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A New Approach to String Theory

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Abstract: In the present paper, we consider quantum theories obtained through the quantization of classical theories with first-class constraints assuming that these constraints form a Lie algebra. We show that in this case, one can construct physical quantities of a new type. We apply this construction to string theory. We find that scattering amplitudes in critical bosonic closed string theory can be expressed in terms of physical quantities of the new type. Our techniques can also be applied to superstrings and heterotic strings.

Keywords: operator formalism; conformal field theory; BRST formalism

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1. Introduction

In BRST formalism, we can construct physical quantities by taking correlation functions of BRST-closed operators in a physical (BRST-closed) state. (These correlation functions can be considered polylinear functions in BRST cohomology.) In the present paper, we consider quantum theories obtained through the quantization of classical theories with first-class constraints assuming that these constraints form a Lie algebra. We show that in this case, one can construct physical quantities of a new type (Section 2). We apply this construction to string theory (Sections 5 and 6). We find that scattering amplitudes in critical bosonic closed string theory can be expressed in terms of physical quantities of the type described in Section 2. Our techniques can also be applied to superstrings and heterotic strings; this will be shown in a separate paper.

Our results on scattering amplitudes in string theory are based on a comparison with the expression of these amplitudes in operator formalism [1,2]. The operator formalism is closely related to Segal's definition of conformal field theory [3]. We recall this definition (or, more precisely, the modification of this definition that is used in operator formalism) and the main ideas of operator formalism (Sections 3 and 4). In Appendix A, we sketch a new, simple approach to operator formalism.

One of the main takeaways from our results is as follows: Knowing the one-string space of states in BRST formalism, one can calculate physical quantities describing interacting strings. Neither multi-string states nor worldsheets with non-trivial topology, which are necessary for other approaches, are fundamental in our approach; we show that they are, in some sense, hidden in one-string space.

The present paper is a byproduct of my attempts to formulate string theory in algebraic and geometric approaches to quantum theory (see [4] and references therein). The results of this paper show the way to solve this problem: it is sufficient to work in the one-string space.

2. General Considerations

For every supermanifold M, we can construct a supermanifold ΠTM by reversing the parity in fibers of tangent bundle TM. If $(x^1, \ldots, x^m)$ are coordinates in M, then the coordinates in ΠTM are $(x^1,\dots,x^m,\xi^1,\dots,\xi^m)$, where the parity of ξ^k is opposite to the parity of x^k. Polynomial functions on ΠTM are identified with differential forms on M, more general functions with pesudodifferential forms. The formula $Q=\xi^k\frac{\partial}{\partial x^k}$ specifies an odd vector field on ΠTM; the anticommutator of this vector field with itself vanishes. We can say that Q specifies a structure of Q-manifold on ΠTM; in other words, Q is a homological vector field. It defines an odd derivation d of the algebra of functions on ΠTM. Operator d obeys $d^2=0$ and can be identified with the de Rham differential.

There exists an invariant definition of ΠTM that shows that the construction of ΠTM is functorial. In other words, a map $M\to N$ induces a map $\Pi TM\to\Pi TN$; the induced map agrees with the de Rham differential. Namely, ΠTM can be identified with the space of maps of $(0,1)$-dimensional superspace $\mathbb{R}^{0,1}$ into M. Every vector field on space $\mathbb{R}^{0,1}$ induces a vector field on the space of maps. The Lie superalgebra of vector fields on $\mathbb{R}^{0,1}$ is $(1,1)$-dimensional; an odd vector field on $\mathbb{R}^{0,1}$ induces a homological vector field Q on the space of maps, and an even vector field induces a grading on the algebra of functions on this space.

This remark allows us to say that $\Pi T\mathfrak{g}$, where $\mathfrak{g}$ is a Lie superalgebra, is equipped with the structure of a differential Lie superalgebra. We denote this Lie superalgebra by $\mathfrak{g}'$ and the differential in it by Q. Sometimes, it is convenient to consider a semi-direct product $\mathfrak{g}''$ of the Lie superalgebra and the Lie superalgebra of vector fields on $\mathbb{R}^{0,1}$.

Similarly, if G is a supergroup, then $G'=\Pi TG$ is also a supergroup (the multiplication in G induces multiplication in the space of maps $\mathbb{R}^{0,1}\to G$). The Lie superalgebra of G' can be identified with $\mathfrak{g}'$, where $\mathfrak{g}$ stands for the Lie superalgebra of G. The homological vector field on G' induces differential Q on $\mathfrak{g}'$.

If $\mathfrak{g}$ is a Lie algebra with generators T_k and commutation relations $[T_k,T_l]=f^r_{kl}T_r$, Lie superalgebra $\mathfrak{g}'$ has even generators T_k, odd generators b_k and commutation relations $[T_k,T_l]=f^r_{kl}T_r$, $[T_k,b_l]=f^r_{kl}b_r$, $[b_k,b_l]_+=0$. Generators T_k are Q-exact (by acting by Q on b_k, we obtain T_k).

Every element of G' can be represented in the form $g\exp(\mu^k b_k)$, where μ_k are odd parameters, $g\in G$ and exp stands for the exponential map of the Lie superalgebra into the corresponding supergroup.

