpha$ is the substitution tensor [4] and the symbol δ_β^α denotes the Kroneker delta ($\delta_\alpha^\alpha = 1$ and $\delta_\beta^\alpha = 0$, if $\alpha \neq \beta$). From this definition one easily finds that $\delta_\beta^\alpha = \delta_\alpha^\beta$.

In general, for the $B^{((\gamma\sigma)0|\mu\nu0)}$ coefficients in the formula, Equation (8) one finds from Equations (9)–(11) two possible situations. First, for $\gamma = p$ and $\sigma = q$ these coefficients are always different from zero. In this 'regular' case we obtain the following formula for space-like contravariant components of the momentum tensor

$$
\begin{aligned}
\pi^{pq} &= \frac{\partial L}{\partial g_{pq,0}} = \frac{1}{2}\sqrt{-g}B^{((pq)0|\mu\nu0)}g_{\mu\nu,0} + \frac{1}{2}\sqrt{-g}B^{((pq)0|\mu\nu k)}g_{\mu\nu,k} \\
&= \frac{1}{2}\sqrt{-g}B^{((pq)0|mn0)}g_{mn,0} + \frac{1}{2}\sqrt{-g}B^{((pq)0|mnk)}g_{mn,k}
\end{aligned}
\tag{12}
$$

for each pair of the spatial (pq)-indexes. In this case the $(pq; mn)$-matrix of the $\sqrt{-g}B^{((pq)0|mn0)} = g^{00}E^{pqmn}$ coefficients, which are located in front of the space-like $g_{mn,0}$ velocities in the right-hand side of this equation, is invertible (see above). Therefore, in this case the field-velocity $g_{mn,0}$ can be expressed as the linear combination of the space-like components π^{pq} of momentum tensor, Equation (11):

$$
\begin{aligned}
g_{mn,0} &= \frac{1}{g^{00}}\left(\frac{2}{\sqrt{-g}}I_{mnpq}\pi^{pq} - I_{mnpq}B^{((pq)0|\mu\nu k)}g_{\mu\nu,k}\right) = \frac{1}{g^{00}}I_{mnpq}\left(\frac{2}{\sqrt{-g}}\pi^{pq}\right. \\
&\quad \left. - B^{((pq)0|\mu\nu k)}g_{\mu\nu,k}\right),
\end{aligned}
\tag{13}
$$

where the Dirac tensor I_{mnpq} is defined by Equation (11). As follows from Equations (12) and (13) for all space-like components of the metric tensor g_{pq} and corresponding momenta π^{mn} one essentially finds no principal difference with those systems in classical mechanics which have Lagrangians written as quadratic functions of the velocities. Indeed, in metric gravity all space-like components of momenta and velocities are always related to each other by a few simple, linear equations, which however, take a multi-dimensional, or matrix form. The method described above is the direct and transparent generalization of Legendre's dual transformation for the tensor fields.

In the second 'singular' case, when $\gamma = 0$ (or $\sigma = 0$) in Equation (8), the first term in the right-hand side of each of these equations equals zero. Therefore, these equations take the from of pure algebraic equations

$$
\pi^{0\sigma} = \frac{\partial L}{\partial g_{0\sigma,0}} = \frac{1}{2}\sqrt{-g}B^{((0\sigma)0|\mu\nu k)}g_{\mu\nu,k}, \quad \text{and} \quad \pi^{\sigma 0} = \frac{\partial L}{\partial g_{\sigma 0,0}} = \frac{1}{2}\sqrt{-g}B^{((\sigma 0)0|\mu\nu k)}g_{\mu\nu,k}
\tag{14}
$$

for $\sigma = 0, 1, \ldots, d-1$. From this equations one finds finds $\pi^{0\sigma} = \pi^{\sigma 0}$. Note also that these equations contain no velocities at all, i.e., we cannot express the $g_{0\sigma,0}$ velocities in terms of the momenta $\pi^{0\sigma}$ and vice versa. Each of the equations, Equation (14), directly determines the momentum $\pi^{0\sigma}$ as a cubic polynomial of the contravariant components of the metric tensor $g^{\alpha\beta}$ which is multiplied by an additional factor $\sqrt{-g}g_{\mu\nu,k}$. In other words, the following d functions

$$
\phi^{0\sigma} = \pi^{0\sigma} - \frac{1}{2}\sqrt{-g}B^{((0\sigma)0|\mu\nu k)}g_{\mu\nu,k} = \phi^{\sigma 0},
\tag{15}
$$

where $\sigma = 0, 1, \ldots, d-1$, must be equal zero during actual physical motions (or time-evolution) of the free gravitational field. In other words, during any actual motion (or time-evolution) of the free gravitational field the d additional conditions (or constraints) $\phi^{0\sigma} = 0$ must be obeyed for the Hamiltonian dynamical variables, since otherwise such a motion is not possible. We have to emphasize that the equations $\phi^{0\sigma} = 0$ are correct only on the true Hamiltonian trajectories (or curves) $(x^0, g_{\alpha\beta}(x^0), \pi^{\mu\nu}(x^0))$ of the free gravitational

field. Outside these trajectories, only some, or even none of these equations are satisfied. In [1] Dirac proposed to write these additional conditions in the symbolic form $\phi^{0\sigma} \approx 0$ (for $\sigma = 0, 1, \ldots, d-1$) with a different sign $\approx$ from the usual (=). These 'weak' equations are the primary constraints of the given Hamiltonian formulation (see, e.g., [1,3]). In other words, during time-evolution of the free gravitational field we always have d primary constraints for the $d(d+1)$ Hamiltonian variables $\{g_{\alpha\beta}, \pi^{\mu\nu}\}$ and this number d never changes if one applies canonical transformations of the Hamiltonian dynamical variables (see below). This is a partial case of the law of inertia for the first-class constraints which is discussed below.

Now, by applying the Legendre transformation to the known $\Gamma - \Gamma$ Lagrangian density L, of the metric gravity, Equation (6), and excluding all space-like velocities $g_{mn,0}$, we can derive the explicit formulas for the total H_t and canonical H_C Hamiltonians of the metric GR. Formally, these quantities are the Hamiltonian densities, but in this study we try to avoid any mention of Hamiltonian densities, since constant play with words 'Hamiltonians' and 'Hamiltonian densities' substantially complicates explanations and often leads to various confusions. In particular, the total Hamiltonian H_t of the gravitational field in metric gravity derived from the $\Gamma - \Gamma$ Lagrangian density L, Equation (5), is written in the form

$$
\begin{aligned}
H_t &= g_{\alpha\beta,0}\pi^{\alpha\beta} - L = g_{pq,0}\pi^{pq} + g_{0\sigma,0}\phi^{0\sigma} - L = g_{pq,0}\pi^{pq} - L + g_{0\sigma,0}\phi^{0\sigma} \\
&= H_C + g_{00,0}\phi^{00} + g_{0k,0}\phi^{0k} + g_{k0,0}\phi^{k0} = H_C + g_{00,0}\phi^{00} + 2g_{0k,0}\phi^{0k},
\end{aligned}
\tag{16}
$$

where $\phi^{0\sigma} = \pi^{0\sigma} - \frac{1}{2}\sqrt{-g}B^{((0\sigma)0|\mu\nu k)}g_{\mu\nu,k}$ are the primary constraints, while $g_{0\sigma,0}$ are the σ velocities (or temporal velocities) and H_C is the canonical Hamiltonian of the metric gravity

$$
H_C = \frac{1}{\sqrt{-g}g^{00}}I_{mnpq}\pi^{mn}\pi^{pq} - \frac{1}{g^{00}}I_{mnpq}\pi^{mn}B^{(pq0|\mu\nu k)}g_{\mu\nu,k}
\tag{17}
$$

$$
+ \frac{1}{4}\sqrt{-g}\left[\frac{1}{g^{00}}I_{mnpq}B^{((mn)0|\mu\nu k)}B^{(pq0|\alpha\beta l)} - B^{\mu\nu k\alpha\beta l}\right]g_{\mu\nu,k}g_{\alpha\beta,l} ,
$$

which does not contain any primary constraint $\phi^{0\sigma}$. The total Hamiltonian $H_t = H_C + g_{0\sigma,0}\phi^{0\sigma}$ is a scalar function, which is defined in the $d(d+1)$ even-dimensional phase space $\{g_{\alpha\beta}, \pi^{\mu\nu}\}$, where all components of the metric $g_{\alpha\beta}$ and momentum $\pi^{\mu\nu}$ tensors have been chosen as the basic Hamiltonian variables. The corresponding $d(d+1)$ dimensional space of Hamiltonian variables must be endowed with a symplectic (or anti-symmetric) bilinear form (Poisson brackets), which turns this space into a symplectic, even-dimensional phase space. The definition of Poisson brackets between all basic dynamical variables, i.e., between coordinates $g_{\alpha\beta}$ and momenta $\pi^{\mu\nu}$, is discussed in the next Section. At the same time the spatial (covariant) components of the metric tensor g_{mn} and spatial (contravariant) components of momenta π^{pq} form another $d(d-1)$ dimensional space, which is also transformed (by the same Poisson brackets) into a symplectic, even-dimensional phase space of the $d(d-1)$ space-like Hamiltonian variables $\{g_{mn}, \pi^{pq}\}$.

4. Poisson Brackets

In general, the Poisson brackets (or PB, for short) are the fundamental and crucially important tools of any correct Hamiltonian theory. The correct definition of these Poisson brackets is the central part of numerous Hamiltonian formulations developed for different physical systems of particles, fields and their combinations. As is well known (see, e.g., [11–14]) the Poisson bracket is an antisymmetric, bi-linear form defined in the $2M$-dimensional phase space which is, in fact, a cotangent space to an M-dimensional manifold located in the position space. More accurate definition of the Poisson brackets can be found, e.g., in [14]. For arbitrary vectors X, Y, Z from this phase space we can define

the bi-linear form which is designated below as $[X, Y]$ and it obeys the four following rules (or axioms)

$$[X, Y] = -[Y, X] \quad \text{(antisymmetry)} ,$$
$$[a_1 X_1 + a_2 X_2, Y] = a_1[X_1, Y] + a_2[X_2, Y] \quad \text{(linearity in either member)} ,$$
$$[XY, Z] = [X, Z]Y + X[Y, Z] \quad \text{(the product law)} ,$$
$$[X, [Y, Z]] + [Y, [Z, X]] + [Z, [X, Y]] = 0 \quad \text{(Jacobi identity)} ,$$

where each of the X, Y and Z vectors belongs to the $2M$ dimensional phase space. In metric gravity the $d(d+1)$-dimensional phase space includes the $\frac{d(d+1)}{2}$ generalized coordinates $g_{\alpha\beta}$ and $\frac{d(d+1)}{2}$ momenta $\pi^{\mu\nu}$, or contravariant components of the momentum tensor π. An additional (or fifth) fundamental rule for the Poisson brackets, which is often called the 'time-evolution' of the Poisson bracket, is written in the form

$$\frac{\partial}{\partial t}[X, Y] = [\frac{\partial X}{\partial t}, Y] + [X, \frac{\partial Y}{\partial t}] , \quad \text{and} \quad \frac{\partial}{\partial b}[X, Y] = [\frac{\partial X}{\partial b}, Y] + [X, \frac{\partial Y}{\partial b}]$$

where t is the temporal variable (or time, for short), while b is an arbitrary numerical parameter.

In the Hamiltonian version of metric gravity the basic dynamical variables are the generalized 'coordinates' $g_{\alpha\beta}$ and momenta $\pi^{\mu\nu}$ defined above. In respect to this the Poisson brackets between the two functions of these dynamical variables are defined as follows:

$$[f_1, f_2] = \frac{\partial f_1}{\partial g_{\alpha\beta}} \frac{\partial f_2}{\partial \pi^{\alpha\beta}} - \frac{\partial f_2}{\partial g_{\alpha\beta}} \frac{\partial f_1}{\partial \pi^{\alpha\beta}} = \frac{\partial f_1}{\partial g_{\alpha\beta}} \frac{\partial f_2}{\partial \pi^{\alpha\beta}} - \frac{\partial f_1}{\partial \pi^{\alpha\beta}} \frac{\partial f_2}{\partial g_{\alpha\beta}} \quad . \tag{18}$$

The Poisson brackets between the generalized coordinates and momenta have the fundamental value for the purposes of this study. They are:

$$[g_{\alpha\beta}, \pi^{\mu\nu}] = \frac{\partial g_{\alpha\beta}}{\partial g_{\gamma\sigma}} \frac{\partial \pi^{\mu\nu}}{\partial \pi^{\gamma\sigma}} - \frac{\partial g_{\alpha\beta}}{\partial \pi^{\gamma\sigma}} \frac{\partial \pi^{\mu\nu}}{\partial g_{\gamma\sigma}} = \Delta_{\alpha\beta}^{\mu\nu} = \frac{1}{2}\left(g_\alpha^\mu g_\beta^\nu + g_\alpha^\nu g_\beta^\mu\right) = \frac{1}{2}\left(\delta_\alpha^\mu \delta_\beta^\nu + \delta_\alpha^\nu \delta_\beta^\mu\right) , \tag{19}$$

where $g_\alpha^\mu = \delta_\alpha^\mu (= \delta_\mu^\alpha)$ is the substitution tensor [4] and symbol δ_β^μ is the Kronecker delta, while the notation $\Delta_{\alpha\beta}^{\mu\nu}$ stands for the gravitational (or tensor) delta-symbol. The three following properties of this delta-symbol are obvious and very useful in calculations of many Poisson brackets: (1) 'horizontal' index symmetry $\Delta_{\alpha\beta}^{\mu\nu} = \Delta_{\alpha\beta}^{\nu\mu} = \Delta_{\beta\alpha}^{\mu\nu} = \Delta_{\beta\alpha}^{\nu\mu}$, (2) 'vertical' index symmetry $\Delta_{\alpha\beta}^{\mu\nu} = \Delta_{\mu\beta}^{\alpha\nu} = \Delta_{\nu\beta}^{\mu\alpha} = \ldots = \Delta_{\mu\nu}^{\alpha\beta}$, and (3) the product property: $\Delta_{\rho\sigma}^{\nu\mu}\Delta_{\alpha\beta}^{\rho\sigma} = \Delta_{\alpha\beta}^{\mu\nu}$. By using these properties we can write that $[g_{\alpha\beta}, \pi^{\mu\nu}] = \Delta_{\alpha\beta}^{\mu\nu} = \Delta_{\mu\nu}^{\alpha\beta} = [g_{\mu\nu}, \pi^{\alpha\beta}]$. Note again that the total number of dynamical Hamiltonian variables in metric gravity is always even and equals $d(d+1)$. The Poisson bracket, Equation (19), explains why in some papers the gravitational momenta $\pi^{\mu\nu}$ are called and considered as conjugate dynamical variables for the corresponding covariant components of metric tensor $g_{\mu\nu}$ (our coordinates).

Other fundamental Poisson brackets between basic dynamical variables of the metric gravity equal zero identically, i.e., $[g_{\alpha\beta}, g_{\mu\nu}] = 0$ and $[\pi^{\alpha\beta}, \pi^{\mu\nu}] = 0$. In general, our dynamical variables depend upon one temporal and $(d-1)$ spatial coordinates $x^0, x^1, \ldots, x^{d-1} = (x_0, \bar{x})$. In this case we have to apply the following definition of the Poisson brackets, e.g.,

$$[g_{\alpha\beta}(\bar{x}, t), \pi^{\mu\nu}(\bar{x}', t)] = \Delta_{\alpha\beta}^{\mu\nu} \delta^{d-1}(\bar{x} - \bar{x}') , \tag{20}$$

where $\delta^{d-1}(\bar{y})$ is the usual delta-function in the position $(d-1)$-space. Such a generalization of the Poisson brackets is straightforward and simple, but in this study we do not want to

complicate our system of notations. In respect to this, below we shall always deal with the Poisson brackets of two quantities taken at the same spatial point.

The explicit form of the fundamental Poisson brackets, Equation (19), allows one to derive the following formulas for slightly different Poisson brackets

$$[g^{\alpha\beta}, \pi^{\mu\nu}] = -\frac{1}{2}\left(g^{\alpha\mu}g^{\beta\nu} + g^{\alpha\nu}g^{\beta\mu}\right) = -g^{\alpha\gamma}\Delta^{\mu\nu}_{\gamma\sigma}g^{\beta\sigma} = -[\pi^{\mu\nu}, g^{\alpha\beta}] \text{ and } [g^{\alpha\beta}, g_{\mu\nu}] = 0. \quad (21)$$

which contain the contravariant components of the metric tensor $g^{\alpha\beta}$. Now, by using the Poisson brackets, Equations (19) and (21), defined above we can determine the Poisson brackets of more complicated quantities and functions. As the first example we calculate the following Poisson bracket

$$[g_{\alpha\beta}g^{\lambda\sigma}, \pi^{\mu\nu}] = [g_{\alpha\beta}, \pi^{\mu\nu}]g^{\lambda\sigma} + g_{\alpha\beta}[g^{\lambda\sigma}, \pi^{\mu\nu}] = \Delta^{\mu\nu}_{\alpha\beta}g^{\lambda\sigma} - \frac{1}{2}g_{\alpha\beta}\left(g^{\lambda\mu}g^{\sigma\nu} + g^{\lambda\nu}g^{\sigma\mu}\right). \quad (22)$$

Let us assume that in this formula $\lambda = \beta$. In this case $g_{\alpha\beta}g^{\beta\sigma} = g^{\sigma}_{\alpha} = \delta^{\sigma}_{\alpha}$ and it is clear that $[g^{\sigma}_{\alpha}, \pi^{\mu\nu}] = 0$. On the other hand, if $\lambda = \beta$, then for the right-hand side of Equation (22) one finds:

$$\Delta^{\mu\nu}_{\alpha\beta}g^{\beta\sigma} - \frac{1}{2}g_{\alpha\beta}\left(g^{\beta\mu}g^{\sigma\nu} + g^{\beta\nu}g^{\sigma\mu}\right) = \frac{1}{2}\left(\delta^{\mu}_{\alpha}g^{\nu\sigma} + \delta^{\nu}_{\alpha}g^{\mu\sigma}\right) - \frac{1}{2}\left(\delta^{\mu}_{\alpha}g^{\nu\sigma} + \delta^{\nu}_{\alpha}g^{\mu\sigma}\right) = 0, \quad (23)$$

which means that for $\lambda = \beta$ the equation, Equation (22), is written in the form $0 = 0$ and we have no contradiction here. Now, consider the following Poisson bracket $[g_{\alpha\beta}g^{\alpha\beta}, \pi^{\mu\nu}]$. As is well known from tensor calculus (see, e.g., [4]) $g_{\alpha\beta}g^{\alpha\beta} = d$, where d is the dimension of tensor space. Therefore, this Poisson bracket is reduced to the equation

$$0 = [g_{\alpha\beta}, \pi^{\mu\nu}]g^{\alpha\beta} + g_{\alpha\beta}[g^{\alpha\beta}, \pi^{\mu\nu}] \text{ or } \Delta^{\mu\nu}_{\alpha\beta}g^{\alpha\beta} - \frac{1}{2}g_{\alpha\beta}\left(g^{\alpha\mu}g^{\beta\nu} + g^{\alpha\nu}g^{\beta\nu}\right) = 0,$$

which is easily transformed to an obvious identity $g^{\mu\nu} - g^{\nu\mu} = 0$. Analogously, it is easy to find a number of remarkable relations between the Poisson brackets $[g_{\alpha\beta}, \pi^{\mu\nu}]$ and $[g^{\alpha\beta}, \pi^{\mu\nu}]$, temporal derivatives of the covariant and contravariant components of the metric tensor and Poisson brackets of these components with the canonical Hamiltonian H_C which directly follows from Equation (22) (see, also our 'technical' Appendix A):

$$[g^{\sigma\gamma}, \pi^{\mu\nu}] = -g^{\alpha\sigma}[g_{\alpha\beta}, \pi^{\mu\nu}]g^{\beta\gamma} = -g^{\alpha\sigma}\Delta^{\mu\nu}_{\alpha\beta}g^{\beta\gamma} = -\frac{1}{2}\left(g^{\sigma\mu}g^{\gamma\nu} + g^{\sigma\nu}g^{\gamma\mu}\right), \quad (24)$$

$$\frac{dg^{\sigma\gamma}}{dt} = -g^{\alpha\sigma}\left(\frac{dg_{\alpha\beta}}{dt}\right)g^{\beta\gamma}, \quad [g^{\sigma\gamma}, H_C] = -g^{\alpha\sigma}[g_{\alpha\beta}, H_C]g^{\beta\gamma}. \quad (25)$$

The second example is of great interest for analytical calculations of the Poisson brackets between components of momenta and some functions of the coordinates only. Let $F(g_{\mu\nu}, g^{\lambda\sigma})$ be an arbitrary analytical function (or functional) of the co- and contravariant components of the metric tensor. The Poisson bracket of this $F(g_{\mu\nu}, g^{\lambda\sigma})$ function and components of momentum tensor $\pi^{\alpha\beta}$ takes the form

$$\begin{aligned} [\pi^{\alpha\beta}, F(g_{\mu\nu}, g^{\lambda\sigma})] &= -\frac{\partial F}{\partial g_{\mu\nu}}[g_{\mu\nu}, \pi^{\alpha\beta}] - \frac{\partial F}{\partial g^{\sigma\lambda}}[g^{\sigma\lambda}, \pi^{\alpha\beta}] \\ &= \frac{1}{2}\left(\frac{\partial F}{\partial g^{\sigma\lambda}}\right)\left(g^{\sigma\alpha}g^{\lambda\beta} + g^{\sigma\beta}g^{\lambda\alpha}\right) - \left(\frac{\partial F}{\partial g_{\mu\nu}}\right)\Delta^{\alpha\beta}_{\mu\nu}. \end{aligned} \quad (26)$$

In particular, for the $F(g_{\mu\nu})$ function (or functional) of the covariant components of the metric tensor only this Poisson bracket can be written in the form

$$[\pi^{\alpha\beta}, F(g_{\mu\nu})] = -\frac{\partial F}{\partial g_{\mu\nu}}[g_{\mu\nu}, \pi^{\alpha\beta}] = -\left(\frac{\partial F}{\partial g_{\mu\nu}}\right)\Delta^{\alpha\beta}_{\mu\nu}. \quad (27)$$

In the case when $F = \Phi(g_{\mu\nu})g_{\rho\sigma,k}$ we obtain

$$[\pi^{\alpha\beta}, \Phi(g_{\mu\nu})g_{\rho\sigma,k}] = -\left(\frac{\partial\Phi}{\partial g_{\mu\nu}}\right)\Delta^{\alpha\beta}_{\mu\nu}g_{\rho\sigma,k} - \Delta^{\alpha\beta}_{\rho\sigma}\left(\Phi\right)_{,k} = -\left(\frac{\partial\Phi}{\partial g_{\alpha\beta}}\right)g_{\rho\sigma,k} - \left(\Phi\right)_{,k}\Delta^{\alpha\beta}_{\rho\sigma} \quad (28)$$

Here we have applied the integration by parts (see, also discussion at the end of this Section). The third example includes Poisson brackets between space-like components of the momenta π^{mn} and components of the Dirac space-like tensor e_{pq} and/or e^{pq}. They are

$$[\pi^{mn}, e_{pq}] = -\Delta^{mn}_{pq} \quad, \text{ and } [\pi^{mn}, e^{pq}] = \frac{1}{2}\left(g^{pm}g^{qn} + g^{pn}g^{qm}\right) \quad. \quad (29)$$

As follows from the first PB in Equation (29) and two other groups of Poisson brackets: $[\pi^{mn}, \pi^{pq}] = 0$, $[e_{mn}, e_{pq}] = 0$ these $d(d-1)$ variables e_{mn} and π^{pq} are the canonical Hamiltonian variables in the $d(d-1)$-dimensional subspace of space-like dynamical variables of metric gravity. These PB play an important role in this study (see below).