Let us now consider a classical system that, after quantization, can be described by Hilbert space $\mathcal{E}$. If a new classical system is obtained from this system by means of constraints obeying a Lie algebra $\mathfrak{g}$ of group G, then the quantized system can be described in BRST formalism by space $\mathcal{E}'$, obtained by adding ghosts to $\mathcal{E}$. (To obtain $\mathcal{E}'$, we take the tensor product of $\mathcal{E}$ by the representation space of canonical anticommutation relations $[\hat{c}^k,\hat{b}_l]_+=\delta^k_l,[\hat{c}^k,\hat{c}_r]_+=0,[\hat{b}_l,\hat{b}_r]_+=0$.) The constraints induce operators T_k in $\mathcal{E}$; BRST operator $\hat{Q}$ has the form $\hat{Q}=T_kc^k+\frac{1}{2}f^r_{kl}\hat{c}^k\hat{c}^l\hat{b}_r$, where f^r_{kl} are structure constants of algebra $\mathfrak{g}$ and $\hat{c}^k,\hat{b}_l$ are ghosts obeying canonical anticommutation relations (in the case of an infinite number of degrees of freedom, we should use normal ordering; this can lead to anomalies). Operators $\hat{T}_k=T_k+f^r_{kl}\hat{c}^l\hat{b}_r$ are BRST-trivial in $\mathcal{E}'$; this follows from relation $\hat{T}_k=[\hat{Q},\hat{b}_k]_+$. Together with operators $\hat{b}_k$, they specify a representation ψ of Lie superalgebra $\mathfrak{g}'$, i.e., a homomorphism of $\mathfrak{g}'$ into space $\mathcal{L}$ of linear operators acting in $\mathcal{E}'$; this homomorphism agrees with differentials (in this statement, space $\mathcal{L}$ is considered a Lie superalgebra).

We assume that representation ψ is integrable (=can be exponentiated), i.e., it can be obtained from a representation Ψ of group G'. (Recall that $\mathfrak{g}$ is the Lie algebra of group G).

Representation Ψ induces a map Ψ^* of $\mathcal{L}^*$ (of the superspace of linear functionals on $\mathcal{L}$) into the space of functions on G' (the space of (pseudo)differential forms on G). This map agrees with differentials; this means, in particular, that it transforms Q-closed element $\sigma\in\mathcal{L}^*$ into a closed (in general, inhomogeneous) form $\Psi^*(\sigma)$ on G. (The BRST operator acts on $\mathcal{L}$ as (anti)commutator with $\hat{Q}$, and this action induces a BRST operator on $\mathcal{L}^*$.)

By integrating $\Psi^*(\sigma)$ over a cycle in G, we obtain a physical quantity (the integral does not change if we add a Q-exact term to σ; hence, it depends only on the BRST cohomology class of σ).

If K is a subgroup of G and the form $\Psi^*(\sigma)$ descends to G/K, we can integrate the form on G/K over a cycle in G/K. (Here, G/K stands for the space of right cosets= space of orbits of left action of K on G.) This construction leads to a more general class of physical quantities.

Let us consider a special case where a Q-closed element $\sigma \in \mathcal{L}^*$ is specified by the formula

$$\sigma(A) = \langle \rho | A | \chi \rangle \tag{1}$$

where $A \in \mathcal{L}$, $\langle \rho | \in (\mathcal{E}')^*$ and $|\chi\rangle \in \mathcal{E}'$ are Q-closed. By taking A as $\Psi(g')$, where $g' \in G'$, we obtain a Q-closed function

$$(\Psi^*\sigma)(g') = \langle \rho | \Psi(g') | \chi \rangle \tag{2}$$

on G' (a non-homogeneous closed form on G). By representing $g' \in G'$ as $g \exp(\mu^k b_k))$, where $g \in G$, we obtain

$$(\Psi^*\sigma)(g \exp(\mu^k b_k)) = \langle \rho | \Psi(g \exp(\mu^k b_k))) | \chi \rangle = \langle \rho | \Psi(g) \exp(\mu^k \hat{b}_k) |) \chi \rangle \tag{3}$$

(we use the fact that G is embedded into G'; hence, Ψ is defined on G).

Function (2) descends to $G'/K' = (G/K)'$ (equivalently, the corresponding closed (pseudo) differential form descends to G/K) if $\langle \rho |$ is a K'-invariant element of $(\mathcal{E}')^*$. (The relation $\langle \rho | \Psi(k') = \langle \rho |$ for $k' \in K'$ implies that $(\Psi^*\sigma)(k'g') = (\Psi^*\sigma)(g')$.)

Homogeneous components of the form shown in (3) are closed forms that can be represented as

$$\langle \rho | \Psi(g) B | \chi \rangle \tag{4}$$

where B is a homogeneous polynomial with respect to $\hat{b}_k$.

Notice that our constructions can be applied to the case where $\mathcal{E}$ and $\mathcal{E}'$ are replaced by their n-th tensor powers; then, groups G and G' should be replaced with the direct products of n copies of these groups.

One can consider a more general situation where we have two subgroups of group G denoted by K and H, element $|\chi\rangle$ is an H'-invariant element of $\mathcal{E}'$ (i.e., $\Psi(h')|\chi\rangle = |\chi\rangle$ for all $h' \in H'$) and element $\langle \rho |$ is a K'-invariant element of the dual space. Then, Function (2) descends to $H' \backslash G' / K'$ (to the space of double cosets).

Our consideration can be generalized to the case where $\mathfrak{g}$ is a Lie algebra of semigroup G. In this case, one should assume that representation ψ is semi-integrable, i.e., it can be obtained from the representation of semigroup G' having Lie algebra $\mathfrak{g}'$.

Another important generalization is the following: It is sufficient to assume that $\langle \rho |$ is $\mathfrak{k}'$-invariant (i.e., $\langle \rho | \psi(\mathfrak{k}') = 0$). Here, $\mathfrak{k}$ is a Lie subalgebra of Lie algebra $\mathfrak{g}$. If $\mathfrak{k}$ is a Lie algebra of a connected subgroup K of semigroup G, this assumption is equivalent to K'-invariance of $\langle \rho |$; we come back to the situation considered above. However, in the situation considered in the next sections, Lie algebra $\mathfrak{k}$ cannot be considered a Lie algebra of some group.

It is easy to check that $\mathfrak{k}'$-invariance of $\langle \rho |$ implies that Function (2) descends to $G'/\mathfrak{k}'$ (equivalently, the corresponding form descends to $G/\mathfrak{k}$).

To define space of cosets $G/\mathfrak{k}$, we consider left action of the Lie algebra $\mathfrak{k}$ on semigroup G. This action specifies a foliation of G; one can define $G/\mathfrak{k}$ as the space of leaves of the foliation.

Alternatively, $G/\mathfrak{k}$ can be defined as a connected manifold M where semigroup G acts transitively with Lie stabilizer $\mathfrak{k}$. (We say that action of G on M is transitive if it induces a surjective map τ_m of Lie algebra $\mathfrak{g}$ to the tangent space of M in any point $m \in M$. The Lie stabilizer at point m is defined as the kernel of τ_m; we assume that there exists a point with a Lie stabilizer $\mathfrak{k}$.)

More generally, if $|\chi\rangle$ is $\mathfrak{h}'$-invariant and $\langle\rho|$ is $\mathfrak{k}'$-invariant, then Function (2) descends to a function on space of double cosets $\mathfrak{h}'\backslash G'/\mathfrak{k}'$ (to a (pseudo)differential form on space of double cosets $\mathfrak{h}\backslash G/\mathfrak{k}$).

In this statement, $\mathfrak{k}$ and $\mathfrak{h}$ are Lie subalgebras of $\mathfrak{g}$. For an appropriate choice of $\mathfrak{g}, \mathfrak{k}$ and $\mathfrak{h}$, this statement can be used to obtain an expression for string amplitudes (Section 6).

3. CFT, TCFT, SCFT, TSFT

Let us start with a reminder of some basic constructions that are used in two-dimensional conformal field theory (CFT) and in operator formalism of string theory.

Recall that two Riemannian manifolds are conformally equivalent (specify the same conformal manifold) if there exists a diffeomorphism between these manifolds preserving the Riemannian metric up to multiplication by a function.

A two-dimensional, oriented conformal manifold can be identified with a complex manifold of complex dimension 1. Maps preserving conformal structure are either holomorphic or antiholomorphic maps of complex manifolds.

We consider moduli space of complex curves (=one-dimensional, compact, connected complex manifolds) of genus g with boundary consisting of n parametrized circles. (We assume that these circles are ordered.) This moduli space denoted by $\mathcal{P}(\mathrm{g}, n)$ can be regarded as an infinite-dimensional complex manifold. Equivalently, one can define $\mathcal{P}(\mathrm{g}, n)$ as the moduli space of complex curves of genus g with n embedded standard discs.

It is easy to construct a natural map $\phi_{m,n} : \mathcal{P}(\mathrm{g}, n) \times \mathcal{P}(\mathrm{g}', n') \to \mathcal{P}(\mathrm{g} + \mathrm{g}', n + n' - 2)$ identifying the last circle in the first factor with the first circle in the second factor. Similarly, one can construct a map $\phi_n : \mathcal{P}(\mathrm{g}, n) \to \mathcal{P}(\mathrm{g} + 1, n - 2)$ identifying two last circles.

In particular, map $\mathcal{P}(0, 2) \times \mathcal{P}(0, 2) \to \mathcal{P}(0, 2)$ specifies a structure of a semigroup on $\mathcal{P}(0, 2)$. This semigroup was introduced independently by Neretin, Konntsevich and Segal; we call it the semigroup of annuli and denote it by $\mathcal{A}$.

The map $\mathcal{P}(0, 2) \times \mathcal{P}(\mathrm{g}, n) \to \mathcal{P}(\mathrm{g}, n)$ specifies an action of $\mathcal{A}$ on $\mathcal{P}(\mathrm{g}, n)$.

Notice that the Lie algebra of $\mathcal{A}$ can be identified with diff (with the complexification of the Lie algebra of vector fields on a circle); in other words, this is a complex Lie algebra with generators l_n obeying $[l_m, l_n] = (m - n)l_{m+n}$.

In Segal's approach, a CFT having central charge $c = 0$ specifies a map $\sigma_{\mathrm{g},n} : \mathcal{P}(\mathrm{g}, n) \to \mathcal{H}^n$, where $\mathcal{H}$ is a vector space equipped with bilinear inner product $\mathcal{H} \otimes \mathcal{H} \to \mathbb{C}$. Using this inner product, one can construct maps $\tilde{\phi}_{m,n} : \mathcal{H}^m \otimes \mathcal{H}^n \to \mathcal{H}^{m+n-2}$ and $\tilde{\phi}_n : \mathcal{H}^n \to \mathcal{H}^{n-2}$. Segal's axioms are compatibility conditions for maps $\sigma_{\mathrm{g},n}, \phi_{m,n}, \phi_n, \tilde{\phi}_{m,n}, \tilde{\phi}_n$.

The action of semigroup $\mathcal{A}$ on $\mathcal{P}(\mathrm{g}, 1)$ and complex conjugate action generate an action of $\mathcal{A} \times \mathcal{A}$ and corresponding Lie algebra diff$\times$diff on $\mathcal{H}$.

A CFT having central charge $c \neq 0$ specifies a map sending a point of $\mathcal{P}(\mathrm{g}, n)$ into a point of $\mathcal{H}^n$ defined up to multiplication by a number. In this case, we have a projective representation of diff$\times$diff in $\mathcal{H}$, i.e., a representation of the central extension of this algebra in $\mathcal{H}$.

The central extension of diff is called Virasoro algebra; we denote it by Vir.

Let us consider CFT with a central charge c. Lie algebra Vir $\times$ Vir acts on its space of states $\mathcal{H}$. In other words, we have operators $L_m, \tilde{L}_n$ obeying

$$[L_m, L_n] = (m - n)L_{m+n} + \frac{c}{12}(m^3 - m)\delta_{m+n},$$

$$[\tilde{L}_m, \tilde{L}_n] = (m - n)\tilde{L}_{m+n} + \frac{c}{12}(m^3 - m)\delta_{m+n}, [L_m, \tilde{L}_n] = 0.$$

There exist many important analogs of these constructions. In particular, one can consider spaces $\mathcal{P}'(\mathrm{g}, n) = \Pi T\mathcal{P}(\mathrm{g}, n)$ instead of $\mathcal{P}(\mathrm{g}, n)$. It is obvious that analogs of maps $\phi_{m,n}$ and ϕ_n exist for these spaces. It follows that $\mathcal{A}' = \mathcal{P}'(0, 2)$ is a semigroup acting on $\mathcal{P}'(\mathrm{g}, n)$.