From the formula, Equation (27), one also finds

$$[\pi^{\alpha\beta}, F(g_{\mu\nu})] = -\left(\frac{\partial F}{\partial g_{\mu\nu}}\right)\Delta^{\alpha\beta}_{\mu\nu} = -\left(\frac{\partial F}{\partial g_{\alpha\beta}}\right)\Delta^{\mu\nu}_{\alpha\beta} = [\pi^{\mu\nu}, F(g_{\alpha\beta})] \quad, \quad (30)$$

or simply $[\pi^{\alpha\beta}, F(g_{\mu\nu})] = [\pi^{\mu\nu}, F(g_{\alpha\beta})]$. The principal moment here is the presence of tensor Δ-symbol in these Poisson brackets. This equality simplifies calculations of a large number of Poisson brackets which are needed to show canonicity of different sets of Hamiltonian dynamical variables. Moreover, by using the formula, Equation (27), we can determine another group of important Poisson brackets between momenta and analytical functions of the fundamental determinant g and its square root $\sqrt{g}$ (or $\sqrt{-g}$ as it is designated in the metric gravity). The general expression for the Poisson bracket between such a $F(g)$ function and $\pi^{\alpha\beta}$ is derived as follows

$$[F(g), \pi^{\alpha\beta}] = \left(\frac{\partial F}{\partial g}\right)\left(\frac{\partial g}{\partial g_{\mu\nu}}\right)[g_{\mu\nu}, \pi^{\alpha\beta}] = \left(\frac{\partial F}{\partial g}\right)gg^{\mu\nu}\Delta^{\alpha\beta}_{\mu\nu} = \left(\frac{\partial F}{\partial g}\right)gg^{\alpha\beta} \quad. \quad (31)$$

or $[\pi^{\alpha\beta}, F(g)] = -\left(\frac{\partial F}{\partial g}\right)gg^{\alpha\beta}$. Now, if $F(g) = g^x$, then one finds $[\pi^{\alpha\beta}, F(g)] = -xF(g)g^{\alpha\beta}$. In particular, if $F(g) = \sqrt{-g}$ and $F(g) = \frac{1}{\sqrt{-g}}$ we obtain from the last equation

$$[\sqrt{-g}, \pi^{\alpha\beta}] = -\frac{1}{2\sqrt{-g}}gg^{\alpha\beta} = \frac{1}{2}\sqrt{-g}g^{\alpha\beta} \text{ and } [\frac{1}{\sqrt{-g}}, \pi^{\alpha\beta}] = -\frac{1}{2\sqrt{-g}}g^{\alpha\beta}, \quad (32)$$

respectively. Another Poisson bracket is often needed in operations with both primary and secondary constraints:

$$[\pi^{0\gamma}, \frac{g^{\sigma\lambda}}{g^{00}}] = \frac{1}{2g^{00}}\left(g^{0\sigma}g^{\gamma\lambda} + g^{\gamma\sigma}g^{0\lambda} - 2g^{\sigma\lambda}g^{0\gamma}\right) \quad. \quad (33)$$

If $\lambda = 0$ here, then one finds

$$[\pi^{0\gamma}, \frac{g^{\sigma 0}}{g^{00}}] = \frac{1}{2g^{00}}\left(g^{\gamma\sigma}g^{00} - g^{0\sigma}g^{0\gamma}\right) \quad. \quad (34)$$

Now, if we assume that $\sigma = 0$ here, then this Poisson bracket equals zero identically (as expected).

To conclude our discussion of the Poisson brackets let us make the two following remarks. First, as it was shown in [6,15] in metric gravity the Poisson bracket(s) between two primary constraints $\phi^{0\sigma}$ and $\phi^{0\gamma}$, Equation (15), are always equal zero, i.e., $[\phi^{0\sigma}, \phi^{0\gamma}] = 0$. In fact, the explicit derivation of this formula is very good and relatively simple exercise in

calculations of Poisson brackets (see, also discussion below). Thus, in the metric gravity all primary constraints commute with each other. This drastically simplifies many important steps of our procedure which is described below. Second, we have to explain calculations of the Poisson brackets between momenta and expressions which include some spatial derivatives of the metric tensor such as $\Phi(g_{\mu\nu})g_{\gamma\lambda,k}$, $\Phi(g_{\mu\nu})g_{\gamma\lambda,k}g_{\rho\sigma,m}$, $g_{\gamma\lambda,k}g_{\rho\sigma,m}$, etc. These and other similar expressions arise very often in actual Hamiltonian formulations, and they can be found, e.g., in operations with both canonical and total Hamiltonians, primary and secondary constraints and other expressions. Analytical calculations of such Poisson brackets by using 'integration by parts' (see, the text around Equation (28)). To explain a few hidden details of such calculations let us consider the following Poisson brackets

$$[\pi^{\alpha\beta}, \Phi(g_{\mu\nu})g_{\gamma\lambda,p}g_{\rho\sigma,q}] = -\frac{\partial\Phi(g_{\mu\nu})}{\partial g_{\mu\nu}}\Delta^{\alpha\beta}_{\mu\nu}g_{\gamma\lambda,p}g_{\rho\sigma,q} + \left[\Phi(g_{\mu\nu})g_{\rho\sigma,q}\right]_{,p}\Delta^{\alpha\beta}_{\gamma\lambda} + \left[\Phi(g_{\mu\nu})g_{\gamma\lambda,p}\right]_{,q}\Delta^{\alpha\beta}_{\rho\sigma}$$

$$= -\frac{\partial\Phi}{\partial g_{\mu\nu}}g_{\gamma\lambda,p}g_{\rho\sigma,q}\Delta^{\alpha\beta}_{\mu\nu} + \Phi_{,p}g_{\rho\sigma,q}\Delta^{\alpha\beta}_{\gamma\lambda} + \Phi g_{\rho\sigma,qp}\Delta^{\alpha\beta}_{\gamma\lambda} + \Phi_{,q}g_{\gamma\lambda,p}\Delta^{\alpha\beta}_{\rho\sigma} + \Phi g_{\gamma\lambda,pq}\Delta^{\alpha\beta}_{\rho\sigma} \ , \tag{35}$$

where $\Phi(x)$ is a scalar function of tensor argument(s). This formula can be simplified even further, but our goal here is to illustrate analytical computations of the Poisson brackets of momenta and some special functions and expressions which contain spatial derivatives of the metric tensor. In particular, the formula, Equation (35), explains the presence of second-order spatial derivatives of the metric tensor in some formulas below.

All Poisson brackets mentioned above are crucially important for the goals of this study, since they define the unique symplectic structure which is closely related to our $d(d+1)$-dimensional (tensor) phase space $\{g_{\alpha\beta}, \pi^{\mu\nu}\}$, which is closely related to the original d-dimensional Riemann space in the metric gravity. In other words, such a simplectic structure is uniformly determined by the Poisson brackets between the covariant components of the fundamental metric tensor $g_{\alpha\beta}$ and contravariant components $\pi^{\mu\nu}$ of the momentum tensor. Finally, we have to note that there is an alternative approach to develop Hamiltonian formulations of the metric gravity which is based on the use of covariant components of momenta $\pi_{\mu\nu}$. In some sense this new approach is simpler than the method discussed above, but its applications lead to re-consideration of fundamental principles of the classical Hamiltonian procedure, operations in the dual phase space and analysis of combinations of both straight and dual phase spaces for the tensor fields. This alternative approach and arising dual phase space are briefly considered below.

Covariant Components of Momenta—On the Dual Phase Space

In actual physical theories of tensor fields an arbitrary tensor can be represented either by its covariant, or contravariant components. For an arbitrary Riemann space relations between co- and contravariant components of the same vector, or tensor-like quantity are determined by the co- and contravariant components of the fundamental metric tensor $g_{\alpha\beta}$ and $g^{\alpha\beta}$. Therefore, one can always represent the same tensor equations in both covariant and contravariant forms. In general, many problems from tensor calculus can be simplified (even substantially), if they are re-written in the contravariant components of the same tensors and vice versa (some examples are considered in [4,5]). The metric gravity can be one of such problems, since both canonical and total Hamiltonians, Equations (17) and (16), contain multiple products of many contravariant components of the fundamental tensor $g^{\alpha\beta}$. Therefore, if we can properly define the covariant components of momenta $\pi_{\lambda\sigma}$, then our original problem can drastically be simplified.

Let us define the covariant components of momenta by the relation $\pi_{\lambda\sigma} = g_{\lambda\mu}\pi^{\mu\nu}g_{\nu\sigma}$. Note here that in any Hamiltonian formulation of the metric gravity, the role of fundamental tensor $g_{\alpha\beta}$ is always twofold. First, it is traditionally used to raise and lower indices in vector and tensor expressions. On the other hand, in all Hamiltonian formulations of the metric gravity the components of the fundamental tensor $g_{\alpha\beta}$ are traditionally chosen as the generalized coordinates, i.e., dynamical variables which are dynamically conjugate

to the corresponding momenta $\pi^{\mu\nu}$. Such a twofold role of the fundamental tensor $g_{\alpha\beta}$ (and $g^{\alpha\beta}$) in Hamiltonian metric gravity leads to an additional problem, since the momenta $\pi^{\mu\nu}$ do not commute with the coordinates $g_{\alpha\beta}$. In turn, this means that the following definitions of covariant components of momenta: $\pi_{\lambda\sigma} = g_{\lambda\mu}g_{\nu\sigma}\pi^{\mu\nu}, \pi_{\lambda\sigma} = g_{\lambda\mu}\pi^{\mu\nu}g_{\nu\sigma}$ and $\pi_{\lambda\sigma} = \pi^{\mu\nu}g_{\lambda\mu}g_{\nu\sigma}$ are not equivalent to each other. Indeed, it is easy to show that $\pi_{\lambda\sigma} = g_{\lambda\mu}g_{\nu\sigma}\pi^{\mu\nu} \neq g_{\lambda\mu}\pi^{\mu\nu}g_{\nu\sigma}$, since $[g_{\nu\sigma}, \pi^{\mu\nu}] = \Delta_{\nu\sigma}^{\mu\nu} = \frac{1}{2}\left(\delta_\nu^\mu \delta_\sigma^\nu + \delta_\sigma^\mu\right) \neq 0$ in the general case. To avoid repetitive discussions of similar problems in this study we shall always define the covariant components of momenta by the relation: $\pi_{\lambda\sigma} = g_{\lambda\mu}\pi^{\mu\nu}g_{\nu\sigma}$ mentioned above.

By using this definition of covariant momenta we can determine the following Poisson brackets

$$[g_{\alpha\beta}, \pi_{\mu\nu}] = \frac{1}{2}\left(g_{\alpha\mu}g_{\beta\nu} + g_{\alpha\nu}g_{\beta\mu}\right) \text{ and } [g^{\alpha\beta}, \pi_{\mu\nu}] = -\frac{1}{2}\left(g_\mu^\alpha g_\nu^\beta + g_\nu^\alpha g_\mu^\beta\right) = -\Delta_{\mu\nu}^{\alpha\beta} \tag{36}$$

and also $[g^{\alpha\beta}, g^{\mu\nu}] = 0$ and $[g_{\alpha\beta}, g^{\mu\nu}] = 0$. The formulas Equations (31), (32) and others can be re-derived for the covariant components of momentum $\pi_{\alpha\beta}$:

$$[F(g), \pi_{\alpha\beta}] = -\left(\frac{\partial F}{\partial g}\right)gg_{\alpha\beta} , \; [\sqrt{-g}, \pi_{\alpha\beta}] = -\frac{1}{2}\sqrt{-g}g_{\alpha\beta} , \; [\frac{1}{\sqrt{-g}}, \pi_{\alpha\beta}] = -\frac{1}{2\sqrt{-g}}g_{\alpha\beta} . \tag{37}$$

These Poisson brackets are also important to perform analytical calculations in the Hamiltonian formulation of the metric gravity. As follows from these formulas the dynamical Hamiltonian variables $\{g^{\alpha\beta}, \pi_{\mu\nu}\}$ form another set of Hamiltonian dynamical variables which is often called the dual set of Hamiltonian (dynamical) variables. In general, this dual set of Hamiltonian variables $\{g^{\alpha\beta}, \pi_{\mu\nu}\}$ can also be used to develop another Hamiltonian formulation of the metric gravity which is simpler than the approach described above. Thus, for the tensor field in metric gravity we always have two sets of canonical Hamiltonian variables: (a) straight (or natural) set $\{g_{\alpha\beta}, \pi^{\mu\nu}\}$, and (b) dual set of dynamical variables $\{g^{\alpha\beta}, \pi_{\mu\nu}\}$ [10]. Further analysis [10] shows that the two similar sets of dynamical Hamiltonian variables (straight and dual sets) always arise and exist in any Hamiltonian formulation of the tensor field theory and they are related to each other by a special canonical transformation. Moreover, there is a beautiful formula [10] for the Poisson brackets which unites both straight and dual sets of dynamical variables defined for the same Hamiltonian system

$$[g_{\alpha\beta}, \pi^{\mu\nu}] = \Delta_{\alpha\beta}^{\mu\nu} = [\pi_{\alpha\beta}, g^{\mu\nu}] \quad . \tag{38}$$

Another interesting Poisson bracket in the metric gravity is

$$[\pi_{\alpha\beta}, \pi^{\mu\nu}] = \frac{1}{2}\left(\delta_\alpha^\mu \pi_\beta^\nu + \delta_\alpha^\nu \pi_\beta^\mu + \delta_\beta^\mu \pi_\alpha^\nu + \delta_\beta^\nu \pi_\alpha^\mu\right) = -[\pi^{\mu\nu}, \pi_{\alpha\beta}] \quad , \tag{39}$$

where $\pi_\kappa^\rho = g^{\rho\lambda}\pi_{\lambda\kappa} = g_{\kappa\lambda}\pi^{\lambda\rho}$. The last equality means that the co- and contravariant components of the momentum tensor do not commute with each other. On the other hand, if they commuted, then the direct and dual sets of simplectic dynamical variables in metric gravity would be equivalent to each other and there would be no real need to keep these two sets of dynamical variables (straight and dual). Indeed, in this case one can always express one set of dynamic variables in terms of another set and vice versa. Such cases include all Hamiltonian theories developed for the truly scalar fields and those fields which are represented by affine vectors and tensors. However, this is not true for the metric gravity and for other theories developed for actual tensor fields in multi-dimensional Riemann spaces of non-zero curvature. In general, to develop the truly correct and covariant Hamiltonian formulation for many dynamical system of tensor fields it is much better to deal with the mixed set of $2d(d+1)$ Hamiltonian dynamical variables. This big set is a unification of the two different $d(d+1)$-dimensional sets of Hamiltonian

dynamical variables: (a) the straight set $\{g_{\alpha\beta}, \pi^{\mu\nu}\}$, and (b) the corresponding dual set $\{g^{\alpha\beta}, \pi_{\mu\nu}\}$. Applications of the two sets of dynamical variables makes our Hamiltonian formulation complete, truly covariant and physically transparent. In addition to this, an instant use of the direct and dual sets of Hamiltonian dynamical variables allows one to write canonical transformations of the Hamiltonian dynamical variables in the most general and powerful form.

5. Hamilton Equations of Motion

The main goal of any Hamiltonian formulation of some physical theory is to derive the correct Hamilton equations of motion by following the well established and physically transparent Hamilton procedure which has its internal logic based on Stokes's theorem in multi-dimensions. In general, the Hamilton method always provides a remarkable simplicity and universality in applications to actual dynamical systems and fields. Each of the Hamilton equations describes the complete time-evolution of one of the Hamiltonian dynamical variables. These correct Hamilton equations (or canonical equations) for the metric gravity are written in the following form

$$\frac{dg_{\alpha\beta}}{dx^0} = [g_{\alpha\beta}, H_t] \quad \text{and} \quad \frac{d\pi^{\mu\nu}}{dx^0} = [\pi^{\mu\nu}, H_t] \quad , \tag{40}$$

where x_0 is the temporal variables and expressions in the right-hand sides of both equations are the Poisson brackets. In other words, the first-order time derivative of each of the Hamiltonian variables is proportional to the corresponding Poisson bracket of this variable with the total Hamiltonian H_t, Equation (16). The explicit form of these Hamiltonian equations and their solutions are discussed in [10]. In particular, for space-like components of the metric tensor g_{ij} one finds the following system of Hamilton equations [10]:

$$\begin{aligned}
\frac{dg_{ij}}{dx^0} &= [g_{ij}, H_t] = [g_{ij}, H_C] = \frac{2}{\sqrt{-g}g^{00}} I_{(ij)pq}\pi^{pq} - \frac{1}{g^{00}} I_{(ij)pq} B^{(pq0|\mu\nu k)} g_{\mu\nu,k} \tag{41} \\
&= \frac{2}{\sqrt{-g}g^{00}} I_{(ij)pq}\left[\pi^{pq} - \frac{1}{2}\sqrt{-g}B^{(pq0|\mu\nu k)} g_{\mu\nu,k}\right] ,
\end{aligned}$$

where the notation $I_{(ij)pq}$ designates the (ij)-symmetrized value of the I_{ijpq} tensor defined in Equation (11), i.e.,

$$I_{(ij)pq} = \frac{1}{2}\left(I_{ijpq} + I_{jipq}\right) = \frac{1}{d-2}g_{ij}g_{pq} - \frac{1}{2}(g_{ip}g_{jq} + g_{iq}g_{jp}) \quad . \tag{42}$$

Now, let us consider the Poisson brackets for the covariant components $g_{0\sigma}$ of the fundamental tensor. It is clear that the Poisson bracket of any $g_{0\sigma}$ component with the canonical Hamiltonian H_C, Equation (17), equals zero identically. Therefore, the Hamilton equations of motion for the covariant $g_{0\sigma}(= g_{\sigma 0})$ components of the metric tensor take the form

$$\frac{dg_{0\sigma}}{dx_0} = [g_{0\sigma}, H_t] = [g_{0\sigma}, H_t - H_C] = g_{0\sigma,0} \tag{43}$$

and analogous equations for the $g_{\gamma 0}$ components. These formulas are, in fact, the definitions of the σ-velocities, where $\sigma = 0, 1, \ldots, d-1$, which essentially coincide with the coefficients in front of the primary constraints in the total Hamiltonian, Equation (16). As follows from Equation (25) there is no need to derive and solve the equations of motion for the covariant components of the metric tensor $g^{\alpha\beta}$. Indeed, if we know the time evolution of all covariant components $g_{\mu\nu}$, then from Equation (25) one easily finds the exact description of time evolution for each $g^{\alpha\beta}$ component.