Let us fix a $\mathbb{Z}_2$-graded vector space $\mathcal{H}$ equipped with inner product and parity reversing differential q respecting this product. Then, topological conformal field theory (TQFT) is specified by maps $\mathcal{P}'(\mathrm{g}, n) \to \mathcal{H}^n$. (Notice that such a map specifies a differential form on $\mathcal{P}(\mathrm{g}, n)$ with values in space $\mathcal{H}^n$. The more standard definition of TQFT is formulated in terms of these forms.) We impose compatibility conditions of these maps with analogs of maps $\phi_{m,n}, \phi_n, \tilde{\phi}_{m,n}, \tilde{\phi}_n$, as well as compatibility conditions with differential q and homological vector field on $\mathcal{P}'(\mathrm{g}, n)$. It follows that semigroup $\mathcal{A}' \times \mathcal{A}'$ and its Lie algebra diff$'$ $\times$ diff$'$ act in $\mathcal{H}$.

By replacing, in the definition of CFT, conformal manifolds with superconformal manifolds, we obtain a definition of superconformal field theory (SCFT). One can also define topological superconformal field theory (TSFT); the modification that leads from SCFT to TSFT is very similar to the modification leading from CFT to TCFT.

4. Subalgebras, Stabilizers, Invariants

Lie algebra diff consists of complex vector fields on a circle. A very general way to construct Lie subalgebras of diff is based on the consideration of the embedding of the circle into a complex manifold M. Then, complex vector fields on the circle that can be holomorphically extended to M constitute a Lie subalgebra of diff. We can obtain a smaller Lie subalgebra assuming that the extended vector field vanishes on some subset of M.

A more concrete realization of this construction can be obtained if we take, as M, a one-dimensional, connected complex manifold (a complex curve) with n parametrized boundary components (n circles $B_1, \ldots, B_n$), p punctures (p deleted points $x_1, \ldots, x_p$) and m marked points $u_1, \ldots, u_m$. (Equivalently, one can consider a complex curve $\underline{M}$ with n embedded disks, p punctures and m marked points; then, we take, as M, curve $\underline{M}$ with deleted disks.) The moduli space of objects of this kind is denoted by $\mathcal{P}(n, p, m)$, and its connected components (labeled by genus g of $\underline{M}$) are denoted by $\mathcal{P}(\mathrm{g}, n, p, m)$. (If $p = 0, m = 0$, we obtain space $\mathcal{P}(\mathrm{g}, n)$, considered in the preceding section.) The direct product of n copies of semigroup $\mathcal{A}$ (hence also the direct product of n copies of Lie algebra diff) acts on these moduli spaces.

Let us fix one of the boundary components (say, the first one) and consider the action of the corresponding semigroup $\mathcal{A}$ on $\mathcal{P}(\mathrm{g}, n, p, m)$. Lie stabilizer $\mathfrak{k}_M \subset$diff at point $M \in \mathcal{P}(\mathrm{g}, n, p, m)$ can be described as the Lie algebra of complex vector fields on boundary component $S = B_1$ that have a meromorphic extension to M with zeros at the marked points and singularities only in the punctures.

By taking the product of n copies of semigroup $\mathcal{A}$ corresponding to all boundary components and considering Lie stabilizer $\mathfrak{k}_M \subset$ diff $\times \ldots \times$diff, we obtain

$$\mathcal{P}(\mathrm{g}, n, p, m) = (\mathcal{A} \times \ldots \times \mathcal{A})/\mathfrak{k}_M.$$

(We used the fact that $\mathcal{A} \times \ldots \times \mathcal{A}$ acts transitively on $\mathcal{P}(\mathrm{g}, n, p, m)$.) Lie stabilizer $\mathfrak{k}_M$ at point $M \in \mathcal{P}(\mathrm{g}, n, p, m)$ consists of vector fields on the boundary of M that have a meromorphic extension to M with zeros at the marked points and singularities only in the punctures.

Let us now consider CFT with central charge $c = 0$ in Segal's approach. In this approach, we assign a vector $\phi_M \in \mathcal{H}^n$ to every point $M \in \mathcal{P}(\mathrm{g}, n)$. Here, $\mathcal{H}$ stands for linear space equipped with anon-degenerate inner product. semigroup $\mathcal{A}^n$, hence its Lie algebra diff^n, acts on $\mathcal{H}^n$. If $f \in$diff is a complex vector field on a circle, then the corresponding operator acting on the i-th factor of $\mathcal{H}^n$ is denoted by $L^{(i)}(f)$. The Virasoro generators acting on the i-th factor (operators corresponding to vector fields $z^{k+1}\frac{d}{dz}$) are denoted by $L_k^{(i)}$. Lie stabilizer $\mathfrak{k}_M \subset$ diff^n consists of complex vector fields on the boundary that can be holomorphically extended to M.

It is easy to check that ϕ_M is $\mathfrak{k}_M$-invariant.

More generally, let us take $M \in \mathcal{P}(g, n, p = 0, m)$. By fixing holomorphic coordinates at marked points (=holomorphic disks with centers at these points), we obtain a point $\tilde{M} \in \mathcal{P}(g, n + m)$ and a vector $\phi_{\tilde{M}} \in \mathcal{H}^{n+m}$.

If $\chi = \chi_1 \otimes \ldots \otimes \chi_m \in \mathcal{H}^m$, we can define $\psi(\chi) \in \mathcal{H}^n$ as the inner product of $\phi_{\tilde{M}}$ and χ. (We use the inner product in $\mathcal{H}$ to calculate the pairing of the last m factors in $\mathcal{H}^{n+m}$ with χ.) If $L_k^{(i)} \chi_i = 0$ for $k \geq 0$, then $\psi(\chi)$ does not depend on the choice of coordinate systems at marked points; it is $\mathfrak{k}_M$-invariant. Here, $\mathfrak{k}_M$ stands for the Lie algebra of complex vector fields on the boundary of M that can be extended to holomorphic vector fields on M vanishing at marked points. (It can also be characterized as the Lie stabilizer of $\mathcal{A}^n$ at point $M \in \mathcal{P}(g, n, p = 0, m)$.)