In general, the Hamilton equations of motion for the contravariant components of momenta $\pi^{\alpha\beta}$ are significantly more complicated (see, e.g., [10]) than analogous equations

for the $g_{\alpha\beta}$ components (our coordinates). However, all these complications are purely technical and they are mainly related to a large number of Poisson brackets which must be determined in order to describe the complete time-evolution of all momentum variables $\pi^{\mu\nu}$. To understand the scale of this problem let us present here the Hamiltonian equations of motion for the contravariant space-like components of the momentum tensor π^{ab}:

$$\frac{d\pi^{ab}}{dx_0} = [\pi^{ab}, H_t] = [\pi^{ab}, H_C] = -\frac{1}{g^{00}}\left[\frac{I_{mnpq}}{\sqrt{-g}}, \pi^{ab}\right]\pi^{mn}\pi^{pq}$$

$$+ \frac{1}{g^{00}}\left[I_{mnpq}, \pi^{ab}\right]\pi^{mn}B^{(pq0|\mu\nu k)}g_{\mu\nu,k} + \frac{1}{g^{00}}I_{mnpq}\pi^{mn}\left[B^{(pq0|\mu\nu k)}, \pi^{ab}\right]g_{\mu\nu,k} + \dots . \tag{44}$$

This formula indicates clearly that in the Hamilton equations in metric gravity which describe time-evolution of momenta are significantly more complicated than analogous equations for time-evolution of the generalized coordinates $g_{\alpha\beta}$. In general, the Poisson bracket $[\pi^{ab}, H_t]$ is determined term-by-term.

As follows from the Hamilton equations presented above the Hamilton method itself has a number of problems when it is applied to the metric gravity, or other dynamical systems with first-class constraints. First, we note that to write Hamilton equations of actual motion we need only the canonical Hamiltonian H_C (not the total Hamiltonian H_t). Indeed, these equations are:

$$\frac{dg_{mn}}{dx_0} = [g_{mn}, H_C] \quad \text{and} \quad \frac{d\pi^{pq}}{dx_0} = [\pi^{pq}, H_C] \tag{45}$$

and there are $d(d-1)$ of these Hamilton equations of actual motion. Second, our Hamilton equations mentioned above, Equation (45), do not contain temporal momenta $\pi^{0\mu}$ and/or $\pi^{\nu 0}$ at all. This means that in these frames we cannot describe time-evolution of the temporal components of metric tensor $g_{0\mu}$ and $g_{\nu 0}$ (our coordinates). Moreover, it is not entirely clear where we can obtain these equations, since all these temporal momenta are included in our Hamiltonian formulation of the metric gravity only as primary constraints. Formally, to solve this problem introduce in the new Hamilton equations either the total Hamiltonian H_t, or the difference $H_t - H_C$, which is a linear combination of the primary constraints $\phi^{0\gamma}$, Equation (15). However, the coefficients in this linear combination are the σ-velocities, which are, in fact, arbitrary parameters of the method, rather than its dynamical variables. These arguments lead to an unambiguous conclusion that the Hamiltonian method itself must substantially be modified, if we want to apply it successfully to constrained dynamical systems, including the metric gravity. Such a modification was carried out by Dirac in his papers [1–3] and will be analyzed in detail in Section 7, but here we just want to mention its main steps.

First of all, Dirac accepted all Hamilton equations from Equations (45) as the equations which correctly describe the actual motions in our dynamical system. Thus, these $d(d-1)$ Hamilton equations have been incorporated in the new Dirac's modification of the classical Hamilton method. At the second step, Dirac rejected the d equations, Equation (43), that are formally correct but practically useless. Instead, these equations have been replaced by an equal number of equations which describe time-evolution of the primary constraints $\phi^{0\sigma}$ and define the new secondary constraints $\chi^{0\sigma}$, i.e., $\frac{d\phi^{0\sigma}}{dx_0} = [\phi^{0\sigma}, H_t] = [\phi^{0\sigma}, H_C] = \chi^{0\sigma}$. Here we apply the fact that all primary constraints commute with each other, i.e., $[\phi^{0\sigma}, \phi^{0\gamma}] = 0$ (see above). At the next (third) step Dirac explicitly derived the Hamilton equations which describe time-evolution of all d secondary constraints $\chi^{0\sigma}$: $\frac{d\chi^{0\sigma}}{dx_0} = [\chi^{0\sigma}, H_t] = [\chi^{0\sigma}, H_C] = D_c^\sigma = a_\mu^\alpha(g)\chi^{0\mu} + b_\mu^\alpha(g)\left(f^k(g)\chi^{0\mu}\right)_{,k}$, where the function (or functional) D_c^σ is the σ-component of Dirac closure which is a quasi-linear combination of the same secondary constraints and total spatial derivatives of expressions which contain the same secondary constraints. All other temporal derivatives of the Dirac closure will produce only similar quasi-linear combinations of secondary constraints and a few total spatial

derivatives of them. The process of time-evolution is formally closed, since we will never see anything new in this chain. Briefly, this means that in Dirac's modification of the classical Hamiltonian method all d primary and d secondary constraints are considered as the new Hamiltonian dynamical variables. Note that the equations which describe the time-evolution of these new dynamical variables are written in a manifestly Hamilton form.

Let us show how this procedure works in the case of metric gravity. The Poisson brackets between the primary constraints $\phi^{0\sigma}$ and canonical Hamiltonian are [6]:

$$
\begin{aligned}
[\quad \phi^{0\sigma}, H_C] = & -\frac{g^{0\sigma}}{2\sqrt{-g}g^{00}} I_{mnpq}\pi^{mn}\pi^{pq} + \frac{g^{0\sigma}}{2g^{00}} I_{mnpq}\pi^{mn}U^{(pq0|\mu\nu k)}g_{\mu\nu,k} + \Big[\pi^{\sigma k}_{,k} + \Big(\pi^{pk}e^{q\sigma} \\
& - \frac{1}{2}\pi^{pq}e^{k\sigma}\Big)g_{pq,k}\Big] - \frac{\sqrt{-g}}{8}\Big(\frac{g^{0\sigma}}{g^{00}} I_{mnpq}B^{((mn)0|\mu\nu k)}B^{(pq0|\alpha\beta t)} - g^{0\sigma}B^{\mu\nu k\alpha\beta t}\Big)g_{\mu\nu,k}g_{\alpha\beta,t} \\
& + \frac{\sqrt{-g}}{4g^{00}} I_{mnpq}B^{((mn)0|\mu\nu k)}g_{\mu\nu,k}g_{\alpha\beta,t}\Big[g^{\sigma t}\Big(g^{00}g^{p\alpha}g^{q\beta} + g^{pq}g^{0\alpha}g^{0\beta} - 2g^{\alpha q}g^{0p}g^{0\beta}\Big) \\
& - g^{\sigma p}\Big(2g^{00}g^{q\alpha}g^{t\beta} - g^{00}g^{\alpha\beta}g^{qt} + g^{\alpha\beta}g^{0q}g^{0t} - 2g^{q\alpha}g^{0\beta}g^{0t} - 2g^{t\alpha}g^{0\beta}g^{0q} + 2g^{qt}g^{0\alpha}g^{0\beta}\Big) \\
& + g^{0\sigma}(2g^{\beta t}g^{\alpha p}g^{0q} - 2g^{p\alpha}g^{q\beta}g^{0t} - 2g^{pq}g^{t\beta}g^{0\alpha} + 2g^{pt}g^{q\beta}g^{0\alpha} + g^{pq}g^{\alpha\beta}g^{0t} - g^{tp}g^{\alpha\beta}g^{0q})\Big] \\
& - \frac{\sqrt{-g}}{4}g_{\mu\nu,k}g_{\alpha\beta,t}\Big[g^{\sigma t}(g^{\alpha\mu}g^{\beta\nu}g^{0k} + g^{\mu\nu}g^{\alpha t}g^{0\beta} - 2g^{\mu\alpha}g^{k\nu}g^{0\beta}) \\
& + g^{0\sigma}(2g^{\alpha t}g^{\beta\mu}g^{\nu k} - 3g^{t\mu}g^{\nu k}g^{\alpha\beta} - 2g^{\mu\alpha}g^{\nu\beta}g^{kt} + g^{\mu\nu}g^{kt}g^{\alpha\beta} + 2g^{\mu t}g^{\nu\beta}g^{k\alpha}) \\
& + g^{\sigma\mu}\Big((g^{\alpha\beta}g^{\nu t} - 2g^{\nu\alpha}g^{t\beta})g^{0k} + 2(g^{\beta\nu}g^{kt} - g^{\beta k}g^{t\nu})g^{0\alpha} + (2g^{k\beta}g^{\alpha t} - g^{\alpha\beta}g^{kt})g^{0\nu}\Big)\Big] \\
& - \frac{\sqrt{-g}g^{00}}{2}E^{pqt\sigma}\Big(\frac{1}{g^{00}} I_{mnpq}B^{((mn)0|\mu\nu k)}g_{\mu\nu,k}\Big)_{,t} - \frac{\sqrt{-g}}{2}B^{((\sigma 0)k|\alpha\beta t)}g_{\alpha\beta,kt} \, ,
\end{aligned}
\tag{46}
$$

where $U^{(pq0|\mu\nu k)}$ is the symmetrized form of the following expression

$$
U^{\alpha\beta 0\mu\nu k} = B^{(\alpha\beta 0|\mu\nu k)} - g^{0k}E^{\alpha\beta\mu\nu} + 2g^{0\mu}E^{\alpha\beta k\nu}
\tag{47}
$$

and $\sigma = 0, 1, \ldots, d-1$. Thus, the corresponding Poisson brackets of the primary constraints with the canonical Hamiltonians H_C are the new functions of generalized coordinates $g_{\alpha\beta}$ (or $g^{\sigma\gamma}$) and momenta $\pi^{\mu\nu}$. In respect to the original terminology introduced by Dirac (see, e.g., [3]) these $\chi^{0\sigma}$ functions are the secondary constraints of this Hamiltonian formulation. Briefly, the definition of secondary constraints is written in the form: $\chi^{0\sigma} = [\phi^{0\sigma}, H_C]$, where $\sigma = 0, 1, \ldots, d-1$. This means that in metric gravity we always have d secondary constraints $\chi^{0\sigma} (= \chi^{\sigma 0})$. On actual Hamiltonian trajectories (and only on these trajectories) of the free gravitational field these secondary constraints must be equal zero, i.e., we can write the following weak equations $\chi^{0\sigma} \approx 0$ for $\sigma = 0, 1, \ldots, d-1$. Note also that the Poisson brackets between the primary and secondary constraints are $[\phi^{0\gamma}, \chi^{0\sigma}] = \frac{1}{2}g^{\gamma\sigma}\chi^{00}$ [10]. It can also be shown that all primary $\phi^{0\lambda}$ and secondary constraints $\chi^{0\sigma}$ which arise during this Hamiltonian formulation of the metric gravity are the first-class constraints [3].

At the next step of the original Dirac procedure [1–3] we have to determine the temporal derivatives of all secondary constraints, i.e., $\frac{d\chi^{0\sigma}}{dx_0} = [\chi^{0\sigma}, H_t] = [\chi^{0\sigma}, H_C] + [\chi^{0\sigma}, g_{00,0}\phi^{00} + 2g_{0k,0}\phi^{0k}]$. The first Poisson brackets is

$$
\begin{aligned}
\frac{d\chi^{0\sigma}}{dx_0} = {} & [\chi^{0\sigma}, H_C] = -\frac{2}{\sqrt{-g}} I_{mnpq}\pi^{mn}\Big(\frac{g^{\sigma q}}{g^{00}}\Big)\chi^{0p} + \frac{1}{2}g^{\sigma k}g_{00,k}\chi^{00} + \delta^\sigma_0\chi^{0k}_{,k} \\
& + \Big[-2\frac{1}{\sqrt{-g}} I_{mnpk}\pi^{mn}\frac{g^{\sigma p}}{g^{00}} + I_{mkpq}\Big(\frac{g^{\sigma m}}{g^{00}}\Big)U^{(pq)0\mu\nu l}g_{\mu\nu,l}\Big]\chi^{0k} \\
& - \Big[g^{0\sigma}g_{00,k} + 2g^{n\sigma}g_{0n,k} + \frac{g^{n\sigma}g^{0m}}{g^{00}}(g_{mn,k} + g_{km,n} - g_{kn,m})\Big]\chi^{0k} = D^\sigma_c \, ,
\end{aligned}
\tag{48}
$$

where D_c^σ is the σ-component of the Dirac closure, $U^{(pq)0\mu\nu k}$ is the quantity $U^{pq0\mu\nu k}$ from Equation (47) which is symmetrized upon all $p \leftrightarrow q$ permutations. The second Poisson bracket in the original expression for $\frac{d\chi^{0\sigma}}{dx_0}$ takes the form

$$[\chi^{0\sigma}, g_{00,0}\phi^{00} + 2g_{0k,0}\phi^{0k}] = -\frac{1}{2}g^{0\sigma}g_{00,0}\chi^{00} - g^{0k}g_{0k,0}\chi^{00} = -\frac{1}{2}\left(g^{0\sigma}g_{00,0} + 2g^{0k}g_{0k,0}\right)\chi^{00} \qquad (49)$$

and it is proportional to the secondary constraint χ^{00}. Thus, the Poisson brackets $[\chi^{0\sigma}, H_t]$ and $[\chi^{0\sigma}, H_C]$ are written as a linear combinations with field-dependent coefficients (we call them quasi-linear combinations) of the secondary constraints $\chi^{0\gamma}$ only. The $[\chi^{0\sigma}, H_C]$ Poisson bracket is called the σ-component of Dirac closure D_c^σ, or the Dirac σ-closure for the Hamiltonian formulation of metric gravity. In some old papers the Dirac closure has been defined as the $[\chi^{0\sigma}, H_t]$ Poisson bracket. The difference between these two definitions is proportional to the secondary constraint χ^{00} (see, Equation (49)), and we do not have any principal contradiction between these two definitions of Dirac's closure. Furthermore, note that this expression for Dirac closure, Equation (48), written in terms of secondary constraints only, is one of three possible results in the original Dirac procedure [2,3]. Briefly, this means that our Hamiltonian formulation of metric gravity does not lead either to any constraints of higher order, e.g., tertiary constraints, or to any inconsistency which can be fatal for the whole theory based on the $\Gamma - \Gamma$ Lagrangian, Equation (5) [3]. Finally, we need to say that in metric gravity the Dirac closure is a d-vector-like quantity in contrast with the Maxwell d-dimensional electrodynamics of the free EM-field, where the Dirac closure is a scalar which equals zero for the free EM-field [3,16].

Thus, in the metric gravity each primary constraint generates one secondary constraint and the Dirac's chain of first-class constraints ends at the secondary constraints. Finally, we have d primary and d secondary first-class constraints, i.e., the total number of the first-class constraints in metric gravity equals $2d$. In this sense there is an obvious similarity between the Hamiltonian approach for the Maxwell theory of multi-dimensional electromagnetic field (see, e.g., [3,16]) and Hamiltonian formulation of the metric gravity. Furthermore, all Hamiltonian formulations of different physical fields, which contain equal numbers of the primary and secondary first-class constraints, are quite similar to each other. The source of such a similarity can be traced back to the fact that the original Lagrangian density ($L_{\Gamma-\Gamma}$, Equation (5), in our case) is written as a quadratic-linear combination of velocities (or field-velocities).

In conclusion we want to note that there is a direct relation which allows one to express the canonical Hamiltonian H_C in terms of the secondary constraints $\chi^{0\sigma}$ and total spatial derivatives

$$H_C = -2g^{0\sigma}\chi^{0\sigma} + \left(2g_{0m}\pi^{mk}\right)_{,k} + \left[\sqrt{-g}\left(g_{0\gamma}B^{((0\gamma)k|\alpha\beta m)} - g_{0n}B^{((nk)0|\alpha\beta m)}\right)g_{\alpha\beta,m}\right]_{,k}. \qquad (50)$$

This formula relates the canonical Hamiltonian H_C which depends upon the space-like momenta π^{mn} (they belong to the pure dynamical $d(d-1)$-dimensional space) and secondary first-class constraints $\chi^{0\sigma}$ which belong to the pure constraint, or non-dynamical $2d$ dimensional subspace (see below). From this point of view the equation, Equation (50), is the 'additionality' relation between the dynamical and constraint parts of the total Hamiltonian of the metric gravity.

6. Canonical Transformations

One of the main advantages of the Hamiltonian formulation(s) of any physical theory is a possibility to apply various canonical transformations of the Hamiltonian dynamical variables. In general, such canonical transformations can be used to simplify either the canonical Hamiltonian H_C, or to reduce this Hamiltonian to some special forms, e.g., to its natural form [10]. In the Hamiltonian formulations of metric gravity the canonical transformations of Hamiltonian dynamical variables are often used to simplify the explicit form of secondary constraints. Indeed, the secondary constraints derived above in the

form of Equation (46) are very complex. Applications and even simple operations with secondary constraints written in such a form are very difficult. For instance, calculations of the Poisson brackets between primary and secondary constraints, of between any pair of secondary constraints produce formulas which are extremely cumbersome. For the first time, this has been noticed by Dirac in his fundamental paper [1]. To resolve these issues he used some canonical transformation of the original (Hamiltonian) dynamical variables which were originally introduced in [15]. At that time nobody performed similar transformations in metric gravity. This explains why Dirac in [1] started his transformations from the original $\Gamma - \Gamma$ Lagrangian density, Equation (5), which is also an ultimate source of the Hamiltonian theory. As is well known in classical mechanics we can always add the total temporal derivative to our original Lagrangian density without any change in the Lagrange equations of motion. The same rule is true in the general relativity and metric gravity, where we have also to take care about general covariance of all our formulas and expressions.

Briefly, the relation between the Dirac Lagrangian density introduced in [1] and our $L_{\Gamma-\Gamma}$ Lagrangian density, Equation (5), is written in the form $L_D = L_{\Gamma-\Gamma} - L^\star$ [17], where the additional Lagrangian density $L^\star$ takes the manifestly covariant form

$$L^\star = \left[\left(\sqrt{-g}g^{00}\right)_{,\alpha}\frac{g^{0\alpha}}{g^{00}}\right]_{,0} - \left[\left(\sqrt{-g}g^{00}\right)_{,0}\frac{g^{0\alpha}}{g^{00}}\right]_{,\alpha} = \left[\left(\sqrt{-g}g^{00}\right)_{,k}\frac{g^{0k}}{g^{00}}\right]_{,0} - \left[\left(\sqrt{-g}g^{00}\right)_{,0}\frac{g^{0k}}{g^{00}}\right]_{,k} . \tag{51}$$

This equation is reduced to the form

$$L^\star = \frac{1}{2}\sqrt{-g}A^{\alpha\beta0\mu\nu k}g_{\alpha\beta,0}g_{\mu\nu,k} = \frac{1}{2}\sqrt{-g}\left(e^{\alpha\beta}e^{k\mu}g^{0\nu} - e^{\mu\nu}e^{k\alpha}g^{0\beta} + e^{k\alpha}\frac{g^{0\mu}g^{0\nu}g^{0\beta}}{g^{00}}\right.$$
$$\left. - e^{k\mu}\frac{g^{0\alpha}g^{0\nu}g^{0\beta}}{g^{00}}\right)g_{\alpha\beta,0}g_{\mu\nu,k} . \tag{52}$$

The $A^{\alpha\beta0\mu\nu k}$ coefficients defined in this equation have a few following symmetries. First, these coefficients are symmetric upon the $\alpha\beta \leftrightarrow \beta\alpha$ and $\mu\nu \leftrightarrow \nu\mu$ permutations. Second, the important property of the $A^{\alpha\beta0\mu\nu k}$ coefficients is their anti-symmetry with respect to interchange of the two pairs of Greek indices, i.e., $A^{\alpha\beta0\mu\nu k} = -A^{\mu\nu0\alpha\beta k}$. Third, these coefficients are linearly related with the coefficients $B^{(\alpha\beta0|\mu\nu k)}$ and Dirac tensor $E^{\alpha\beta\gamma\sigma}$ (both defined in Section 3). The explicit form of this relation is

$$A^{\alpha\beta0\mu\nu k} = B^{(\alpha\beta0|\mu\nu k)} - g^{0k}E^{\alpha\beta\mu\nu} + 2g^{0\mu}E^{\alpha\beta k\nu} . \tag{53}$$

Finally, the relation between the Dirac's Lagrangian L_D and our original $L_{\Gamma-\Gamma}$ Lagrangian of the metric gravity (see above) is written in the form [17]

$$L_D = L_{\Gamma-\Gamma} - L^\star = L_{\Gamma-\Gamma} - \frac{1}{2}\sqrt{-g}A^{\alpha\beta0\mu\nu k}g_{\alpha\beta,0}g_{\mu\nu,k} , \text{ where } L^\star = \frac{1}{2}\sqrt{-g}A^{\alpha\beta0\mu\nu k}g_{\alpha\beta,0}g_{\mu\nu,k} . \tag{54}$$

From this equation one easily finds the following expression for the momenta $p^{\gamma\sigma}$ in the Dirac Hamiltonian formulation of the metric gravity

$$\frac{\partial L_D}{\partial g_{\gamma\sigma,0}} = \frac{\partial L_{\Gamma-\Gamma}}{\partial g_{\gamma\sigma,0}} - \frac{\partial L^\star}{\partial g_{\gamma\sigma,0}} , \text{ or } p^{\gamma\sigma} = \pi^{\gamma\sigma} - \frac{1}{2}\sqrt{-g}A^{(\gamma\sigma)0\mu\nu k}g_{\mu\nu,k} , \tag{55}$$

where $p^{\gamma\sigma}$ are the new momenta (or Dirac's momenta), while $\pi^{\gamma\sigma}$ are the old momenta defined above in Section 3. The last equation in Equation (55) is, in fact, the explicit definition of the Dirac's momenta which is conveniently to write in the two following forms:

$$p^{pq} = \pi^{pq} - \frac{1}{2}\sqrt{-g}A^{(pq)0\mu\nu k}g_{\mu\nu,k} \text{ and } p^{0\sigma} = \pi^{0\sigma} - \frac{1}{2}\sqrt{-g}A^{(0\sigma)0\mu\nu k}g_{\mu\nu,k} , \tag{56}$$

where $A^{(0\sigma)0\mu\nu k}g_{\mu\nu,k} = \frac{1}{2}\left(B^{(\alpha\beta0|\mu\nu k)} + B^{(\beta\alpha0|\mu\nu k)}\right)$ and $p^{0\sigma} = p^{\sigma0}$. Thus, we have the two sets of Hamiltonian dynamical variables for the two different Hamiltonian formulations of the metric gravity: $\{g_{\alpha\beta}, \pi^{\mu\nu}\}$ (the old set) and $\{g_{\alpha\beta}, p^{\mu\nu}\}$ (the new set). Since these two sets of dynamical variables are related to each other by a canonical transformation, then the three following conditions for the Poisson brackets must be obeyed: $[g_{\alpha\beta}, g_{\mu\nu}] = 0, [g_{\alpha\beta}, p^{\mu\nu}] = \Delta^{\mu\nu}_{\alpha\beta}$ and $[p^{\alpha\beta}, p^{\mu\nu}] = 0$, where all new variables are written in terms of the old variables. For old variables we already know that the following equations are true: $[g_{\alpha\beta}, g_{\mu\nu}] = 0, [g_{\alpha\beta}, \pi^{\mu\nu}] = \Delta^{\mu\nu}_{\alpha\beta}$ and $[\pi^{\alpha\beta}, \pi^{\mu\nu}] = 0$. In reality, applications of these canonicity conditions needs some additional explanations, since for all Hamiltonian systems such conditions are always derived and formulated in a different form which is based on the 'alternative' Laplace (not Poisson!) brackets. Here we have to make one step aside and discuss the general canonicity conditions for an arbitrary transformation of the Hamiltonian dynamical variables.

6.1. General Conditions of Canonicity for Transformations of the Dynamical Variables

Let us assume that some Hamiltonian system is described by the $2n$ independent dynamical variables $\{q_k, p_k\}$, where $k = 1, \ldots, n$. In general, it is possible to replace these 'old' dynamical variables by the new dynamical variables $\{\tilde{q}_i, \tilde{p}_i\}$, where $i = 1, \ldots, n$:

$$\tilde{q}_i = \phi_i(t, q_k, p_k) \quad \tilde{p}_i = \psi_i(t, q_k, p_k) \ , \tag{57}$$

but after such a transformation of variables we want to be sure that the new Hamiltonian system will be 'dynamically equivalent' to our original Hamiltonian system. Transformations of the dynamical variables each of which transforms one Hamiltonian system into another Hamiltonian system, which is completely and unambiguously 'dynamically equivalent' to the original system, are defined as the canonical transformations. In general, all canonical transformations of any Hamiltonian system form the closed algebraic structure, or group, for short (see, e.g., [10,11]). It was shown (by Jacobi) that for any time-dependent canonical transformation of the dynamical variables, Equation (57), the following canonicity condition (below, the main canonicity condition) must be obeyed

$$\sum_{k=1}^{n} \tilde{p}_k d\tilde{q}_k - \tilde{H}\delta t = c\left(\sum_{k=1}^{n} p_k dq_k - H\right)\delta t - \delta F(t, q_k, p_k) \ , \tag{58}$$

where $c(\neq 0)$ is some real number which does not depend upon the time t. The function $F(t, q_k, p_k)$ is the Jacobi generating function, i.e., the function which generates this canonical transformation. Vice versa, one can easily show that, if Equation (58) holds for some transformation of the dynamical variables, then this transformation is canonical. For better understanding of equations from this subsection we use the explicit sign of summation.

Moreover, since the valence $c(\neq 0)$ does not depend upon the time, then by establishing the criteria of canonicity, we can always restrict ourselves (for more details, see, e.g., [11]) to the time-independent canonical transformations only, i.e.,

$$\tilde{q}_i = \phi_i(q_k, p_k) \quad \tilde{p}_i = \psi_i(q_k, p_k) \ . \tag{59}$$

For a canonical time-independent transformation the main condition, Equation (58), is written in the form

$$\sum_{k=1}^{n} \tilde{p}_k d\tilde{q}_k = c\sum_{k=1}^{n} p_k dq_k - \delta K(q_k, p_k) \ , \tag{60}$$

where $\tilde{q}_k, \tilde{p}_k$ $(k = 1, \ldots, n)$ are the new generalized coordinates and momenta, while q_i, p_i $(i = 1, \ldots, n)$ are the old coordinates and momenta (old dynamical variables). Moreover, in this equation $K(q_k, p_k) = F(\bar{t}, q_k, p_k)$, i.e., it is a short Jacobi generating function of

the coordinates and momenta only, which coincides with the Jacobi generating function $F(\bar{t}, q_k, p_k)$ taken at some fixed time $t = \bar{t}$. The variation of $K(q_k, p_k)$ is written in the form

$$\delta K = -\sum_{i=1}^{n}\left(\Phi_i \delta q_i + \Psi_i \delta p_i\right) \ . \tag{61}$$

On the other hand, by using the formula $\delta \tilde{q}_k = \frac{\partial \tilde{q}_k}{\partial q_i}\delta q_i + \frac{\partial \tilde{q}_k}{\partial p_i}\delta p_i$ in Equation (60) one finds the following expression for the $\delta K(q_k, p_k)$ variation:

$$\delta K = -\sum_{i=1}^{n}\left[\sum_{k=1}^{n}\left(\tilde{p}_k \frac{\partial \tilde{q}_k}{\partial q_i}\right)-cp_i\right]\delta q_i - \sum_{i=1}^{n}\left[\sum_{k=1}^{n}\left(p_k\frac{\partial \tilde{q}_k}{\partial p_i}\right)\right]\delta p_i \ . \tag{62}$$

By comparing Equations (61) and (62) one finds

$$\Phi_i = \sum_{k=1}^{n}\tilde{p}_k\frac{\partial \tilde{q}_k}{\partial q_i} - cp_i \quad \text{and} \quad \Psi_i = \sum_{k=1}^{n}p_k\frac{\partial \tilde{q}_k}{\partial p_i} \ . \tag{63}$$

For canonical transformation(s) the expression in the left side of Equation (61) must be a total differential. From here one finds three following conditions:

$$\frac{\partial \Phi_i}{\partial q_j} = \frac{\partial \Phi_j}{\partial q_i} \quad , \quad \frac{\partial \Psi_i}{\partial p_j} = \frac{\partial \Psi_j}{\partial p_i} \quad , \quad \frac{\partial \Phi_i}{\partial p_j} = \frac{\partial \Psi_j}{\partial q_i} \quad , \qquad . \tag{64}$$

By substituting the functions Φ_i and Ψ_i in these equations by their expressions from Equation (63) one finds after a few additional and simple transformations:

$$\sum_{k=1}^{n}\left(\frac{\partial \tilde{q}_k}{\partial q_i}\frac{\partial \tilde{p}_k}{\partial q_j} - \frac{\partial \tilde{q}_k}{\partial q_i}\frac{\partial \tilde{q}_k}{\partial q_j}\right)= 0 \ , \ \text{or} \ \ \{q_i, q_j\} = 0 \ , \tag{65}$$

$$\sum_{k=1}^{n}\left(\frac{\partial \tilde{q}_k}{\partial p_i}\frac{\partial \tilde{p}_k}{\partial p_j} - \frac{\partial \tilde{q}_k}{\partial p_i}\frac{\partial \tilde{q}_k}{\partial p_j}\right)= 0 \ , \ \text{or} \ \ \{p_i, p_j\} = 0 \ , \tag{66}$$

$$\sum_{k=1}^{n}\left(\frac{\partial \tilde{q}_k}{\partial q_i}\frac{\partial \tilde{p}_k}{\partial p_j} - \frac{\partial \tilde{q}_k}{\partial p_i}\frac{\partial \tilde{q}_k}{\partial q_j}\right)= c\delta_{ij} \ , \ \text{or} \ \ \{q_i, p_j\} = c\delta_{ij} \ , \tag{67}$$

where δ_{ij} is the Kronecker symbol and c is some numerical constant. The constructions (or sums) which appear in these three equations are the Laplace brackets which are well known in classical mechanics (see, e.g., [11,12]). The standard notation for the Laplace brackets (see, e.g., [11,12]) is $\{,\}$. Each of these sums includes $2n$ functions ($\tilde{q}_k$ and $\tilde{p}_k$) and two variables only, e.g., either q_i, q_j, or p_i, p_j, or q_i, p_j. As follows from Equations (65)–(67) some transformation of the dynamical variables will be canonical if (and only if) the three groups of following conditions are obeyed: $\{q_i, q_j\} = 0, \{p_i, p_j\} = 0$ and $\{q_i, p_j\} = c\delta_{ij}$, where $c \neq 0$ and $(i, j) = 1, \ldots, n$, for all $2n$ new dynamical variables $\tilde{q}_k, \tilde{p}_k$ ($k = 1, \ldots, n$).

In reality, the original Laplace brackets are not convenient in applications. However, as follows from the Appendix B these brackets can be replaced by the Poisson brackets, each of which is the adjoint to the corresponding Laplace bracket. In terms of the Poisson brackets the same criteria of canonicity are written in a different form (for more details, see our Appendix B): $[\tilde{q}_i, \tilde{q}_j] = 0 \ , \ [\tilde{p}_i, \tilde{p}_j] = 0 \ , \ [\tilde{q}_i, \tilde{p}_j] = c\delta_{ij}$, where $(i, j) = 1, \ldots, n$ and c is the valence of this canonical transformation. These numerical values of the Poisson brackets taken for $c = 1$ are used below as the criteria of canonicity for the transformation of dynamical variables. To simplify the text below we shall call these brackets by the canonical, univalent set of the Poisson brackets, or CUSPB, for short.

6.2. Applications to the Metric Gravity

Let us apply the formulas derived above to the metric gravity by considering a transformation from the old set of dynamical variables $\{g_{\alpha\beta}, \pi^{\mu\nu}\}$ to the new set of such variables

$\{g_{\alpha\beta}, p^{\mu\nu}\}$. First, we note that the generalized coordinates $g_{\alpha\beta}$ are identical in both sets. For the brackets defined in the previous subsection this means that $\{g_{\alpha\beta}, g_{\gamma\rho}\} = 0$ and $[g_{\alpha\beta}, g_{\gamma\rho}] = 0$. Furthermore, for the univalent ($c = 1$) transformations of dynamical variables we have for the new momenta $p^{\mu\nu} = \pi^{\mu\nu} + f^{\mu\nu}(g_{\gamma\rho})$, where $f^{\mu\nu}(g_{\gamma\rho})$ is a tensor function of generalized coordinates only. From here one finds that $[g_{\alpha\beta}, p^{\mu\nu}] = [g_{\alpha\beta}, \pi^{\mu\nu}] = \Delta^{\mu\nu}_{\alpha\beta}$. In other words, the first and last Poisson brackets from CUSPB are obeyed automatically for our transformation of the Hamiltonian dynamical variables. The only non-trivial bracket in CUSPB (see, Equation (A17) in the Appendix B) is the second Poisson bracket between two new momenta which takes the following form in our tensor notations:

$$[p^{\alpha\beta}, p^{\mu\nu}] = 0 \ , \ \text{or} \ [\pi^{\alpha\beta} - \frac{1}{2}\sqrt{-g}A^{(\alpha\beta)0\sigma\rho k}g_{\sigma\rho,k}, \pi^{\mu\nu} - \frac{1}{2}\sqrt{-g}A^{(\mu\nu)0\lambda\kappa k}g_{\lambda\kappa,k}] = 0 \ , \quad (68)$$

which is instantly reduced to the equation

$$[\pi^{\alpha\beta}, \sqrt{-g}A^{(\mu\nu)0\sigma\rho m}g_{\sigma\rho,m}] = [\pi^{\mu\nu}, \sqrt{-g}A^{(\alpha\beta)0\sigma\rho m}g_{\sigma\rho,m}] \ . \quad (69)$$

The transformation of dynamical variables will be canonical, if (and only if) this equation is obeyed. To prove the validity of this equation one has to perform direct calculations of the Poisson brackets in both sides of Equation (69). In reality, both sides of Equation (69) are compared with each other and identical terms (in both sides) are cancelled. Finally, this equation is reduced to the form of identity such as $0 = 0$.

In those cases when either $\alpha = 0$, or $\beta = 0$ (or both) one obtains from Equations (53) and (54) the following equation:

$$p^{0\gamma} = \pi^{0\gamma} - \frac{1}{2}\sqrt{-g}B^{((0\gamma)0|\mu\nu k)}g_{\mu\nu,k}, \quad (70)$$

which defines the momenta with one (or two) temporal component(s), or primary constraints $p^{0\gamma} \approx 0$ in the Dirac's Hamiltonian formulation of the metric gravity. For these momenta the canonicity conditions, Equation (68), must also be obeyed. After a few simple transformations the essential canonicity conditions for the $p^{0\gamma}$ and $p^{0\sigma}$ momenta take one of the following forms

$$[p^{0\gamma}, p^{0\sigma}] = 0 \ , \ \text{or} \ [\pi^{0\gamma}, \sqrt{-g}B^{((0\sigma)0|\mu\nu k)}g_{\mu\nu,k}] = [\pi^{0\sigma}, \sqrt{-g}B^{((0\gamma)0|\mu\nu k)}g_{\mu\nu,k}] \ , \quad (71)$$

which simply means that all primary constraints in the Dirac's Hamiltonian formulations commute with each other. The same statement is true for the original Hamiltonian formulation of metric GR [6] discussed above. This fact has been checked in [15]. On the other hand, we have to note that the fact that all primary constraints in the metric gravity commute with each other follows directly from the canonicity of the complete Dirac's set of Hamiltonian dynamical variables.

At this point it is very convenient to introduce the universal notation $\phi^{\mu\nu}$ for the momenta, or for contravariant components of the momenta. In Dirac's Hamiltonian formulation these momenta are $\phi^{\mu\nu} = p^{\mu\nu}$, while in the Hamiltonian formulation from [6] these momenta are $\phi^{\mu\nu} = \pi^{\mu\nu} - \frac{1}{2}\sqrt{-g}A^{(\mu\nu)0\alpha\beta k}g_{\alpha\beta,k}$. In this notation the canonical Hamiltonians H_C in both formulations of metric gravity are represented in the same 'universal' form [17]

$$H_C = \frac{1}{\sqrt{-g}g^{00}}I_{mnpq}\phi^{mn}\phi^{pq} - \frac{1}{g^{00}}\phi^{mn}\left(g^{0l}g_{mn,l} - 2g^{0\alpha}g_{\alpha n,m}\right) \quad (72)$$

$$+ \frac{1}{4}\sqrt{-g}\left[\frac{1}{g^{00}}\left(g^{0k}E^{(mn)\mu\nu} - 2g^{0\mu}E^{(mn)k\nu}\right)\left(g^{0l}g^{\alpha}_m g^{\beta}_n - 2g^{0\alpha}g^l_m g^{\beta}_n\right) - B^{\mu\nu k\alpha\beta l}\right]g_{\mu\nu,k}g_{\alpha\beta,l} \ ,$$

where $g_\beta^\alpha = \delta_\beta^\alpha$ is the substitution tensor defined above. In both formulations the primary constraints commute with each other, i.e., $[\phi^{0\gamma}, \phi^{0\sigma}] = 0$. The knowledge of the canonical Hamiltonian H_C and all primary constraints allows one to restore the total Hamiltonian H_t:

$$H_t = H_C + g_{00,0}\phi^{00} + 2g_{0k,0}\phi^{0k} = H_C + g_{00,0}p^{00} + 2g_{0k,0}p^{0k} \ . \tag{73}$$

In particular, in the Dirac's Hamiltonian formulation we obtain $H_t = H_C + g_{00,0}p^{00} + 2g_{0k,0}p^{0k}$. It appears that the total Hamiltonian H_t does not change during canonical transformations of the dynamical variables, i.e., $H_t^{K\&K}(g_{\alpha\beta}, \pi^{\mu\nu}) = H_t^{Dir}(g_{\alpha\beta}, p^{\mu\nu})$ [17]. In other words, the total Hamiltonian is an obvious and unique invariant of this theory. In respect to this, the Hamilton equations do not change its form during canonical transformations and we can write, e.g.,

$$g_{\alpha\beta,0} = [g_{\alpha\beta}, H_t^{K\&K}] \ , \ \pi^{\mu\nu}_{,0} = [\pi^{\mu\nu}, H_t^{K\&K}] \ \Leftrightarrow \ g_{\alpha\beta,0} = [g_{\alpha\beta}, H_t^{Dir}] \ , \ p^{\mu\nu}_{,0} = [p^{\mu\nu}, H_t^{Dir}] \tag{74}$$

i.e., these two sets of Hamilton equations are equivalent to each other. In other words, the Hamilton equations conserve their form during canonical transformation of the dynamical variables. In fact, this was the first definition (or criterion) of canonicity for the transformations of dynamical variables which has been formulated by Sir William R. Hamilton himself in 1834 and 1835. We have shown that his criterion works for the Hamiltonian approach to the metric gravity. However, the metric gravity is a dynamical system with constraints. It is clear that the Hamilton criterion, as well as other criteria of canonicity known for the transformations of dynamical variables in classical mechanics, must be supplemented by some statement(s) about the algebra of constraints (see below). Therefore, we need to derive the explicit expressions for the secondary constraints, their Poisson brackets with the canonical and/or total Hamiltonians, primary constraints, etc.

All secondary constraints in the Dirac's Hamiltonian formulation are derived from the equations $\chi^{0\sigma} = [\phi^{0\sigma}, H_t] = [\phi^{0\sigma}, H_C]$. The explicit expressions are

$$\chi^{0\sigma} = -\frac{g^{0\sigma}}{2\sqrt{-g}g^{00}}I_{mnpq}\phi^{mn}\phi^{pq} + g_m^\sigma\left[(\phi^{mk})_{,k} + \left(\phi^{pk}e^{qm} - \frac{1}{2}\phi^{pq}e^{km}\right)g_{pq,k}\right]$$
$$+\frac{1}{2}\sqrt{-g}g^{0\sigma}\left[-g_{mn,kl}E^{mnkl} + \frac{1}{4}g_{mn,k}g_{pq,l}\left(-E^{mnpq}e^{kl} + 2E^{klpn}e^{mq} + E^{pqnl}e^{mk}\right)\right] \ . \tag{75}$$

This formula is very compact and contains only two lines (compare with the formula, Equation (46)). It indicates clearly that Dirac's idea to apply canonical transformations of the Hamiltonian dynamical variables in order to simplify secondary first-class constraints works perfectly. Now, by using the explicit form of the primary $\phi^{0\gamma} = p^{0\gamma}$ and secondary $\chi^{0\sigma}$ constraints in Dirac's formulation one finds

$$[\phi^{0\gamma}, \chi^{0\sigma}]_{Dirac} = [p^{0\gamma}, \chi^{0\sigma}]_{Dirac} = \frac{1}{2}g^{\gamma\sigma}\chi^{00}_{Dirac} \ , \tag{76}$$

i.e., the formula which exactly coincides (by its form) with the formula $[\phi^{0\gamma}, \chi^{0\sigma}]_{K\&K} = \frac{1}{2}g^{\gamma\sigma}\left(\chi^{00}\right)_{K\&K}$ mentioned above. It is very interesting, since the explicit forms of all primary and secondary constraints are substantially different in these two formulations. This and other similar facts directly follow from the canonicity of our transformation of the Hamiltonian dynamical variables. The time-evolution of the secondary constraints leads to the following formula

$$\frac{d\chi^{0\sigma}}{dx_0} = \chi^{0\sigma}_{,0} = [\chi^{0\sigma}, H_C] = -\left[\frac{2}{\sqrt{-g}g^{00}}I_{pqmk}g^{\sigma m}\phi^{pq} + g^{0\sigma}g_{00,k} + 2g^{\sigma p}g_{0p,k}\right.$$
$$+ \left.\left(\frac{g^{\sigma p}g^{0q}}{g^{00}}\right)\left(g_{pq,k} + g_{qk,p} - g_{pk,q}\right)\right]\chi^{0k} - g_0^\sigma(\chi_{0k})_{,k} + \frac{1}{2}g^{\sigma k}g_{00,k}\chi^{00} = D_c^\sigma \ , \tag{77}$$

where D_c^σ is the σ-component of the Dirac closure derived in the Dirac's Hamiltonian formulation of the metric gravity. All components of the Dirac closure are quasi-linear combination of secondary constraints and some total spatial derivatives of these secondary constraints. Again, this formula is very compact and transparent. Finally, we want to present the formula, which allows one to express the canonical Hamiltonian H_C in terms of secondary constraints and some (total) spatial derivatives, Equation (50), can also be derived in the Dirac Hamiltonian formulation. The formula takes the form, which is slightly different from Equation (50) above:

$$
\begin{aligned}
H_C = -2g_{0\lambda}\chi^{0\lambda} \quad + \quad & \left(2g_{0m}\phi^{mk}\right)_{,k} - \Big[\sqrt{-g}E^{mnpq}g_{mn,q} \\
& - \sqrt{-g}g_{\mu\nu,k}\left(\frac{g^{0\mu}}{g^{00}}\right)\left(g^{\nu p}g^{0k} - g^{\nu k}g^{0p}\right)\Big]_{,p}
\end{aligned}
\tag{78}
$$

This formula also represents the canonical Hamiltonian H_C (in the Dirac formulation) written as a quasi-linear combination of the secondary constraints $\chi^{0\sigma}$ and a few total spatial derivatives of some expressions which include the same secondary constraints.