Let us formulate a similar statement in the case where we work with TCFT instead of CFT. In this case, we have maps $\mathcal{P}'(g, n) \to \mathcal{H}^n$, where $\mathcal{P}'(g, n) = \Pi T\mathcal{P}g, n)$ and $\mathcal{H}$ is equipped by a differential q. Algebra diff$'$ is represented in $\mathcal{H}$ by operators $L(f), b(f)$, where $f \in$diff. They obey $[L(f), L(g)] = L([f, g], [L(f), b(g)] = b([f, g]), [b(f), b(g)]_+ = 0, L(f) = [q, b(f)]_+$. This action induces an action of diff$'^n$ on $\mathcal{H}^n$; the operators acting on the i-th factor are denoted by $L^{(i)}(f), b^{(i)}(f)$ or by $L_k^{(i)}, b_k^{(i)}$ if $f = z^{k+1} \frac{d}{dz}$.

Let us consider $M \in \mathcal{P}(g, n, p = 0, m)$, a vector $\kappa = \kappa_1 \otimes \ldots \otimes \kappa_m \in \mathcal{H}^m$ obeying $q\kappa_i = 0$ and

$$L_k^{(i)} \kappa_i = 0, b_k^{(i)} \kappa_i = 0 \tag{5}$$

for $k \geq 0$. Then, by slightly modifying the above construction, we can define a vector $\tau(\kappa) \in \mathcal{H}^n$. This vector is $\mathfrak{k}'_M$-invariant, where $\mathfrak{k}_M$ is the Lie stabilizer of $\mathcal{A}^n$ at point $M \in \mathcal{P}(g, n, p = 0, m)$. (By considering M a point of $\mathcal{P}'(g, n, p = 0, m)$, we can say that $\mathfrak{k}'_M$ is the Lie stabilizer of $\mathcal{A}'^n$ at this point.)

Using the inner product in $\mathcal{H}$, we can define bra-state $\langle \tau(\kappa)|$. This state is also $\mathfrak{k}'_M$-invariant.

5. String Theory

Let us consider classical CFT, which allows CFT to have central charge c after quantization. To obtain the corresponding string theory, we impose constraints $L_n = 0, \tilde{L}_n = 0$, where $L_n, \tilde{L}_n$ are classical analogs of Virasoro generators. Using the general construction of Section 2, we see that one can obtain the space of states of string theory (more precisely, one-string space in BRST formalism) by adding ghosts. In other words, we should take the tensor product of Hilbert space $\mathcal{E}$ of CFT by space of ghosts $\mathcal{E}_{gh}$, which can be considered a space of states of CFT with central charge $c_{gh} = -26$. (The space of ghosts is a tensor product of spaces of states of bc-system and $\tilde{b}\tilde{c}$-system.) We obtain space $\mathcal{E}' = \mathcal{E} \otimes \mathcal{E}_{gh}$. Let us consider critical closed bosonic strings. This means that we assume that $c = 26$. Then, space $\mathcal{E}'$ is a space of states of CFT with zero central charge. The generators of Virasoro algebra of this CFT are denoted by $\hat{L}_n, \hat{\tilde{L}}_m$. We need the following relations among operators $\hat{L}_n, \hat{\tilde{L}}_m, b_n, \tilde{b}_n, Q$ acting in this space:

$$\begin{aligned}
[\hat{L}_m, \hat{L}_n] &= (m - n)\hat{L}_{m+n} \\
[\hat{\tilde{L}}_m, \hat{\tilde{L}}_n] &= (m - n)\hat{\tilde{L}}_{m+n} \\
[\hat{L}_m, b_n] = (m - n)b_{m+n}, [b_m, b_n]_+ &= 0 \\
[\hat{\tilde{L}}_m, \tilde{b}_n] = (m - n)\tilde{b}_{m+n}, [\tilde{b}_m, \tilde{b}_n]_+ &= 0 \\
\hat{L}_n = [Q, b_n], \hat{\tilde{L}}_n = [Q, \tilde{b}_n], [Q, Q]_+ &= 0
\end{aligned} \tag{6}$$

These relations indicate that by adding ghosts to CFT with critical central charge $c = 26$, we obtain TCFT (topological CFT) on space $\mathcal{E}'$. Our results can be extended to any TCFT; the assumption that TCFT is obtained from CFT by adding ghosts is irrelevant.

Lie superalgebra diff$'$ is represented in $\mathcal{E}'$ by linear operators $L(\mathbf{v}), b(\mathbf{v})$ obeying

$$[L(\mathbf{v}), L(\mathbf{v}')] = L([\mathbf{v}, \mathbf{v}']), [L(\mathbf{v}), b(\mathbf{v}')] = b([\mathbf{v}, \mathbf{v}']), [b(\mathbf{v}), b(\mathbf{v}')]_+ = 0.$$

Operators $\tilde{L}(\mathbf{v}), \tilde{b}(\mathbf{v})$ obey similar relations; they give a second representation of diff$'$, commuting with the first one. (Here, $\mathbf{v}, \mathbf{v}'$ are complex-valued vector fields on circle: $\mathbf{v}, \mathbf{v}' \in$ diff.)

The first four lines of (6) describe the representation of generators of diff$' \times$ diff$'$ in $\mathcal{E}'$. This representation can be extended to a representation of diff$'' \times$ diff$''$. (Recall that one can obtain diff$''$ by adding a nilpotent generator and a ghost number to generators L_n, b_n of diff$'$.)

Let us consider the diagonal part of Lie algebra diff$' \times$ diff$'$ (the Lie subalgebra generated by operators $L_n + \tilde{L}_n, b_n + \tilde{b}_n$).

We assume that the action of the diagonal part of Lie algebra diff$' \times$ diff$'$ in $\mathcal{E}'$ can be integrated and gives an action of $\mathcal{A}'$ on $\mathcal{E}'$ (a homomorphism Ψ of semigroup $\mathcal{A}'$ into space $\mathcal{L}$ of linear operators in $\mathcal{E}'$).