Now, we can complete our discussion of canonical transformations in the metric gravity. There are three general rules which regulate changes in the primary and secondary constraints during such transformations. The first rule is simple and it is called the law of inertia for the first-class constraints. Indeed, by performing a number of canonical transformations between different sets of dynamical variables we have found that the total number of the primary $\phi^{0\sigma}$ constraints N_p never changes during such transformations. The same statement is true for the total number of secondary $\chi^{0\sigma}$ constraints N_s and for the sum $N_p + N_s$. We have to emphasize here that all primary and secondary constraints which arise in the Hamiltonian formulations of metric gravity are first-class. The second rule of 'form-invariance' is even simpler: the internal structure of all first-class constraints must be conserved during canonical transformations of the Hamiltonian dynamical variables. The preservation of form-invariance for all first-class constraints is crucial to prove that any two Hamiltonian formulations developed for the same constrained dynamical system are equivalent to each other. The third rule essentially follows from the second rule: all Poisson brackets between the first-class constraints and canonical/total Hamiltonians, other constraints, etc, must also be form-invariant during canonical transformations. For simple Poisson brackets, e.g., for the $[\phi^{0\sigma}, \chi^{0\gamma}]$ brackets, this rule leads to the exact coincidence of corresponding expressions. The three rules mentioned here essentially mean preservation of the algebra of first-class constraints. Thus, the canonical transformations in the metric gravity must guarantee a complete preservation of the form-invariance for the total Hamiltonian H_t and for the algebra of first-class constraints.

The formulas derived in this Section allow one to apply the Dirac's Hamiltonian formulation of metric gravity to analyze and solve various gravitational problems. In some cases, however, one needs to know analytical expressions for other Poisson brackets, e.g., the Poisson bracket between two secondary constraints $[\chi^{0\gamma}, \chi^{0\sigma}]$ is of great interest, but it has never been obtained in previous papers. This Poisson bracket is determined in our 'technical' Appendix A. In general, calculations of this and other Poisson brackets can be performed with the use of our formulas and method described in Section 4.

7. Dirac's Modifications of the Classical Hamilton Method

In this Section we want to reconsider modifications which were made by Dirac in the classical Hamilton method [1–3]. This will eventually lead us to the new universal criterion of canonicity for Hamiltonian formulations of metric gravity. First, we note again that in any Hamiltonian formulation of metric gravity we always have $\frac{d(d+1)}{2}$ generalized coordinates $g_{\alpha\beta}$ and $\frac{d(d+1)}{2}$ momenta $p^{\mu\nu}$. These coordinates and momenta are the Hamiltonian dynamical variables of our problem (metric gravity). The total number of these variables equals $d(d+1)$ which is an even number for any d-dimensional Riemann space-time. Note

also that our original Lagrangian $L_{\Gamma-\Gamma}$, Equation (6), is a quadratic function upon velocities of the space-like components of the metric tensor g_{mn}, i.e., upon $g_{mn,0}$. On the other hand, the same $L_{\Gamma-\Gamma}$ Lagrangian is a linear function of the d remaining $g_{0\gamma,0}$ ($= g_{\gamma 0,0}$) velocities which are also called the temporal velocities.

By using the standard Legendre transformation (see Section 3) one can pass from the Lagrangian $L_{\Gamma-\Gamma}$ to the Hamiltonian H_C which is quadratic function of space-like momenta π^{mn}. This Hamiltonian is called the canonical Hamiltonian, and it is an explicit function of the space-like dynamical $\{g_{mn}, p^{pq}\}$ variables (there are $d(d-1)$ of such variables) and d 'temporal' coordinates $g_{0\sigma}$ only. The Hamiltonian H_C is of great interest for the whole metric gravity, since it describes the actual motions of a free gravitational field. However, we have to note that this Hamiltonian H_C does not depend upon any of the temporal momenta, i.e., it does not include any of the $p^{0\sigma}$ (or $p^{\sigma 0}$) momenta. This means that all Poisson brackets such as $[g_{0\sigma}, H_C] = 0$ and $[g_{\sigma 0}, H_C] = 0$ equal zero identically, and canonical Hamiltonian H_C does not describe time-evolution of the temporal coordinates $g_{0\sigma}$ and/or $g_{\sigma 0}$ coordinates, in principle. Briefly, we can say that in the canonical Hamiltonian H_C these temporal coordinates $g_{0\mu}$ (and $g_{\mu 0}$) are rather parameters than actual dynamical variables.

For normal applications of the Hamilton method we must have a Hamiltonian that contains all momenta, including temporal ones. Such a complete, or total Hamiltonian H_t will describe time-evolution of all $d(d+1)$ dynamical variables of the problem $\{g_{\alpha\beta}, p^{\mu\nu}\}$ and all functions of these variables, including the canonical Hamiltonian H_C, new coordinates and momenta, which can be introduced by some canonical transformations, etc. Formally, this total Hamiltonian can be derived from our quadratic-linear Lagrangian $L_{\Gamma-\Gamma}$ by using the Legendre transform which is described in detail in Section 3. However, for our quadratic-linear Lagrangian $L_{\Gamma-\Gamma}$, Equation (6), the Legendre transform works with some singularities. The two main singularities must be mentioned here, since they play crucial roles in Dirac's modification of the classical Hamilton method. First, as follows from the definition of momenta and from the general technique of Legendre transformations, we cannot obtain, in principle, the explicit expressions of the velocities $v_\gamma (= g_{0\gamma,0})$ written in terms of momenta $p^{0\gamma}$ and vice versa. Instead, we obtain the following algebraic equations: $p^{0\gamma} \approx f(g_{\alpha\beta}, g^{0\gamma})$, or $\phi^{0\gamma} = p^{0\gamma} - f(g_{\alpha\beta}, g^{0\gamma}) \approx 0$ which are called the primary constraints (see, e.g., [3] and references therein). Second, in respect to the procedure of Legendre transformation, this moment must be multiplied by the corresponding velocity $g_{0\gamma,0} (= v_\gamma)$, which is not a dynamical variable of our Hamiltonian method. This velocity is rather a parameter (arbitrary parameter) of the updated Legendre procedure.

Thus, we have derived the total Hamiltonian H_t, Equation (16), which is written as the sum of the canonical Hamiltonian H_C and primary constraints $\phi^{0\alpha}$. The coefficients in front of the primary constraints equal to the corresponding velocities v_α, i.e., $H_t = H_C + g_{00,0}\phi^{00} + 2g_{0k,0}\phi^{0k} = H_C + v_\alpha \phi^{0\alpha}$. Now, the time-evolution of any dynamical variable, or any function/functional, or quantity, which depends upon the complete set of dynamical variables $\{g_{\alpha\beta}, p^{\mu\nu}\}$, are determined by the Poisson bracket of this variable (or function) with the total Hamiltonian H_t, e.g., $g_{\alpha\beta,0} = [g_{\alpha\beta}, H_t]$. Note that this new (total) Hamiltonian of the metric gravity acts in the $d(d+1)$ dimensional space of the dynamical variables, in contrast with the canonical Hamiltonian H_C which formally operates in the $d(d-1)$ dimensional space of the space-like dynamical variables. In general, introduction of the new Hamiltonian H_t always brings some new motions that did not exist in the original Hamiltonian system with the canonical Hamiltonian H_C.

Immediately, the two following questions arise: (1) what is the sense of these 'additional' motions, and (2) how can they affect the actual motions of our field, which are determined by the canonical Hamiltonian H_C. To understand this and answer the questions raised, let us consider the time-evolution of the canonical Hamiltonian H_C. First of all, we can write the following general formula which describes time-evolution of the canonical Hamiltonian

$$H_C(t+\Delta) = H_C(t) + \frac{\Delta}{1!}\left(\frac{dH_C}{dt}\right) + \frac{\Delta^2}{2!}\left(\frac{d^2H_C}{dt^2}\right) + \frac{\Delta^3}{3!}\left(\frac{d^3H_C}{dt^3}\right) + \cdots , \tag{79}$$

where Δ is a small time interval. In this equation the first-order time derivative of H_C is written in the form:

$$\frac{dH_C}{dt} = [H_C, H_t] = [H_C, H_C + v_\alpha \phi^{0\alpha}] = v_\alpha [H_C, \phi^{0\alpha}] = -v_\alpha \, \chi^{0\alpha} \ , \tag{80}$$

where $\phi^{0\alpha}$ and $\chi^{0\alpha}$ are the primary and secondary first-class constraints, respectively. The explicit formulas for the $\phi^{0\alpha}$ and $\chi^{0\alpha}$ constraints are presented above (see, Equations (70) and (75)). Moreover, in this equation and everywhere below the notation $v_\alpha (= g_{0\alpha,0})$ is an arbitrary, in principle, velocity of the temporal (0α)-component of the metric tensor. In Dirac's theory this and other similar velocities, e.g., $v_\beta (= g_{0\beta,0})$, $v_\gamma (= g_{0\gamma,0})$, etc, are considered as arbitrary parameters of the method.

The second time-derivative of the canonical Hamiltonian H_C is

$$\frac{d^2 H_C}{dt^2} = \left[\frac{dH_C}{dt}, H_t \right] = [-v_\alpha \chi^{0\alpha}, H_C + v_\beta \phi^{0\beta}] = -v_\alpha D_c^\alpha - \frac{1}{2} \, v_\alpha v_\beta \, g^{\alpha\beta} \chi^{00} \tag{81}$$

where D_c^α is the α component of the Dirac closure (see, Equation (77)), while χ^{00} is the secondary constraint defined above (see, Equation (75)). The third time-derivative of the canonical Hamiltonian takes the form

$$\frac{d^3 H_C}{dt^3} = [\frac{d^2 H_C}{dt^2}, H_t] = -v_\alpha [D_c^\alpha, H_C] \quad + \quad \frac{1}{2} v_\alpha v_\beta [g^{\alpha\beta} \chi^{00}, H_C] + \frac{1}{2} v_\alpha v_\beta v_\gamma [g^{\alpha\beta} \chi^{00}, \phi^{0\gamma}]$$
$$- \quad v_\alpha v_\gamma [D_c^\alpha, \phi^{0\gamma}] \ . \tag{82}$$

In principle, such a chain of time derivatives $\frac{d^n H_C}{dt^n}$ is infinite (in contrast with the n-dimensional Maxwell electrodynamics [16]), but we have to note that all values in the right-hand sides of these equations are always represented as finite, linear (or quasi-linear) combinations of the secondary, first-class constraints only. Furthermore, the coefficients in front of each term in these expressions depends upon the $v_\alpha, v_\beta, v_\gamma$ and other similar velocities, which "are completely arbitrary and at our disposal" [3]. In other words, these velocities are arbitrary parameters in the Dirac's modification of the classical Hamilton method. It is clear that similar transformations which depend upon arbitrary parameters cannot affect the actual (Hamiltonian) motion of the original dynamical system, e.g., a free gravitational field, in our case. Instead, they produce some changes in the Hamiltonian dynamical variables, which do not correspond to a change of physical state. Generators of such 'fictional' transformations are the secondary first-class constraints (as Dirac predicted in [3]). This follows directly from Equations (80)–(82) and other similar equations for higher-order derivatives in that chain. In field theory similar transformations of the dynamical variables are well known, and in earlier papers they were called gauge transformations, or simply gauges. Dirac could obtain and write (see, e.g., [3]) all essential equations for the actual motion and for the corresponding gauge generators in the united form of Hamilton equations. This explains the overall significance of Dirac's modification of the classical Hamilton method.

In Dirac method the complete system of Hamiltonian equations for the metric gravity is written in the form

$$\frac{dg_{pq}}{dt} = g_{pq,0} = [g_{pq}, H_C] \quad \text{and} \quad \frac{dp^{mn}}{dt} = \left(p^{mm} \right)_{,0} = [p^{pq}, H_C] \ . \tag{83}$$

These $d(d-1)$ Hamilton equations describe the actual motion of a free gravitational field. Solutions of these equations cannot become v-dependent at any moment of time-evolution (see discussion above). In addition to these equations we also have d Hamilton equations which describe time-evolution of the primary constraints:

$$\frac{d\phi^{0\alpha}}{dt} = [\phi^{0\alpha}, H_C] = \chi^{0\alpha} \ , \tag{84}$$

where $\chi^{0\alpha}$ are the secondary constraints. The following group of d Hamilton equations describe time-evolution of the secondary constraints:

$$\frac{d\chi^{0\alpha}}{dt} = [\chi^{0\alpha}, H_C] = D_c^\alpha \ ,\tag{85}$$

where $D_c^\alpha = a_\mu^\alpha(g)\chi^{0\mu} + b_\mu^\alpha(g)\left(f^k(g)\chi^{0\mu}\right)_{,k}$ is the α-component of the Dirac closure, which is a quasi-linear combination of the secondary constraints $\chi^{0\sigma}$ and some total spatial derivatives of expressions which also include the same secondary constraints. All classical theory of the free gravitational field (in metric gravity) is summed up in these Dirac's equations, Equations (83)–(85), written here in a manifestly Hamiltonian form. Note also that there is a simple procedure which allows one to simplify (drastically) the explicit form of Dirac closure in the metric gravity. Indeed, in metric gravity on the shell of primary constraints we can always determine d field-dependent coefficients $\mathcal{C}_\beta^\alpha(g)$ for which the following equations are satisfied:

$$[\chi^{0\alpha} + \mathcal{C}_\beta^\alpha\phi^{0\beta}, H_C] = \Lambda_\alpha\chi^{0\alpha} \ ,\tag{86}$$

where $\alpha = 0, 1, \ldots, d-1, \beta = 0, 1, \ldots, d-1$ and $\Lambda_\alpha(g)$ are some algebraic, field-dependent expressions, which are often called 'eigenvalues' (or factor-eigenvalues) of the Dirac closure. Derivation of equations for the unknown $\mathcal{C}_\beta^\alpha(g)$ coefficients in Equation (86) is straightforward. In this procedure the Dirac closure becomes 'diagonal' and each component of Dirac closure, e.g., D_c^α always contains only one secondary constraint, e.g., $\chi^{0\alpha}$ in Equation (86). All secondary constraints in this procedure are uniquely determined as factor-eigenvectors which are defined on the shell of primary constraints. In this version of Dirac's approach we do not need to say many words to describe the internal structure of Dirac closure.

It is also important to remember that in metric gravity we always have $[\phi^{0\alpha}, \phi^{0\beta}] = 0$, which means the pair-wise commutativity of the primary constraints. These generalized Hamilton equations, Equations (83)–(85), form a complete and unambiguous set of equations, which govern the behaviour of a free gravitational field in the $d(d+1)$-dimensional space of dynamical variables, or in the original d-dimensional Riemann space. The Hamiltonian equations from the first group, Equation (83), are the canonical Hamilton equations of actual motion for true dynamical variables. Analogous equations from the second group, Equations (84) and (85), are the Hamilton equations for gauge generators. These equations determine the actual gauge generators for the given dynamical system, i.e., for the free gravitational field in our case. All equations from the second group describe certain changes in the dynamical variables, i.e., coordinates and momenta, which do not affect the real physical state.

Thus, our original $d(d+1)$-dimensional space of dynamical variables in the metric gravity splits into the $d(d-1)$-dimensional space of dynamical variables, which describe actual motions, and 2 d-dimensional space of variables, which are transformed in some way with time, but this does not make any changes in the real physical state. Formally, we can write this in the form: $S[d(d+1)] = S[d(d-1)] \oplus S[2\ d]$, where all spaces are even-dimensional. If an additional temporal variable t is introduced in our analysis, then all these three spaces become odd-dimensional and Hamilton method works perfectly in each of these spaces. Note that the Hamilton equations in the form of Equations (83)–(85) are more useful and informative for the field people, than the equivalent original system of the $d(d+1)$ Hamilton equations:

$$\frac{dg_{\alpha\beta}}{dt} = g_{\alpha\beta,0} = [g_{\alpha\beta}, H_t] \quad \text{and} \quad \frac{dp^{\mu\nu}}{dt} = \left(p^{\mu\nu}\right)_{,0} = [p^{\mu\nu}, H_t] \quad .\tag{87}$$

The replacement of this system of Hamilton equations by much more useful system of slightly different Hamilton equations, Equations (83)–(85), is the main advantage of the Dirac's modifications made in the classical Hamilton method. Another advantage follows

from the fact that the governing equations for all gauge generators are also written in the form of Hamilton equations. The third advantage is obvious: from now on all calculations in the metric gravity are reduced to analytical calculations of the Poisson brackets only.

Finally, we can formulate the complete and pure formal criterion of canonicity for some transformation between any two equivalent Hamiltonian formulations of the metric gravity. Based on arguments and equations presented in this and previous Sections, the universal criterion of canonicity for the metric gravity can be formulated in the following form. Some transformation of the dynamical Hamilton variables in metric gravity is canonical if (and only if) it transforms our original system of Hamilton equations, Equations (83)–(85), into a new system of similar Hamilton equations:

$$\frac{d\tilde{g}_{pq}}{dt} = \tilde{g}_{pq,0} = [\tilde{g}_{pq}, \tilde{H}_C] \quad \text{and} \quad \frac{d\tilde{p}^{mn}}{dt} = \left(\tilde{p}^{mm}\right)_{,0} = [\tilde{p}^{pq}, \tilde{H}_C] \ , \tag{88}$$

$$\frac{d\tilde{\phi}^{0\alpha}}{dt} = [\tilde{\phi}^{0\alpha}, \tilde{H}_C] = \tilde{\chi}^{0\alpha} \quad \text{and} \quad \frac{d\tilde{\chi}^{0\alpha}}{dt} = [\tilde{\chi}^{0\alpha}, \tilde{H}_C] = \tilde{D}_c^{\alpha} \ , \tag{89}$$

where the sign ˜ means the new variable and/or function, while all new functions $\tilde{H}_C, \tilde{\chi}^{0\alpha}$ and $\tilde{D}_c^{\alpha} = \tilde{a}_{\mu}^{\alpha}(\tilde{g})\tilde{\chi}^{0\mu} + \tilde{b}_{\mu}^{\alpha}(\tilde{g})\left(\tilde{f}^{k}(g)\tilde{\chi}^{0\mu}\right)_{,k}$, which appear in these equations, must have the same structure as the old functions $H_C, \chi^{0\alpha}$ and D_c^{α} in Equations (83)–(85). This new system of equations represents the form-invariance of the Hamilton equations derived by Dirac for the metric gravity, which is a constrained dynamical system with the first-class constraints only. Furthermore, for the true canonical transformation in the metric gravity the following equations for the Poisson brackets must be obeyed:

$$[\tilde{g}_{\alpha\beta}, \tilde{g}_{\mu\nu}] = 0 \ , \quad [\tilde{g}_{\alpha\beta}, \tilde{p}^{mn}] = \Delta_{\alpha\beta}^{mn} \ , \quad [\tilde{p}^{mn}, \tilde{p}^{pq}] = 0 \ , \quad [\tilde{\phi}^{0\gamma}, \tilde{\phi}^{0\sigma}] = 0$$

$$[\tilde{g}_{\mu\nu}, \tilde{\phi}^{0\sigma}] = \Delta_{\mu\nu}^{0\sigma} \ , \quad [\tilde{p}^{mn}, \tilde{\phi}^{0\sigma}] = 0 \ . \tag{90}$$

This criterion of canonicity can be generalized to other Hamiltonian dynamical systems with first-class constraints.

Note also that in this Section we discuss only one version of the complete Dirac's approach [1], which has been developed to deal with a free gravitational field in the metric gravity. Generalization of this procedure to other fields with non-trivial gauge invariance is also possible. For instance, the same approach works perfectly for a free electromagnetic field even in multi-dimensions (see, e.g., [3,7,16]). Our preliminary results indicate clearly that the quantum version of this approach is applicable (with some changes) to the modern united electroweak theory.

On Complete Reverse Recovery of the Original Field Equations

In the previous Sections, we have carefully derived the Hamiltonian equations for a free gravitational field and all primary and secondary first-class constraints. The main purpose of our analysis was to obtain the correct equations of motion (or time-evolution) of a free gravitational field and obtain all important gauge conditions. Here the following question immediately arises: what are these correct Hamiltonian equations of motion? Where and how was the criterion of correctness established? The answer is clear, and we have to recognize as correct only such Hamiltonian equations and first-class constraints which uniformly lead us back to the original (or maternal) equations of motion already known for our field. For a free gravitational field the maternal field equations are the Einstein's equations $G_{\alpha\beta} = 0$ (or $R_{\alpha\beta} = 0$) mentioned in Section 2. For a free electromagnetic field (or EM-field, for short) the maternal equations are the Maxwell equations in vacuum. Therefore, any correct Hamiltonian approach for EM-field must be able to produce the governing Maxwell equations at any spatial point **x** and at any moment of time t. To explain how this works, let us consider the Hamiltonian formulation for the Maxwell electromagnetic field in the $(n+1)$-dimensional (flat) space-time.

We can start directly form the explicit form of the corresponding *EM*-Hamiltonian (all missing details, definitions and notations can be found in [3,16]). Moreover, form the definition of momenta $B^\mu = F^{\mu 0}$ to this point we already have one primary constraint $\phi = B^0 \approx 0$, since both $F^{\mu\nu}$ and $F_{\mu\nu}$ tensors are always antisymmetric. The fundamental Poisson brackets are: $[A_\mu(\mathbf{x}), B^\nu(\mathbf{x}')] = g_\mu^\nu \delta^n(\mathbf{x} - \mathbf{x}')$, $[A_\mu(\mathbf{x}), A_\nu(\mathbf{x}')] = 0$ and $[B^\mu(\mathbf{x}), B^\nu(\mathbf{x}')] = 0$. The Hamiltonian of a free electromagnetic field in the $(n+1)$-dimensional space-time is written in the form [3,16]:

$$H = \int \left(\frac{1}{4} F^{pq} F_{pq} - \frac{1}{2} F^{p0} F_{p0} + F^{q0} A_{0,q} \right) d^n \mathbf{x} = \int \left(\frac{1}{4} F^{pq} F_{pq} + \frac{1}{2} B^p B^p - A_0 B^p_{,p} \right) d^n \mathbf{x} \ , \quad (91)$$

where all notations are exactly the same as in [3,10]. The corresponding Hamiltonian density takes the form

$$\mathcal{H} = \frac{1}{4} F^{pq} F_{pq} + \frac{1}{2} B^p B^p - A_0 B^p_{,p} \ . \quad (92)$$

Integration by parts of the first term in the Hamiltonian, Equation (91), leads to the following expression for the Hamiltonian density Equation (92):

$$\mathcal{H} = -\frac{1}{2} \left(F^{pq} \right)_q A_p + \frac{1}{2} B^p B^p - A_0 B^p_{,p} = \frac{1}{2} \left(\frac{\partial^2 A_q}{\partial x_p \partial x_q} - \frac{\partial^2 A_p}{\partial x_q \partial x_q} \right) A_p + \frac{1}{2} B^p B^p - A_0 B^p_{,p} \ , \quad (93)$$

where $p = 1, 2, \ldots, n$ and $q = 1, 2, \ldots, n$, i.e., all these indexes are space-like. First of all, by determining the Poisson bracket $[B^0, \mathcal{H}] = B^p_{,p}$ one finds the secondary constraint $\chi = B^p_{,p} \approx 0$. In standard notation this means that $\frac{\partial}{\partial t} \left(\frac{\partial A_p}{\partial x_p} \right) \approx 0$, or $\frac{\partial A_p}{\partial x_p} = C$, where C is a numerical constant which does not depend upon $\mathbf{x}$ and/or t. This secondary constraints χ commute with the Hamiltonian density $\mathcal{H}$ and Dirac closure equals zero identically.

By using the Hamiltonian density $\mathcal{H}$, Equation (93), we obtain the following system of canonical Hamilton equations

$$\frac{d A_p}{dt} = [A_p, \mathcal{H}] = \frac{\partial \mathcal{H}}{\partial B^p} = \frac{1}{2} \left(2 B^p \right) = B^p \quad (94)$$

and

$$\frac{d B^p}{dt} = [B^p, \mathcal{H}] = -\frac{\partial \mathcal{H}}{\partial A_p} = -\frac{1}{2} \left[2 \left(\frac{\partial^2 A_q}{\partial x_q \partial x_p} - \frac{\partial^2 A_p}{\partial x_q \partial x_q} \right) \right] = \frac{\partial^2 A_p}{\partial x_q \partial x_q} - \frac{\partial^2 A_q}{\partial x_q \partial x_p} \ . \quad (95)$$

Combination of these two equations one finds

$$\frac{d^2 A_p}{dt^2} = \frac{\partial^2 A_p}{\partial x_q \partial x_q} - \frac{\partial^2 A_q}{\partial x_q \partial x_p} \ . \quad (96)$$

Now, taking into account the condition which follows from secondary constraint: $\frac{\partial A_q}{\partial x_q} = C$, we can reduce this equation to the form of n-dimensional wave equation:

$$\frac{\partial^2 A_p}{\partial t^2} - \frac{\partial^2 A_p}{\partial x_q \partial x_q} = 0 \ , \text{ or } \ \frac{\partial^2 \mathbf{A}}{\partial t^2} - \Delta \mathbf{A} = 0 \ , \quad (97)$$

where $\mathbf{A} = (A_1, A_2, \ldots, A_n)$ is the n-dimensional vector potential of the EM-field. This is the Maxwell equations for a free electromagnetic field in the $(n+1)$-dimensional space-time. The n-dimensional Laplace operator Δ in this equation is $\Delta = \frac{\partial^2}{\partial x_q \partial x_q}$. Thus, we have recovered all Maxwell equations of the free radiation field. Note here that if someone does not recognize any constraints at all, or these constraints were determined with mistakes, then such a Hamiltonian formulation of the electromagnetic theory does not allow one to

recover the corresponding Maxwell equations. Formally, in this case any relation with the original Maxwell theory of radiation will be lost. In reality, this means that our Hamiltonian, Equation (91), does not describe the Maxwell EM-field in vacuum, or this Hamiltonian formulation is not valid for the Maxwell electromagnetic field and cannot be used in applications to this field. This is the principle of complete reverse recovery of the original field equations in application to the Maxwell theory of EM-field.

Now, consider the Hamiltonian formulation of the metric gravity which has been developed by Dirac in [1]. In contrast with a free Maxwell EM-field, for a free gravitational field, everything becomes significantly more complicated, but our principle of the complete reverse recovery works in this case too. Recently, we have shown that the equations for the second-order temporal derivatives of the space-like co-variant components g_{mn} of the metric tensor, which follow from the Hamilton equations obtained in the Dirac Hamiltonian formulation of the metric gravity, essentially coincide with the corresponding Einstein's equations for the same components. However, at that time this article was already completed and it was not possible to add a few new chapters into it. In addition to this, it takes a long time to transform a set of difficult formulas into a logically perfect text. Therefore, our results in this direction will be published some time later and elsewhere. Here we just want to present a few important details of the procedure used.

Note that there are three peculiarities in the Einstein's equations for a free gravitational field ($R_{\alpha\beta} = 0$ plus d additional conditions $R_{\alpha,\gamma}^{\gamma} = \frac{1}{2}\frac{\partial R}{\partial x^{\alpha}}$), which are crucially important for our present purposes. First, these Einstein's equations are written as a system of differential equations which contains the first- the second-order temporal derivatives of the metric tensor $g_{\alpha\beta}$. Second, all second-order temporal derivatives from the $g_{0\beta}$ and $g_{\beta 0}$ components of metric tensor cancel out from these Einstein's equations. Third, the temporal second-order derivatives of the spatial components of metric tensor are explicitly included in the Einstein's equations. In fact, each second-order derivative $\frac{d^2 g_{mn}}{dt^2} = \frac{d^2 g_{mn}}{dx_0^2}$, arises in Einstein's equations only from the R_{0m0n} components of the Riemann curvature tensor. From here it is easy to find that all these second-order temporal derivatives $\frac{d^2 g_{mn}}{dt^2}$ are included in Einstein's equations only as separated terms, and each of these terms has the same numerical coefficient $-\frac{1}{2}$ in front. As follows from here one can reduce the Einstein's equations for the g_{mn} components to the form $\frac{d^2 g_{mn}}{dt^2} = Q(g_{pq}, \frac{dg_{pq}}{dt}, g_{0\alpha}, \frac{dg_{0\alpha}}{dt})$. These equations must coincide (or be equivalent) with the analogous equations for the $\frac{d^2 g_{mn}}{dt^2}$ derivatives which follow from the Hamilton equations in the Dirac formulation of the Hamiltonian metric gravity. In fact, we have to calculate the Poisson bracket $\frac{d^2 g_{mn}}{dt^2} = [[g_{mn}, H_t], H_t] = [[g_{mn}, H_C], H_t]$ and exclude all momenta by using the Hamilton equations, primary and secondary first-class constraints derived in the Dirac Hamiltonian formulation (see above). Such calculations are quite complex, extremely time-consuming and very sensitive, since any mistake made either in the Hamiltonian, or in one of the first-class constraints substantially complicates further calculations. After such a mistake one can lose any relation with the original (or maternal) theory and cannot move forward, until this mistake is found and corrected. Nevertheless, after many weeks of calculations, we are happy to report that Hamiltonian formulation of the metric gravity, which has been developed by Dirac in [1], successfully passed this our test of recovery (at least partially). Now, by using this Hamiltonian formulation we are able to recover the original field equations for all covariant space-like components g_{mn} of the metric tensor. An alternative Hamiltonian formulation of metric gravity which obviously fails this test is mentioned in Section 9.

To conclude this discussion we have to note that any correct Hamiltonian formulation of an arbitrary, in principle, field theory must reproduce (exactly and unambiguously) the original governing equations of this field. The correct Hamilton equations of motions and explicit form of all first-class constraints are crucially important to reach this goal. If this is not the case, then such a Hamiltonian theory is wrong and has nothing to do with the maternal field theory.

8. Invariant Integrals of the Metric Gravity

This Section is a central part of our study, since here we define a number of integral invariants of the metric gravity, i.e., we reach an absolute top of the classical Hamilton mechanics. Obviously, in one Section we cannot even outline the main problems which exist in the theory of integral invariants and its applications to the Hamiltonian metric gravity. Therefore, the following presentation of this theory will be very brief. Moreover, here we restrict ourselves only to a description of the extension of integral invariants for dynamical (Hamiltonian) systems with constraints. More details of the theory of integral invariants and its applications to the Hamiltonian formulations of metric gravity will be presented in our next article [18].

First of all we need to define the one-dimensional, or line integrals in the metric gravity. In general, the one-dimensional integral in multi-dimensional Riemann spaces is defined as follows

$$
\begin{aligned}
\int \pi^{\alpha\beta} \, dg_{\alpha\beta} &= \int \pi^{\alpha\beta} \left(\frac{\partial g_{\alpha\beta}}{\partial x^\beta} \right) dx^\beta = \int \pi^{\alpha\beta} \left(\Gamma_{\alpha,\beta\gamma} + \Gamma_{\beta,\alpha\gamma} \right) dx^\gamma \\
&= \int \pi^{\alpha\beta} \left(g_{\alpha\lambda} \Gamma^\lambda_{\beta\gamma} + g_{\beta\lambda} \Gamma^\lambda_{\alpha\gamma} \right) dx^\gamma \ ,
\end{aligned}
\tag{98}
$$

where $\Gamma_{\alpha,\beta\gamma}$ are the Cristoffel symbols of the first kind, while $\Gamma^\alpha_{\beta\gamma}$ are the Cristoffel symbols of the second kind (see, e.g., [4]). The integrand in this integral is not a tensor. This means that the line (or one-dimensional) integral substantially depends on the curve along which it is calculated and also on the initial and final points chosen on this curve. If the start and end points coincide with each other, then such an integral is called a closed loop integral, or an integral taken along a closed loop. Below, similar closed loop integrals are designated by the sign $\oint$. In general, the complete theory of line (or one-dimensional) integrals in multi-dimensional Riemann spaces is very complex. However, for our current analysis of the integral invariants in the Hamiltonian formulations of metric gravity we do need to use the formula, Equation (98). In fact, all line integrals can be considered in the $\frac{d(d+1)}{2}$-dimensional pseudo-Euclidean (orthogonal) space, which is formally identical (or isomorphic) to our original d-dimensional Riemann space-time (for more details, see, e.g., [4,5]).

This is very good news, since all integrals and integral forms defined in multi-dimensional pseudo-Euclidean spaces can be handled in a familiar way (see, e.g., [19,20]). In particular, by applying the usual definition of the closed loop integrals in pseudo-Euclidean spaces we can consider the two following integrals in the $\frac{d(d-1)}{2}$-dimensional space:

$$
I = \oint \left[\pi^{mn} \, dg_{mn} - H_C dt \right] \quad \text{and} \quad I_P = \oint \pi^{mn} \, dg_{mn} \ ,
\tag{99}
$$

where π^{mn} are the space-like components of momenta, while H_C is the canonical Hamiltonian which has been defined in Section 3. For now we restrict ourselves to the consideration of the space-like components of momenta π^{mn} and coordinates g_{mn}. In other words, below we shall deal with the $\frac{d(d-1)}{2}$ dimensional Euclidean position space (and $d(d-1)$-dimensional phase space) instead of the original $(d-1)$-dimensional sub-space in our original d-dimensional Riemann space-time. The coordinates in this position space coincide with the covariant components of the fundamental space-like tensor g_{mn}. The first integral in Equation (99) is called the Poincare–Cartan integral invariant, while the second integral is the Poincare integral invariant [21,22] which is often called the main integral invariant of mechanics (or Hamiltonian mechanics). This Poincare integral invariant has a fundamental value for the Hamiltonian formulation of metric gravity as well as for the general theory of canonical transformations in metric gravity and for analysis and solution of other gravitational problems. Indeed, it is relatively easy to prove the following statement. If for

some system of first-order differential equations written for the space-like components of the metric tensor g_{mn} and momenta π_{mn}:

$$\frac{dg_{mn}}{dt} = Q_{mn}(t, g_{ab}, \pi^{pq}) \quad , \quad \frac{d\pi^{mn}}{dt} = P^{mn}(t, g_{ab}, \pi^{pq}) \tag{100}$$

the Poincare integral I_P, Equation (99), is invariant, then this system of equations, Equation (100), is Hamiltonian in the moment of time t which is located between $t - \delta$ and $t + \delta$, where δ is a very small positive number. This term 'Hamiltonian' means here that the functions $Q_{mn}(t, g_{ab}, \pi^{pq})$ and $P^{mn}(t, g_{ab}, \pi^{pq})$ from the right-hand side of Equations (100) are represented as the partial derivatives (or Poisson brackets) of some scalar function H, i.e.,

$$Q_{mn}(t, g_{ab}, \pi^{pq}) = \frac{\partial H}{\partial \pi^{mn}} = [g_{mn}, H] \quad , \quad P^{mn}(t, g_{ab}, \pi^{pq}) = -\frac{\partial H}{\partial g_{mn}} = [\pi^{mn}, H] \quad , \tag{101}$$

where the notation $[a, b]$ stands for the Poisson bracket defined by Equation (18). A uniform reconstruction of the explicit form of this function H (or Hamiltonian) is not an easy task, but if we know that the Poincare–Cartan integral is also an integral invariant, then the unknown Hamiltonian exactly coincides [21] with the canonical Hamiltonian H_C mentioned in the first integral from Equation (99).

Let us discuss the following fundamental question. We shall assume that the integral I, Equation (99), is an integral invariant for our dynamical system and H_C is our Hamiltonian which describes the actual motion of this system, i.e., the equations of time-evolution take the form of Hamilton equations, Equation (83), for our $d(d-1)$ dynamical variables $\{g_{mn}, \pi^{pq}\}$, where $[g_{mn}, g_{pq}] = 0, [\pi^{mn}, \pi^{pq}] = 0$ and $[g_{mn}, \pi^{pq}] = \Delta^{pq}_{mn}$. Now, we want to extend our phase space by adding a set of $2d$ new dynamical variables $\{g_{0\gamma}, \pi^{0\gamma}\}$. Here we assume the usual permutation symmetry for all 'additional' coordinates and momenta: $g_{0\gamma} = g_{\gamma 0}$ and $\pi^{0\gamma} = \pi^{\gamma 0}$. The total dimension of this new phase space will be $d(d+1)$, which corresponds to the d-dimensional Riemann space-time, and it is the main working space for the general relativity and metric gravity. We require that in this new extended phase space the integral defined by the last expression in Equation (102) must also be an integral invariant with the new Hamiltonian H_t. Furthermore, the new 'extended' Hamiltonian H_t must be closely related with the canonical Hamiltonian H_C from Equation (99).

Based on the formulas derived in Section 3 we can transform the Poincare–Cartan integral invariant from Equation (99) into the following form

$$
\begin{aligned}
I = \oint \left[\pi^{mn} \, dg_{mn} - H_C dt \right] &= \oint \left\{ \left[\pi^{mn} \, dg_{mn} + \pi^{0\gamma} \left(\frac{dg_{0\gamma}}{dt} \right) dt \right] - \left[H_C + \pi^{0\gamma} \left(\frac{dg_{0\gamma}}{dt} \right) \right] dt \right\} \\
&= \oint \left[\pi^{\alpha\beta} \, dg_{\alpha\beta} - \left(H_C + v_\gamma \pi^{0\gamma} \right) dt \right] = \oint \left[\pi^{\alpha\beta} \, dg_{\alpha\beta} - H_t dt \right] \quad ,
\end{aligned}
\tag{102}
$$

where $H_t = H_C + v_\gamma \pi^{0\gamma} = H_C + g_{0\gamma,0} \pi^{0\gamma}$ and $g_{0\gamma,0} = v_\gamma$ ($\gamma = 0, 1, \ldots, d-1$) are the corresponding velocities, while $\pi^{0\gamma}$ are the temporal momenta which must be equal zero along each Hamilton trajectory of the actual motion. In other words, we have d primary constraints $\pi^{0\gamma} \approx 0$ in the metric gravity. Otherwise, i.e., if $\pi^{0\gamma} \neq 0$ for some γ, then Equation (102) does not hold. Another crucial fact which has been used to transform, Equation (102), follows from the formulas for canonical Hamiltonian H_C, Equations (17) and (72), which do not contain any of the temporal momenta $\pi^{0\gamma}$, but it may include some of the $g_{0\mu}$ and/or $g_{\mu 0}$ coordinates. If these conditions are obeyed, then from Equation (102) we can derive the following equality:

$$\oint \left[\pi^{mn} \, dg_{mn} - H_C dt \right] = I = \oint \left[\pi^{\alpha\beta} \, dg_{\alpha\beta} - H_t dt \right] \quad , \tag{103}$$

which means that both of these integrals, i.e., integrals on the left and right sides of this equation, are the true integral invariants and their numerical values equal to each other. In other words, these two integral invariants coincide with each other, i.e., they are not independent, and for constrained dynamical systems we always have to deal with this complication.

The formula, Equation (103), has a number of consequences for Hamiltonian formulations of the metric gravity, but here we consider just one of them. First, as follows from the left-hand side of Equation (103) the set of dynamical variables $\{g_{mn}, \pi^{pq}\}$ will be canonical, if the Poisson brackets between these dynamical variables coincide with CUSPB, i.e., $[g_{mn}, g_{pq}] = 0, [g_{mn}, \pi^{pq}] = \Delta_{mn}^{pq}, [\pi^{mn}, \pi^{pq}] = 0$ for all possible (mn)- and (pq)-pairs. In other words, it is a necessary and sufficient condition of canonicity in this case. However, if we apply the same arguments to the integral in right-hand side of Equation (103) we can only say that the exact coincidence of the Poisson brackets between dynamical variables $[g_{\alpha\beta}, g_{\mu\nu}] = 0, [g_{\alpha\beta}, \pi^{\mu\nu}] = \Delta_{\alpha\beta}^{\mu\nu}, [\pi^{\alpha\beta}, \pi^{\mu\nu}] = 0$ with the standard CUSPB values is only necessary (but not sufficient!) condition of canonicity. In order to obtain the sufficient conditions of canonicity we must also guarantee that all temporal momenta $\pi^{0\gamma}$ and/or $\pi^{\gamma 0}$ do not change with time t along the true Hamilton trajectories. This means that all time-derivatives of the temporal momenta must be equal zero at all times, i.e., we have a number of additional equations such as $\pi^{0\gamma} \approx 0, [\pi^{0\gamma}, H_t] \approx 0, [[\pi^{0\gamma}, H_t], H_t] \approx 0$, etc. To obey all these equations for the primary, secondary and other constraints we have to follow Dirac's modifications made in the classical Hamilton method for constrained dynamical systems (see above). Otherwise, if some of these conditions do not hold, then the numerical value of the integral in the right-hand side of Equation (103) will be different from I, i.e., this integral is not invariant in this case and we have an obvious contradiction here. This explains why the criteria of canonicity derived for constrained dynamical systems always include two parts: (a) coincidence of the Poisson brackets between dynamical variables with the standard CUSPB values, and (b) conservation of the algebra of first-class constraints. This presumes the form-invariance of all first-class constraints and Poisson brackets of these constraints with each other, and with the canonical/total Hamiltonians and other essential functions of dynamical variables.