This is a standard assumption that lies at the basis of Segal's definition of CFT (see Section 3).

We can apply general considerations of Section 2 by taking $G = \mathcal{A}$.

Let us consider the case where a form on $G = \mathcal{A}$ (a function on $G' = \mathcal{A}'$) is specified by (3). Semigroup $\mathcal{A}$ is homotopy-equivalent to S^1; therefore, an integral of the closed form shown in (3) over any cycle of dimension > 1 vanishes. To obtain non-trivial physical quantities, we construct the form shown in (3) in such a way that it descends to $G/\mathfrak{k}$, where $\mathfrak{k}$ is an appropriate Lie subalgebra of Lie algebra diff of semigroup $G = \mathcal{A}$ (or, more generally, a Lie subalgebra of Lie algebra diff^n of semigroup $G = \mathcal{A}^n$).

Examples of subalgebras $\mathfrak{k}$ and corresponding quotient spaces are constructed in Section 3.

6. String Amplitudes

We start the construction of string amplitudes by fixing a one-dimensional compact complex manifold $P_0 \in \mathcal{P}(\mathrm{g}, 1)$ (a complex curve of genus g with parametrized boundary diffeomorphic to a circle S^1). Let us denote by $\mathfrak{k}$ a Lie algebra consisting of vector fields on the boundary that can be extended to holomorphic vector fields on P_0. Semigroup $\mathcal{A}$ acts on moduli space $\mathcal{P}(\mathrm{g}, 1)$; hence, we can consider the corresponding action of its Lie algebra diff on this space. Lie algebra $\mathfrak{k}$ can be characterized as a Lie stabilizer of this action at P_0. The action of $\mathcal{A}$ on $\mathcal{P}(\mathrm{g}, 1)$ is transitive; hence, $\mathcal{P}(\mathrm{g}, 1)$ can be identified with $\mathcal{A}/\mathfrak{k}$. Lie stabilizer $\mathfrak{k}_P$ of diff at point $P \in \mathcal{P}(\mathrm{g}, 1)$ is a Lie subalgebra of diff consisting of vector fields that can be holomorphically extended from the boundary to P.

This construction can be generalized to the case where $P_0 \in \mathcal{P}(\mathrm{g}, n)$ (i.e., it has a boundary consisting of n parametrized circles; we assume that the orientation of boundary circles agrees with the orientation of P_0). Group $\mathcal{A}^n$ and its Lie algebra diff^n (the direct sum of n copies of Lie algebra diff) act on $\mathcal{P}(\mathrm{g}, n)$. Lie algebra $\mathfrak{k}_P$ can be defined as the Lie stabilizer of this action at P; if $P = P_0$, we use the notation $\mathfrak{k}_P = \mathfrak{k}$. Lie algebra $\mathfrak{k}_P$ consists of complex vector fields on the boundary that can be holomorphically extended from the boundary to P. The action of $\mathcal{A}^n$ on $\mathcal{P}(\mathrm{g}, n)$ is transitive; hence, $\mathcal{P}(\mathrm{g}, n)$ can be identified with $\mathcal{A}^n/\mathfrak{k}$.

All these statements are particular cases of the statements formulated in Section 3.

Notice that these objects appear in operator formalism in string theory. The main object of operator formalism is an element of $\mathcal{E}'$ depending on $P \in \mathcal{P}(\mathrm{g}, 1)$ (more generally, we have a map $\mathcal{P}(\mathrm{g}, n) \to \mathcal{E}'^n$, where $\mathcal{E}'^n$ stands for the tensor product of n copies of $\mathcal{E}'$). In the notations of [1], this map sends P into ϕ_P.

It is well known that ϕ_P *is* $\mathfrak{k}'_P$*-invariant* (see formula (5.1) of [1] or formula (7.33) of [5]). Notice that ϕ_P also appears in Segal's approach to CFT; the $\mathfrak{k}'_P$-invariance of ϕ_P follows immediately from this approach (see Section 4).

In what follows, we apply the considerations of Section 2 to the case where $G = \mathcal{A}^n$, $P \in \mathcal{P}(\mathrm{g}, n)$ and Ψ_n denotes the map of $\mathcal{A}'^n$ into the space of linear operators in $\mathcal{E}'^n$. (We have a representation ψ_n of Lie algebra $(\mathrm{diff}' \oplus \mathrm{diff}')^n$ in this space. This representation is a homomorphism ψ_n of Lie algebra $(\mathrm{diff}' \oplus \mathrm{diff}')^n$ into the space of linear operators in $\mathcal{E}'^n$ considered Lie algebra. It is obtained as the tensor product of n copies of homomorphism ψ of $\mathrm{diff}' \oplus \mathrm{diff}'$ into the space of linear operators in $\mathcal{E}'$. On the diagonal part of $(\mathrm{diff}' \oplus \mathrm{diff}')^n$, homomorphism ψ_n is specified by operators $L_k(\mathbf{v}) + \tilde{L}_k(\mathbf{v}), b_k(\mathbf{v}) + \tilde{b}_k(\mathbf{v})$, where $k = 1, \ldots, n$. Representation ψ_n can be integrated to give a representation Ψ_n of the diagonal part of $\mathcal{A}'^n \times \mathcal{A}'^n$; later, we use the notations $\mathcal{A}^n$ and $\mathcal{A}'^n$ for diagonal parts.)

One can verify that $P = gP_0$, where $g \in \mathcal{A}^n$ implies

$$\phi_P = (\Psi_n(g))(\phi_{P_0}) \tag{7}$$

The CFT with space of states $\mathcal{E}'$ has central charge $c = 0$. Map $\mathcal{P}(\mathrm{g}, n) \to \mathcal{E}'^n$ of operator formalism is Segal's map $\sigma_{\mathrm{g},n} : \mathcal{P}(\mathrm{g}, n) \to \mathcal{H}^n$ in the case $\mathcal{H} = \mathcal{E}'$. Formula (7) immediately follows from Segal's axioms.

Let us consider the form shown in (3) obtained from (1), where $\langle\rho|$ is $\mathfrak{k}'$-invariant. As a $\mathfrak{k}'$-invariant element $\langle\rho|$, we take the bra-state corresponding to ϕ_{P_0} where $P_0 \in \mathcal{P}(\mathrm{g}, n)$.

Then, the expression in (3) looks as follows:

$$\begin{aligned}(\Psi_n^*\sigma)(g\exp(\mu_k^r(b_r^{(k)} + \tilde{b}_r^{(k)}))) &= \\ \langle\rho|\Psi_n(g)\Psi_n(\exp(\mu_k^r(b_r^{(k)} + \tilde{b}_r^{(k)})|\chi\rangle &= \\ \langle\phi_P|\exp(\mu_k^r(b_r^{(k)} + \tilde{b}_r^{(k)}))|\chi\rangle\end{aligned} \tag{8}$$

(we used (7)).

The expression in (8) can be considered an inhomogeneous closed differential form on $G = \mathcal{A}^n$; it descends to $G/\mathfrak{k} = \mathcal{P}(\mathrm{g}, n)$ because ϕ_P is $\mathfrak{k}'_P$-invariant.

Homogeneous components of the form shown in (8) are closed forms on $G/\mathfrak{k} = \mathcal{P}(\mathrm{g}, n)$ that can be represented as

$$\langle\phi_P B|\chi\rangle \tag{9}$$

where B is a homogeneous polynomial with respect to $b_r^{(k)} + \tilde{b}_r^{(k)}$

The expression in (9) coincides with formulas of operator formalism. Let us show that the differential form in (9) descends to some quotients of G; by integrating with respect to cycles in the quotients, we obtain string amplitudes.

It follows from the considerations above that this expression descends to a closed form on $\mathcal{P}(\mathrm{g}, n)$. Moreover, by imposing some conditions on χ, one can prove that it further descends to a closed form ω_B on $\hat{\mathcal{P}}(\mathrm{g}, n) = \mathcal{P}(\mathrm{g}, n)/(S^1)^n$. (The action of group $(S^1)^n$ on $\mathcal{P}(\mathrm{g}, n)$ is defined in terms of rotations of boundary circles.) Namely, we should assume that $\chi \in \mathcal{E}'^n$ can be represented as a tensor product $\chi = \chi^{(1)} \otimes \ldots \otimes \chi^{(n)}$, where $(L_0^{(k)} - \tilde{L}_0^{(k)})\chi^{(k)} = 0$, $(b_0^{(k)} - \tilde{b}_0^{(k)})\chi^{(k)} = 0$. This condition means that we can apply the statement at the very end of Section 2 by taking the Lie algebra of group $(S^1)^n$ as $\mathfrak{h}$.

There exists a natural map $\hat{\mathcal{P}}(\mathrm{g}, n) \to \mathcal{M}(\mathrm{g}, n)$, where $\mathcal{M}(\mathrm{g}, n)$ is the moduli space of complex curves (one-dimensional, compact complex manifolds) of genus g with n marked points. This map is a homotopy equivalence; hence, it induces an isomorphism of homology groups. This allows us to integrate forms ω_B over homology classes of $\mathcal{M}(\mathrm{g}, n)$. (Of course, we can obtain a non-zero answer only if the dimension of the form is equal to the dimension of the homology class. Notice that equivalently, we can integrate the original non-homogeneous form;

the answer depends only on the homogeneous component of degree equal to the dimension of the integration cycle.)

We obtain a formal expression for string amplitudes by integrating ω_B over the fundamental homology cycle of $\mathcal{M}(g,n)$ (one should take B having degree equal to the dimension of $\mathcal{M}(g,n)$). This is a formal divergent expression; the physical explanation of divergence is the presence of tachyons in the spectrum of bosonic strings. From a mathematical viewpoint, the problem lies in the non-compactness of $\mathcal{M}(g,n)$ (a fundamental homology class is a locally finite cycle; to guarantee convergence, we should integrate over a finite cycle or to work with Deligne–Mumford compactification). However, integrals of form ω_B over genuine homology classes of $\mathcal{M}(g,n)$ exist. (Notice that these forms where used in string field theory [5].)

7. Conclusions and Modifications

In the present paper, we have shown that by starting with the one-string space of states in BRST formalism, one can obtain an expression for string amplitudes: one should integrate (9) over some cycles in appropriate quotients of $G = \mathcal{A}^n$.

The above constructions can be modified in various ways.

Our considerations are based on the statement at the end of Section 2: we assumed that $G = \mathcal{A}^n$, Lie subalgebra $\mathfrak{k}$ is a Lie stabilizer of G at the point of $\mathcal{P}(g,n)$ and Lie subalgebra $\mathfrak{h}$ is the Lie algebra of $(S^1)^n$. One can take other subalgebras $\mathfrak{k},\mathfrak{h}$.; in particular, one can take one or both of these subalgebras as Lie stabilizers of G at the points of $\mathcal{P}(g,n,p,m)$. (For example, in the situation described at the end of Section 4, we can take $\mathfrak{k} = \mathfrak{k}_M$ and $\langle\rho| = \langle\tau(\kappa)|$.)

One can hope to obtain closed forms with integrals related to interesting physical quantities (for example, to inclusive cross-sections or to mass renormalization [6]).

One more way to obtain new quantities is based on the remarks at the end of the appendix, where it is shown that one can construct an analog of operator formalism in terms of L-functionals.

Other modifications allow us to consider scattering in superstrings and heterotic strings. They are based on the consideration of superconformal manifolds and the supersymmetric analog of semigroup $\mathcal{A}$. Notice that in the present paper, we tacitly assumed that we consider left–right symmetric conformal field theories; of course, when considering heterotic strings and other theories with independent left and right sectors, we should drop this assumption. (In these cases, it is useful to apply the ideas of [7].) More details will be given in the follow-up paper entitled "A new approach to superstring".