9. Conclusions

Thus, we have investigated the two different Hamiltonian formulations [1,6] of the metric gravity in d-dimensional Riemann space, where $d \geq 3$. These two Hamiltonian formulations are related to each other by a canonical transformation of dynamical variables in the $d(d+1)$-dimensional phase space and each of them allows one to restore the complete d-dimensional diffeomorphism as the correct (and well-known) gauge invariance of the free gravitational field in the metric gravity. By using the known canonical transformation between these two Hamiltonian formulations of the metric gravity we have investigated the basic properties of other similar canonical transformations and derived some useful criteria of canonicity for an arbitrary transformation of the Hamiltonian dynamical variables in the $d(d+1)$-dimensional phase space. The results of our study are important in numerous applications, since in metric gravity canonical transformations of Hamiltonian dynamical variables are often used to simplify either the canonical H_C and/or total Hamiltonian(s) H_t, or secondary constraints, or to reduce the canonical Hamiltonian H_C to some special form, e.g., to its normal form, which is well known in classical mechanics.

In general, all criteria of canonicity for transformations of dynamical variables in the metric gravity require the exact coincidence of the Poisson (or Laplace) brackets for the new and old dynamical (Hamiltonian) variables. Briefly, if the Poisson brackets of the new dynamical variables (expressed in the old dynamical variables) do not coincide with their canonical values, then such a transformation is not canonical. This is the universal criterion of canonicity which is known from classical mechanics of Hamiltonian dynamical systems. However, in all Hamiltonian formulations of metric gravity we always deal with the constrained dynamical systems. Therefore, all criteria of canonicity, which are

valid for such systems, must contain the second part which deals with the algebra of first-class constraints, form-invariance of the canonical/total Hamiltonian(s) and/or form-invariance of the Hamilton equations. For instance, the true canonical transformation in the metric gravity must keep form-invariance of the Hamilton equations derived in the Dirac's modification of the classical Hamilton method. It can be illustrated by the transformation of Equations (83)–(85) into Equations (88) and (89) during our canonical transformation of the Hamiltonian dynamical variables. This is the first criterion of canonicity in the metric which is relatively simple and ready to be applied to actual problems. The second similar criterion [17] requires the exact coincidence of the total Hamiltonian H_t and preservation of the algebra of constraints for both (old and new) Hamiltonian formulations of the metric gravity.

We have also reconsidered modifications made by Dirac [1–3] in the classical Hamilton approach. It is shown that these modifications are crucial to improve overall efficiency of the new Hamiltonian method for dynamical systems with constraints, including various physical fields with additional gauge conditions, or gauges, for short. The main advantage of the new Dirac's approach is a possibility to write all governing equations in the united, manifestly Hamilton form (see Equations (83)–(85) and Equations (88) and (89) above). The original Dirac's idea that all motions in Hamiltonian dynamical systems with first-class constraints can always be separated into actual motions and special motions along constraints (or gauge-consistent motions) was extremely productive. Now, by using this Dirac's modification of the classical Hamilton method we can describe time-evolution of a large number of actual and model fields. Furthermore, we can make a conjecture that the free fields which represents all currently known fundamental interactions can unambiguously be described by the this version of Hamiltonian method, which was originally developed and later modified by Dirac.

In this study we have also considered the method of integral invariants and applied it to investigate canonical transformations between different Hamiltonian formulations of the metric gravity. This method was originally proposed and developed by Poincare and Cartan [21]. Since then it was transformed into a very powerful approach, which currently is an absolute tool in the Hamilton classical mechanics. In reality, the invariance of the Poincare–Cartan integral can be chosen as a foundation of the whole Hamiltonian mechanics. Indeed, if this integral is invariant for some dynamical system, then such a system is Hamiltonian and its time-evolution is described by a system of Hamilton equations. For Hamiltonian dynamical systems with constraints the general theory of integral invariants must be modified, but its overall power still remains outstanding.

Unfortunately, the limited space of this article did not allow us to discuss other important directions of the Hamiltonian formulations of metric gravity. In particular, we could not consider the explicit derivation of the gauge generators which are defined by chain of the first-class constraints [6,23] (see also [10]). Moreover, in this study, we did not even mention various non-canonical quasi-Hamiltonian formulations of the metric gravity. However, we can make a reference to an excellent review article [24] which contains a detailed analysis of this problem and a large number of references to papers published up to the beginning of 2011. Here we want to note that any of these non-canonical Hamiltonian formulations uses a set of dynamical ADM-variables, which were introduced in [25]. This fact has been noticed and criticized by Bergmann, Dirac and many others. Our calculations of the corresponding Poisson brackets can be found in [10,17]. However, since early 1960's there were no explanations of this remarkable fact and its consequences neither from ADM people, nor from their followers (see, e.g., [26–28]). Then, in 1985 it was suddenly detected that ADM formulation of the Hamiltonian metric gravity cannot restore, in principle, the total four-dimensional diffeomorphism [29] which is the correct and well-known gauge symmetry of a free gravitational field in four-dimensional space-time. Recently, we have found another crucial problem for ADM gravity and similar non-canonical 'Hamiltonian' formulations. Indeed, in Dirac's Hamiltonian formulation, we could restore the original Einstein's equations for a free gravitational field. Analogous ADM

Hamiltonian formulation uses, in part, the same dynamical variables (12 of 20 variables), but there are some fundamental mistakes in all secondary constraints. Therefore, the extra terms which present in the restored field equations for ADM formulation do not cancel each other (as they do in Dirac's formulation), but remain and even multiply. Finally, in Dirac's Hamiltonian formulation we obtain the maternal Einstein equations with no additional terms, while for ADM Hamiltonian formulation we have similar equations with many extra terms in them. As follows from this fact the ADM Hamiltonian formulation either describes some different (i.e., non-Einstein's) field, or it is an absolutely wrong theoretical construction which does not represent and real and/or model field (if the arising system of extended Einstein-like equations is not closed). In the future, under better circumstances, we plan to discuss these (and other) issues which currently exist in the Hamiltonian formulations of the metric gravity.

Conflicts of Interest: The author declares no conflict of interest.

Appendix A. Some Useful Relations in the Hamiltonian Version of Metric Gravity

In this 'pure technical' Appendix we derive a few equations and relations which are crucially important for our Hamiltonian formulation of the metric gravity. First, let us define the space-like tensor I_{mnpq} by the equation $I_{mnpq} = ag_{mn}g_{pq} - g_{mp}g_{nq}$, Here we want to show that $I_{mnpq}E^{pqkl} = \delta_m^k \delta_n^l$ in the case when $a = \frac{1}{d-2}$. First of all, we note that $g_{\alpha\beta}g^{\alpha\beta} = d$, where d is the total dimension of our Riemann space. From here one finds

$$g_{\alpha\beta}e^{\alpha\beta} = g_{\alpha\beta}g^{\alpha\beta} - g_{\alpha\beta}\frac{g^{0\alpha}g^{0\beta}}{g^{00}} = d - g_\alpha^0 \frac{g^{0\alpha}}{g^{00}} = d - 1 \ . \tag{A1}$$

On the other hand, since $e^{0\beta} = 0$, $e^{\alpha 0} = 0$ and $e^{00} = 0$, we can write the last equality in a different form

$$d - 1 = g_{\alpha\beta}e^{\alpha\beta} = g_{pq}e^{pq} + g_{p0}e^{p0} + g_{0q}e^{0q} + g_{00}e^{00} = g_{pq}e^{pq} \ . \tag{A2}$$

In other words, we obtain $g_{pq}e^{pq} = d - 1 = g_{\alpha\beta}e^{\alpha\beta}$. Furthermore, one can derive a similar rule to lower the index in the $e^{\alpha\beta}$ tensor: $g_{\alpha\beta}e^{\beta\gamma} = e_\alpha^\gamma = g_{pq}e^{qm} = e_p^m = g_p^m = \delta_p^m$.

Now, we can prove the statement formulated above. The formula for the $I_{mnpq}E^{pqkl}$ product takes the form

$$
\begin{aligned}
I_{mnpq}E^{pqkl} &= \left(ag_{mn}g_{pq} - g_{mp}g_{nq}\right)\left(e^{pq}e^{kl} - e^{pk}e^{ql}\right) = a(d-1)g_{mn}e^{kl} - g_{mp}\delta_n^p e^{kl} \\
&- ag_{mn}e^{pk}\delta_p^l + \delta_m^k\delta_n^l = [a(d-1) - 1 - a]g_{mn}e^{kl} + \delta_m^k\delta_n^l \ .
\end{aligned}
\tag{A3}
$$

From here one finds that if $a(d-2) = 1$ (or $a = \frac{1}{d-2}$), then the first term in the last equation equals zero identically and $I_{mnpq}E^{pqkl} = \delta_m^k\delta_n^l$.

Another important relation which we want to prove here is the connection between the $[g_{\alpha\beta}, \pi^{\mu\nu}]$ and $[g^{\alpha\beta}, \pi^{\mu\nu}]$ Poisson brackets. Formally, the covariant components of the metric tensor $g_{\alpha\beta}$ are the only generalized coordinates in the metric gravity. However, there is an obvious relation between covariant and contravariant components of the metric tensor: $g_{\alpha\beta}g^{\beta\gamma} = g_\alpha^\gamma = \delta_\alpha^\gamma$. From here one finds the following relation between the two Poisson brackets

$$[\pi^{\mu\nu}, g_{\alpha\beta}]g^{\beta\gamma} + g_{\alpha\beta}[\pi^{\mu\nu}, g^{\beta\gamma}] = 0 \text{ or } [g_{\alpha\beta}, \pi^{\mu\nu}]g^{\beta\gamma} = -g_{\alpha\beta}[g^{\beta\gamma}, \pi^{\mu\nu}] \ . \tag{A4}$$

By multiplying both sides of this equation by the tensor $g^{\alpha\sigma}$ we obtain the following relation

$$g^{\alpha\sigma}[g_{\alpha\beta}, \pi^{\mu\nu}]g^{\beta\gamma} = -g_{\alpha\beta}g^{\alpha\sigma}[g^{\beta\gamma}, \pi^{\mu\nu}] = -g^{\sigma}_{\beta}[g^{\beta\gamma}, \pi^{\mu\nu}] = -[g^{\gamma\sigma}, \pi^{\mu\nu}] \ , \qquad (A5)$$

or

$$[g^{\gamma\sigma}, \pi^{\mu\nu}] = -g^{\alpha\sigma} \, \Delta^{\mu\nu}_{\alpha\beta} \, g^{\beta\gamma} = -\frac{1}{2}\left(g^{\mu\sigma}g^{\nu\gamma} + g^{\mu\gamma}g^{\nu\sigma}\right) = -[\pi^{\mu\nu}, g^{\gamma\sigma}] \ , \qquad (A6)$$

i.e., the result which exactly coincides with the formulas, Equations (21) and (24), from the main text. The derivation of two other formulas from Equations (25) is absolutely analogous.

The last formula, which we want to derive in this Appendix, is the Poisson brackets between two secondary, fist-class constraints of the metric gravity, i.e., $[\chi^{0\sigma}, \chi^{0\gamma}]$. This formula is needed to complete our Hamiltonian formulation of the metric gravity. Furthermore, it is of great interest in a number of gravitational problems. This formula has never been produced in earlier studies, since its direct derivation is not an easy task. Below, we apply a different approach [10] which is based on the Jacobi identity:

$$\begin{aligned} [\quad \chi^{0\sigma}, \chi^{0\gamma}] &= [\chi^{0\sigma}, [\phi^{0\gamma}, H_C]] = -[\phi^{0\gamma}, [H_C, \chi^{0\sigma}]] - [H_C, [\chi^{0\sigma}, \phi^{0\gamma}]] \\ &= [\phi^{0\gamma}, [\chi^{0\sigma}, H_C]] + [H_C, [\phi^{0\gamma}, \chi^{0\sigma}]] = [\phi^{0\gamma}, D^{\sigma}_C] - \frac{1}{2}[g^{\gamma\sigma}, H_C] \ , \qquad (A7) \end{aligned}$$

where D^{σ}_C is the σ-component of the Dirac's closure, Equation (77), while H_C is the canonical Hamiltonian of the metric gravity, Equation (72). Here we apply the expressions derived in the Dirac's Hamiltonian formulation of metric gravity.

Analytical calculations of both terms in the last equation from Equation (A7) are relatively easy, since only a few Poisson brackets really contribute. Here we just present the final result for the second term in Equation (A7):

$$\begin{aligned} -\frac{1}{2}[g^{\gamma\sigma}, H_C] &= \frac{1}{4\sqrt{-g}g^{00}} I_{mnpq}\left[\left(g^{\gamma m}g^{\sigma n} + g^{\gamma n}g^{\sigma m}\right)\phi^{pq} + \phi^{mn}\left(g^{\gamma p}g^{\sigma q} + g^{\gamma q}g^{\sigma p}\right)\right] \\ &\quad - \frac{1}{4g^{00}}\left(g^{\gamma m}g^{\sigma n} + g^{\gamma n}g^{\sigma m}\right)\left(g^{0k}g_{mn,k} - 2g^{0\alpha}g_{\alpha n,m}\right) \ . \qquad (A8) \end{aligned}$$

This expression can be simplified, e.g., by using the identities such as $g_{mn}g^{m\gamma} = g_{\alpha n}g^{\alpha\gamma} - g_{0n}g^{0\gamma} = g^{\gamma}_n - g_{0n}g^{0\gamma}$, but here we do not want to make similar simplifications.

In order to determine the first Poisson bracket in Equation (A7) we note that the σ-component of the Dirac closure can be written in the compact form of quasi-linear combination of secondary first-class constraints $D^{\sigma}_C = V^{\sigma}_{\lambda}\chi^{0\lambda} = V^{\sigma}_k\chi^{0k} + V^{\sigma}_0\chi^{00}$, where $V^{\sigma}_{\lambda} = V^{\sigma}_{\lambda}(g_{pq}, \phi^{mn}, g^{0\mu})$ is the structure functions (or functional) [17]. Therefore, for the Poisson bracket in Equation (A8) we can write $[\phi^{0\gamma}, D^{\sigma}_C] = [\phi^{0\gamma}, V^{\sigma}_{\lambda}] \, \chi^{0\lambda} + V^{\sigma}_{\lambda} \, [\phi^{0\gamma}, \chi^{0\lambda}]$, where $\lambda = (0, k)$. In these notations the explicit formula for the second term is written in the form:

$$\begin{aligned} V^{\sigma}_{\lambda}[\phi^{0\gamma}, \chi^{0\lambda}] &= -\Big[\frac{g^{\sigma m}}{\sqrt{-g}g^{00}} I_{pqmk}\phi^{pq} + \frac{1}{2}g^{\sigma 0}g_{00,k} + g^{\sigma p}g_{0p,k} + \frac{g^{\sigma p}g^{0q}}{2g^{00}}\left(g_{pq,k} + g_{qk,p}\right. \\ &\quad - \left. g_{pk,q}\right)\Big]g^{\gamma k}\chi^{00} - \frac{1}{2}g^{\sigma}_0(g^{\gamma k}\chi^{00})_{,k} + \frac{1}{2}g^{0\gamma}g^{\sigma k}g_{00,k}\chi^{00} \ , \qquad (A9) \end{aligned}$$

where $g^{\alpha}_{\beta} = \delta^{\alpha}_{\beta}$ is the substitution tensor. An analogous formula for the first term is derived either by applying the following expression $[\phi^{0\gamma}, V^{\sigma}_{\lambda}] = -\frac{\partial V^{\sigma}_{\lambda}}{\partial g_{0\gamma}}$, or directly. The final expression takes the form

$$
\begin{aligned}
{[\phi^{0\gamma}, V^{\sigma}_{\lambda}]\chi^{0\lambda}} &= \left[\frac{1}{\sqrt{-g}g^{00}} \left(g^{0\sigma}g^{\gamma m} + g^{\gamma\sigma}g^{0m} - 3g^{0\gamma}g^{\sigma m} \right) I_{pqmk}\phi^{pq} \right]\chi^{0k} \\
&\quad - \delta^{\gamma}_0 \left(g^{0\sigma}\chi^{0k} \right)_{,k} - \delta^{\gamma}_p \left(g^{\sigma p}\chi^{0k} \right)_{,k} + \frac{1}{2}\delta^{\gamma}_0 \left(g^{\sigma k}\chi^{0k} \right)_{,k}
\end{aligned} \tag{A10}
$$

The complete formula for the $[\chi^{0\sigma}, \chi^{0\gamma}]$ Poisson bracket is the algebraic sum of three expressions from Equations (A8)–(A10).

Appendix B. On Canonicity of the Hamiltonian Dynamical Variables

Let us briefly discuss the criteria of canonicity for the different sets of dynamical variables which describe the same Hamiltonian system. In almost all applications of similar criteria in classical mechanics it is important to know that the new set of dynamical variables will be canonical, if the old set of dynamical variables was canonical. For simplicity, below the new dynamical variables $\tilde{q}_i$ and $\tilde{p}_j$ are designated by the upper signs $\tilde{\ }$, while old dynamical variables are denoted as q_i and p_j. Now, we can define the following $2n \times 2n$ Jacobi matrix $\hat{M}$ of some transformation of the dynamical variables:

$$
\hat{M} = \begin{pmatrix}
\frac{\partial \tilde{q}_1}{\partial q_1} & \cdots & \frac{\partial \tilde{q}_1}{\partial q_n} & \frac{\partial \tilde{q}_1}{\partial p_1} & \cdots & \frac{\partial \tilde{q}_1}{\partial p_n} \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
\frac{\partial \tilde{q}_n}{\partial q_n} & \cdots & \frac{\partial \tilde{q}_n}{\partial q_n} & \frac{\partial \tilde{q}_n}{\partial p_1} & \cdots & \frac{\partial \tilde{q}_n}{\partial p_n} \\
\frac{\partial \tilde{p}_1}{\partial q_1} & \cdots & \frac{\partial \tilde{p}_1}{\partial q_n} & \frac{\partial \tilde{p}_1}{\partial p_1} & \cdots & \frac{\partial \tilde{p}_1}{\partial p_n} \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
\frac{\partial \tilde{p}_n}{\partial q_n} & \cdots & \frac{\partial \tilde{p}_n}{\partial q_n} & \frac{\partial \tilde{p}_n}{\partial p_1} & \cdots & \frac{\partial \tilde{p}_n}{\partial p_n}
\end{pmatrix}
= \begin{pmatrix}
\frac{\partial \tilde{\mathbf{q}}}{\partial \mathbf{q}} & \frac{\partial \tilde{\mathbf{q}}}{\partial \mathbf{p}} \\
\frac{\partial \tilde{\mathbf{p}}}{\partial \mathbf{q}} & \frac{\partial \tilde{\mathbf{p}}}{\partial \mathbf{p}}
\end{pmatrix}. \tag{A11}
$$

where $\frac{\partial \tilde{\mathbf{q}}}{\partial \mathbf{q}}, \frac{\partial \tilde{\mathbf{q}}}{\partial \mathbf{p}}, \frac{\partial \tilde{\mathbf{p}}}{\partial \mathbf{q}}$ and $\frac{\partial \tilde{\mathbf{p}}}{\partial \mathbf{p}}$ are the $n \times n$ matrices. Another matrix which we need here is the unit simplectic $2n \times 2n$ matrix $\hat{J}$:

$$
\hat{J} = \begin{pmatrix}
0 & \cdots & 0 & -1 & \cdots & 0 \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
0 & \cdots & 0 & 0 & \cdots & -1 \\
1 & \cdots & 0 & 0 & \cdots & 0 \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
0 & \cdots & 1 & 0 & \cdots & 0
\end{pmatrix}
= \begin{pmatrix}
0 & -E \\
E & 0
\end{pmatrix}. \tag{A12}
$$

where E is the $n \times n$ unit matrix. It easy to show that the matrix $\hat{J}$ is invertable and it obeys the following equation: $\hat{J}^2 = -\hat{J}\hat{J}^{-1}$, or $\hat{J}^{-1} = -\hat{J}$. This fundamental property of the unit simplectic matrix $\hat{J}$ substantially determines many known properties of the canonical transformations and predicts a number of necessary steps in the Hamilton method. As follows from Equation (A12) the matrix $\hat{J}$ is not self-adjoint, but its product with the imaginary unit ι, i.e., the matrix $\iota\hat{J}$ is a truly self-adjoint matrix. This fact is crucial for correct definition of the Poisson brackets in quantum mechanics (see below).

It can be shown that for the matrix $\hat{M}$, which represents some canonical transformation of the Hamilton dynamical variables, the following condition is always obeyed: $\hat{M}'\hat{J}\hat{M} = c\hat{J}$, where $c(\neq 0)$ is a real and/or complex number which is called the valence of this transformation. Additionally, in this equation $\hat{J}$ is the unit simplectic matrix and $\hat{M}'$ is the matrix adjoint to the original matrix $\hat{M}$. Vice versa, if some matrix of differential transformation of variables $\hat{M}$ obeys the equation $\hat{M}'\hat{J}\hat{M} = c\hat{J}$, then such a differential transformation of dynamical variables, which is represented by the matrix $\hat{M}$, is canonical with the valence c. Thus, we have formulated the criterion of canonicity in terms of the

Jacobi matrix which is easily determined for an arbitrary transformation of the Hamiltonian dynamical variables. Let us discuss this matrix criterion with some additional details. First of all, by using the properties of the $\hat{J}$ matrix mentioned above it is possible to show (in two steps) that the two following equations (or conditions) follow from each other

$$\hat{M}'\hat{J}\hat{M} = c\hat{J} \iff \hat{M}\hat{J}\hat{M}' = c\hat{J} \ . \tag{A13}$$

which means that these two equations are equivalent to each other. Indeed, at the first step we multiply both sides of the equation $\hat{M}'\hat{J}\hat{M} = c\hat{J}$ by the $[\hat{M}']^{-1}$ matrix from the left and by the $[\hat{M}]^{-1}$ matrix from the right. These operations lead to the new equation $\frac{1}{c}\hat{J} = [\hat{M}']^{-1}\hat{J}[\hat{M}]^{-1}$. At the second step we just need to reverse both sides of this equation. By taking into account that $\hat{J}^{-1} = -\hat{J}, [\hat{M}^{-1}]^{-1} = \hat{M}$ and $\{[\hat{M}']^{-1}\}^{-1} = \hat{M}'$, we find the equality: $c\hat{J} = \hat{M}\hat{J}\hat{M}'$, which exactly coincides with the second equation in Equations (A13). On the other hand it is easy to check that, if we start from the second equation in Equations (A13), then the two analogous and simple steps allow one to derive the first equation in Equations (A13).

The matrix $\hat{M}$ (and/or $\hat{M}'$) which obeys any of these equations, Equations (A13), is called the true simplectic matrix with numerical valence c. In general, all even-dimensional (non-singular) simplectic matrices form the closed simplectic group $Sp(k, R)$ (or $Sp(k, C)$), where $k = 2n$. Finally, we arrive to the following theorem: some non-singular transformation of Hamilton dynamical variables $\tilde{q}_i = \tilde{q}_i(t, q_i, p_i), \tilde{p}_i = \tilde{p}_i(t, q_i, p_i)$, where $i = 1, \ldots, n$, will be canonical, if (and only if) its Jacobi matrix $\hat{M}$, Equation (A11), is the true simplectic matrix with the valence c. In this case the condition, Equation (A13), must be obeyed identically for all old dynamical variables and time t.

Further investigation shows that the matrix elements of the $\hat{M}'\hat{J}\hat{M}$ matrix coincide with the corresponding Lagrange brackets written in old coordinates and momenta, while the matrix elements of the $\hat{M}\hat{J}\hat{M}'$ matrix coincide with the corresponding Poisson brackets which are also written in old coordinates and momenta. Thus, we have an obvious duality between the Lagrange and Poisson brackets, which can be illustrated by a simple rule which is applied to form the (ij)-matrix elements of the adjoint matrix $\hat{M}'$ (from $\hat{M}$) and vice versa. This rule is simple: in each partial derivative, which is included in the Jacobi matrix of the canonical transformation (or its adjoint), the letters and indices at the top and bottom are swapped, while the symbol ˜ always stays at the top. Let us consider the following example. As mentioned above the matrix equation $\hat{M}'\hat{J}\hat{M} = c\hat{J}$ for the canonical transformation is equivalent to the following system of equations:

$$\{q_i, q_j\} = 0 \ , \ \ \{p_i, p_j\} = 0 \ , \ \ \{q_i, p_j\} = c\delta_{ij} \ , \tag{A14}$$

where the notation $\{a, b\}$ means the Laplace bracket defined in the main text. In other words, the matrix elements of the $\hat{M}'\hat{J}\hat{M}$ matrix always coincide with the corresponding Laplace brackets, while numerical values of these brackets are determined from the matrix equation $\hat{M}'\hat{J}\hat{M} = c\hat{J}$. Now, by taking adjoint of Equation (A14) one finds the following matrix equation: $\hat{M}\hat{J}\hat{M}' = c\hat{J}$, which leads to the three group of equalities for the adjoint Laplace brackets:

$$\{q_i, q_j\}^\star = 0 \ , \ \ \{p_i, p_j\}^\star = 0 \ , \ \ \{q_i, p_j\}^\star = c\delta_{ij} \ , \tag{A15}$$

where the sign '$\star$' means that inside of Lagrange brackets we have to apply our 'swap of variables' described above. By using the explicit formulas for the Lagrange brackets and our recipes to construct the adjoint matrix we can write

$$\{q_i, q_j\}^\star = \left[\sum_{k=1}^n \left(\frac{\partial \tilde{q}_k}{\partial q_i}\frac{\partial \tilde{p}_k}{\partial q_j} - \frac{\partial \tilde{q}_k}{\partial q_j}\frac{\partial \tilde{p}_k}{\partial q_i}\right)\right]^\star = \sum_{k=1}^n \left(\frac{\partial \tilde{q}_i}{\partial q_k}\frac{\partial \tilde{q}_j}{\partial p_k} - \frac{\partial \tilde{q}_j}{\partial q_k}\frac{\partial \tilde{q}_i}{\partial p_k}\right) = [\tilde{q}_i, \tilde{q}_j] \tag{A16}$$

where the notation $[a, b]$ stands for the 'regular' Poisson bracket. Analogous expressions can be derived for other fundamental Lagrange brackets $\{q_i, p_j\}^\star = [\tilde{q}_i, \tilde{p}_j]$ and $\{p_i, p_j\}^\star = [\tilde{p}_i, \tilde{p}_j]$, where all Poisson brackets are calculated in the old dynamical variables. These formulas indicate clear that the adjoint of the Laplace bracket equals (remarkably and unambiguously) to the corresponding Poisson bracket. Briefly, this means that we have reduced calculations of the adjoints of the Laplace brackets to computations of the corresponding Poisson brackets. Note also that the right-hand sides of Equations (A15) do not change during the $\star$ procedure and we obtain the following numerical values for the fundamental Poisson brackets

$$[\tilde{q}_i, \tilde{q}_j] = 0 \ , \ \ [\tilde{p}_i, \tilde{p}_j] = 0 \ , \ \ [\tilde{q}_i, \tilde{p}_j] = c\delta_{ij} \ , \tag{A17}$$

which coincide with the expected numerical values. Thus, we have shown that the Poisson brackets coincide with the adjoints of the corresponding Laplace brackets (and vice versa). In other words, these two systems of brackets are closely related to each other and each of these brackets can equally be used to check and prove the canonicity of some new set of Hamilton dynamical variables. Based on these facts it is relatively easy (for simplicity, we choose $c = 1$ here) to prove the following theorem [12]: if $u_1, u_2, \ldots, u_{2n}$ are the $2n$ independent functions of the variables $q_1, \ldots, q_n, p_1, \ldots, p_n$, then the two equations

$$\sum_{k=1}^{2n} \{u_k, u_i\}[u_k, u_j] = \sum_{k=1}^{2n} \{u_i, u_k\}[u_j, u_k] = \delta_{ij} \ , \tag{A18}$$

are always obeyed for these functions. This equation(s) explicitly shows a very close relation between the Laplace and Poisson brackets and they essentially follow from the definitions of these two brackets.

The last remark, which we want to make here, describes the main difference which arises in definitions of the canonical transformations in the classical and quantum mechanics. The Jacobi matrix $\hat{M}$, which describes the canonical transformation of the Hamiltonian dynamical variables, can also be defined (with a few additional tricks) in quantum mechanics. However, the governing equation Equation (A11) for the Jacobi $\hat{M}$ matrix in quantum mechanics includes the self-adjoint unit matrix $\imath\hat{J}$, which is not the unit simplectic $\hat{J}$ matrix as it was in classical mechanics. This means that in quantum mechanics both newly-defined Poisson brackets and their numerical values will always include (explicitly) the imaginary unit $\imath$. In the fundamental Poisson bracket we can introduce the 'new' momenta in the coordinate representation by including $\imath$ in its definition. On the other hand, the numerical value of the corresponding Poisson bracket must also include the imaginary unit $\imath$. Let us consider the following example. Suppose we have a point non-relativistic particle with the mass m which is located at the point with the Cartesian coordinates (x, y, z) and has the velocity (v_x, v_y, v_z). One can introduce the momenta for this particle $p_i = mv_i$, where $i = (x, y, z)$. In classical mechanics we have three fundamental Poisson brackets: $[x_i, p_j] = \delta_{ij}, [x_i, x_j] = 0$ and $[p_i, p_j] = 0$. The same momenta and numerical values of all non-zero Poisson brackets in quantum mechanics are defined as follows: $p_k = -\imath\hbar\frac{\partial}{\partial x_k}$ and $[p_i, x_j] = -\imath\hbar\delta_{ij}$, respectively [30]. Here $(i, j) = (x, y, z)$ and $\hbar = \frac{h}{2\pi} \approx 1.054571817 \cdot 10^{-34}$ $J \cdot sec$ is the reduced Planck constant which is also called the Dirac constant. An additional trick in this case is the explicit form of the coordinate operator x_k in momentum representation: $x_k = -\imath\hbar\frac{\partial}{\partial p_k}$, or $\frac{\partial}{\partial p_k} = \frac{\imath}{\hbar}x_k$, where $k = x, y, z$.

Note that both quantum momenta and numerical values of the Poisson brackets contain the imaginary unit $\imath$. Actual presence of the self-adjoint matrix $\imath\hat{J}$ in the governing equation $\hat{M}'(\imath\hat{J})\hat{M} = \imath\hat{J}$ for the quantum Jacobi matrix $\hat{M}$ allows one to define the canonical transformations in quantum mechanics as the unitary transformations. In other words, canonical transformations of the Hamiltonian dynamical variables in quantum mechanics are represented by the unitary matrices $\hat{M}$ and $\hat{M}'$ only. Thus, we have found the following relation between the classical and quantum Poisson brackets $[f, g]_{class} = \frac{\imath}{\hbar}[\hat{f}, \hat{g}]_{quant}$. This

relation can be considered as a canonical transformation between the classical and quantum mechanics with the imaginary valence $c = \frac{t}{\hbar}$. The imaginary valence of this canonical transformation means that all dynamical variables in quantum mechanics must be represented by self-adjoint operators and canonical transformations between different sets of dynamical variables in quantum mechanics are always performed by the unitary matrices, which differ from the simplectic matrices used for the same purposes in classical mechanics.

Appendix C. Variational Derivation of Integral Invariants for Hamiltonian Systems

As is well known, the Hamilton method has a number of significant advantages over other methods which are used to solve the same problems in classical mechanics, e.g., over the Lagrange method. One of these advantages is the method of integral invariants which is a 'hidden' part of any true Hamilton approach. The method of integral invariants allows one to analyze and solve many problems in mechanics. Here we describe a variational derivation of integral invariants for arbitrary, in principle, Hamiltonian dynamical systems. First, let us consider variations of the two following actions (or action integrals) W_L and W_H:

$$W_L = \int_{t_0}^{t_1} L\Big(t, q_i(t, \alpha), \dot{q}_i(t, \alpha)\Big) dt \ \text{ and } \ W_H = \int_{t_0(\alpha)}^{t_1(\alpha)} L\Big(t, q_i(t, \alpha), \dot{q}_i(t, \alpha)\Big) dt \,, \quad \text{(A19)}$$

where the notation α stands for the parameter. All coordinates $q_i(t, \alpha)$ and velocities $\dot{q}_i(t, \alpha)$ in these two actions depend on this parameter. In the second action (W_H) the lower and upper limits in the time-integral also depend upon this parameter α. During variations of these two integrals we can always interchange the sings of variations δ and time derivative $\frac{d}{dt}$, since we can write

$$\delta \dot{q}_i = \delta \Big(\frac{d}{dt} q_i(t, \alpha) \Big) = \frac{\partial}{\partial \alpha} \Big[\frac{d}{dt} q_i(t, \alpha) \Big] \delta \alpha = \frac{d}{dt} \Big[\frac{\partial}{\partial \alpha} q_i(t, \alpha) \delta \alpha \Big] = \frac{d}{dt} \delta q_i(t, \alpha) \quad \text{(A20)}$$

By using this equation we can derive the following formulas for variations of these two actions W_L and W_H:

$$\delta W_L = \int_{t_0}^{t_1} \sum_{i=1}^{n} \Big[\frac{\partial L}{\partial q_i} - \frac{d}{dt} \Big(\frac{\partial L}{\partial \dot{q}_i} \Big) \Big] \delta q_i dt \quad \text{(A21)}$$

and

$$\delta W_H = \Big\{ \sum_{i=1}^{n} \Big(\frac{\partial L}{\partial \dot{q}_i} \Big) \delta q_i - \Big[\sum_{i=1}^{n} \Big(\frac{\partial L}{\partial \dot{q}_i} \Big) \dot{q}_i - L \Big] \delta t \Big\} \Big|_{t_0(\alpha)}^{t_1(\alpha)} + \int_{t_0(\alpha)}^{t_1(\alpha)} \sum_{i=1}^{n} \Big[\frac{\partial L}{\partial q_i} - \frac{d}{dt} \Big(\frac{\partial L}{\partial \dot{q}_i} \Big) \Big] \delta q_i dt \,. \quad \text{(A22)}$$

Now, by introducing the following notations: $\frac{\partial L}{\partial \dot{q}_i} = p_i$ and $\sum_{i=1}^{n} \Big(\frac{\partial L}{\partial \dot{q}_i} \Big) \dot{q}_i - L = H$ we can re-write the last expression in the form:

$$\delta W_H = \Big[\sum_{i=1}^{n} p_i \delta q_i - H \delta t \Big] \Big|_{t_0(\alpha)}^{t_1(\alpha)} + \int_{t_0(\alpha)}^{t_1(\alpha)} \sum_{i=1}^{n} \Big[\frac{\partial L}{\partial q_i} - \frac{d}{dt} \Big(\frac{\partial L}{\partial \dot{q}_i} \Big) \Big] \delta q_i dt \,, \quad \text{(A23)}$$

where the function H is Hamiltonian of the system, while p_i ($i = 1, \ldots, n$) are the momenta.

Let us define the optimal (or shortest) path between two spatial points M_0 and M_1. The optimal path satisfies the Lagrange equation $\frac{\partial L}{\partial q_i} - \frac{d}{dt} \Big(\frac{\partial L}{\partial \dot{q}_i} \Big) = 0$ and variation of the Lagrange action W_L is always equal zero for the shortest path. It is clear that in this case the optimal path does not depend upon the parameter α, i.e., for this path we can write the equation $q_i = q_i(t)$, where $i = 1, \ldots, n$ and all $q_i(t)$ functions do not depend on the parameter α. If we consider the variation of the Hamilton action W_H, then the corresponding optimal path is also determined by the Lagrange equation. However, since

in this case $t_0 = t_0(\alpha)$ and $t_1 = t_1(\alpha)$, then we obtain the whole α-dependent family of optimal paths $q_i = q_i(t, \alpha)$, where $i = 1, \ldots, n$. On each of these optimal α-dependent paths, the variation of W_H is written in the form

$$\delta W_H(\alpha) = \delta \int_{t_0(\alpha)}^{t_1(\alpha)} L\left(t, q_i(t, \alpha), \dot{q}_i(t, \alpha)\right) dt = \left(\frac{\partial W_H(\alpha)}{\partial \alpha}\right) \delta \alpha = \left[\sum_{i=1}^{n} p_i \delta q_i - H \delta t\right]\Bigg|_{t_{0(\alpha)}}^{t_1(\alpha)} . \quad \text{(A24)}$$

A different derivation of this formula can be found, e.g., [31]. For actual (or optimal) motion of the Hamiltonian system the variation of W_H must be equal zero, i.e., $\delta W_H(\alpha) = 0$.

Instead of the extended $(n + 1)$-dimensional coordinate space $\{q_i(\alpha), t(\alpha)\}$, let us introduce the extended $(2n + 1)$-dimensional phase space of the generalized coordinates q_i, momenta p_i and time t which we shall designate as the $\{q_i(\alpha), p_i(\alpha), t(\alpha)\}$ space. In this phase space, we choose an arbitrary closed curve C_0, which is described by the equations: $q_i(0) = q_i(t_0, \alpha), p_i(0) = p_i(t_0, \alpha), t = t_0(\alpha)$, where $i = 1, \ldots, n$ and $0 \leq \alpha \leq \ell$. Note that for $\alpha = 0$ and for $\alpha = \ell$ we have the same point on the curve. This means that $q_i(0) = q_i(\ell)$ and $p_i(0) = p_i(\ell)$. At the next step, from each point of this closed curve C_0, as from the initial point, we draw the corresponding optimal path. Such an optimal path is uniquely determined (after we set the initial point on the curve C_0) from the system of Hamiltonian canonical equations. By choosing different initial points on the closed curve C_0 we obtain a tube of optimal (or shortest) paths each of which is also α-dependent. The explicit equations for this tube of optimal paths are: $q_i = q_i(t, \alpha), p_i = p_i(t, \alpha)$, where $i = 1, \ldots, n, 0 \leq \alpha \leq \ell$ and also $q_i(t, 0) = q_i(t, \ell)$ and $p_i(t, 0) = p_i(t, \ell)$ for $i = 1, \ldots, n$. Formally, each optimal path on the surface of this tube can be considered as some generatrix of the tube.

On this tube, we select the second closed curve C_1, which also covers the tube and has only one common point with each generatrix. The equations of this closed curve C_1 are written in the form: $q_i(1) = q_i(t_1, \alpha), p_i(1) = p_i(t_1, \alpha), t = t_1(\alpha)$, where $i = 1, \ldots, n$ and $0 \leq \alpha \leq \ell$. Now, by integrating the equation $\delta W_H(\alpha) = 0$ over the parameter α in the range from $\alpha = 0$ to $\alpha = \ell$, one finds

$$\begin{aligned} 0 &= W_H(\ell) - W_H(0) = \int_0^\ell \left[\sum_{i=1}^n p_i \delta q_i - H \delta t\right]\Bigg|_{t_0}^{t_1} = \int_0^\ell \left[\sum_{i=1}^n p_i^1 \delta q_i^1 - H_1 \delta t_1\right] \\ &- \int_0^\ell \left[\sum_{i=1}^n p_i^0 \delta q_i^0 - H_0 \delta t_0\right] = \oint_{C_1}\left[\sum_{i=1}^n p_i \delta q_i - H \delta t\right] - \oint_{C_0}\left[\sum_{i=1}^n p_i \delta q_i - H \delta t\right] . \quad \text{(A25)} \end{aligned}$$

or

$$\oint_{C_1}\left[\sum_{i=1}^n p_i \delta q_i - H \delta t\right] = \oint_{C_0}\left[\sum_{i=1}^n p_i \delta q_i - H \delta t\right] . \quad \text{(A26)}$$

This means that the closed loop integral $I = \oint \left[\sum_{i=1}^n p_i \delta q_i - H \delta t\right]$ taken along an arbitrary closed contour does not change its value during an arbitrary displacement along the tube of straight paths. In other words, this integral I is invariant, which is called the integral invariant of Poincare–Cartan (see, Equation (99), in the main text) for the Hamiltonian dynamical system with the Hamiltonian H.

In pure mathematical language we have created and investigated the rotor tube for the differential form $\omega^1 = \mathbf{p}\, d\mathbf{q} - H dt$, where $\mathbf{p} = (p_1, \ldots, p_n)$ and $\mathbf{q} = (q_1, \ldots, q_n)$. The rotor lines, each of which is the generatrix of this tube, are uniformly determined by the canonical Hamiltonian equations. In other words, the true Hamiltonian trajectories (or integral curves of the canonical Hamilton equations) are the generatrixes of the rotor tube for the differential form $\omega^1 = \mathbf{p}\, d\mathbf{q} - H dt$. Note that about integral invariants for Hamiltonian systems it is possible to talk endlessly, but for now it is better to stop here.

The last thing which we want to present in this Appendix is the explicit formula for the differential of the ω^1 form, which is the 2-form $d\omega^1$ [20]:

$$d\omega^1 = \sum_{i=1}^{n} \left(dp_i \wedge dq_i - \frac{\partial H}{\partial p_i} dp_i \wedge dt - \frac{\partial H}{\partial q_i} dq_i \wedge dt \right) = 0 \ , \tag{A27}$$

where $\wedge$ is the standard notation of exterior product [19,20]. The matrix $\hat{A}$ of this 2-form $d\omega^1$ in the $(2n+1)$-dimensional phase space is:

$$\hat{A} = \begin{pmatrix} 0 & -E & \frac{\partial H}{\partial \mathbf{p}} \\ E & 0 & \frac{\partial H}{\partial \mathbf{q}} \\ -\frac{\partial H}{\partial \mathbf{p}} & -\frac{\partial H}{\partial \mathbf{q}} & 0 \end{pmatrix} \ , \ \text{where} \ \begin{pmatrix} 0 & -E \\ E & 0 \end{pmatrix} = \hat{J} \tag{A28}$$

and $\hat{J}$ is the unit simplectic $2n \times 2n$ matrix which has been defined and discussed in the Appendix B. The rank of this matrix $\hat{A}$ equals $2n$ (the rank of $\hat{J}$ matrix), i.e., it has one zero-eigenvalue. The corresponding eigenvector is $(-\frac{\partial H}{\partial \mathbf{q}}, \frac{\partial H}{\partial \mathbf{p}}, 1)$ and it determines the direction of rotor lines for the differential form $\omega^1 = \mathbf{p} \, d\mathbf{q} - H dt$. More about the uniqueness of integral invariants, their classification and integral invariants of higher orders can be found, e.g., in [32].

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