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Appendix A

Let us start with some general remarks about quantization of symplectic vector spaces. In appropriate coordinates, we can write the symplectic form on such a space either as $\omega = \sum dp_k dq^k$ (real Darboux coordinates p_k, q^k) or as $\omega = \sum da_k^* da_k$ (complex Darboux coordinates a_k^*, a_k). (Notice that our considerations can also be applied in the case where the number of indices is infinite or, more generally, in the case where k takes values in some measure space; in the latter case, one should replace sums with integrals.) In real Darboux coordinates, we can represent a quantum state as a vector (or, more precisely, as a ray) in the Hilbert space of square integrable functions of q_k (coordinate representation) or of p_k (momentum representation); these representations are related by the Fourier transform. In a complex

Darboux representation, we represent a state as a vector in Fock space $\mathcal{F}$ (in a representation of canonical commutation relations

$$[\hat{a}_k, \hat{a}_l] = \delta_{k,l}, [\hat{a}_k, \hat{a}_l] = [\hat{a}^*_k, \hat{a}^*_l] = 0 \tag{A1}$$

where there exists a cyclic vector θ obeying $\hat{a}_k\theta = 0$). Notice that the choice of Darboux coordinates is not unique; different Darboux coordinates are related by linear canonical transformations:

$$\tilde{p}_k = A^l_k p_l + B_{kl} q^l, \tilde{q}^k = C^{kl} p_l + D^k_l q^l$$

in the real case and

$$\tilde{a}_k = \Phi^l_k a_l + \Psi^l_k a^*_l, \tilde{a}^*_k = \overline{\Phi}^l_k a^*_l + \overline{\Psi}^l_k a_l$$

in the complex case. (Recall that by definition, canonical transformations preserve Poisson brackets in classical mechanics and commutation relations after quantization.)

Let us concentrate our attention on the complex case. One says that the canonical transformation

$$\tilde{\hat{a}}_k = \Phi^l_k \hat{a}_l + \Psi^l_k \hat{a}^*_l, \tilde{\hat{a}}^*_k = \overline{\Phi}^l_k \hat{a}^*_l + \overline{\Psi}^l_k \hat{a}_l$$

is proper if there exists a unitary operator U obeying

$$\tilde{\hat{a}}_k = U a_k U^{-1}, \tilde{\hat{a}}^*_k = U a^*_k U^{-1}.$$

In the case of a finite number of degrees of freedom, all canonical transformations are proper; hence, a Hilbert spaces constructed by means of different Darboux coordinates can be identified (up to a constant factor, because U is defined up to such a factor). It is easy to check that a canonical transformation is proper iff there exists a vector $\tilde{\theta}$ in Fock space $\mathcal{F}$ obeying $\tilde{\hat{a}}_k\tilde{\theta} = 0$ (see [8] for more details). Vector $\tilde{\theta}$ corresponds to a Lagrangian subspace W in the complexification of symplectic vector space V; subspace W is defined by equations

$$\Phi^l_k a_l + \Psi^l_k a^*_l = 0.$$

Conversely, a Lagrangian subspace W in the complexification of V specifies a vector θ_W in $\mathcal{F}$; this vector is defined by equations

$$\hat{w}_k \theta_W = 0 \tag{A2}$$

where w_k stands for a basis of W. Notice that (A2) does not always have a solution, but if the solution exists, it is defined up to a constant factor. The solution is not necessarily normalizable (if W is real, θ_W is always non-normalizable).

In general, Lagrangian submanifolds correspond to vectors in Hilbert spaces (in the framework of semiclassical approximation). This correspondence is ambiguous, but for linear symplectic spaces and linear Lagrangian submanifolds (the case we consider), the quantization is a well-defined procedure.

The same construction works if the canonical commutation relations in (A1) are replaced with the canonical anticommutation relations

$$[\hat{a}_k, \hat{a}_l]_+ = \delta_{k,l}, [\hat{a}_k, \hat{a}_l]_+ = [\hat{a}^*_k, \hat{a}^*_l]_+ = 0 \tag{A3}$$

and the bosonic Fock space is replaced with the fermionic Fock space.

The coordinates in the analog of the symplectic vector space are regarded to be odd (anticommuting) variables.

Let us now consider an oriented compact manifold M with the boundary being represented as a disjoint union of two parts: outgoing part ∂M_+, with orientation agreeing with the orientation of M, and incoming part ∂M_-, with opposite orientation. Let us fix an action functional S on fields defined on M. Then, variation δS of functional S can be written in the form

$$\delta S = \int_M EM + \alpha_+ - \alpha_- \tag{A4}$$

The first summand contains integration over the whole manifold, and it vanishes if the fields obey the equations of motion. The second and third summands contain integration over the outgoing boundary (α_+) and the incoming boundary (α_-). We can consider all summands in (A4) to be one-forms on the space of fields. Let us restrict (A4) to space $\mathcal{E}$ of fields satisfying the equations of motion $EM = 0$. Then, the first summand disappears, and the difference $\alpha_+ - \alpha_-$ is equal to exact form δS. This means that two-forms $\delta\alpha_+$ and $\delta\alpha_-$ coincide on $\mathcal{E}$. (We use the notation δ for the de Rham differential on infinite-dimensional spaces.) We obtain a closed two-form on $\mathcal{E}$; if this form is non-degenerate, we can consider $\mathcal{E}$ a symplectic manifold; in general, $\mathcal{E}$ is a presymplectic manifold.

Let us consider in more detail the case where M is a two-dimensional manifold. Then, the boundary of M consists of disjoint circles. By applying the above construction to an annular neighborhood of a circle (considering the space of solutions of equations of motion on an annulus), we obtain a presymplectic manifold; let us assume that this manifold is symplectic. We identify it with the phase space and denote it by $\mathcal{P}$.

Let us assume that that the boundary of M consists of n outgoing circles (the incoming boundary is empty). By restricting the solutions of equations of motion on M to the annular neighborhoods of boundary circles, we obtain a map of space $\mathcal{E}$ of solutions on M into the n-th power of phase space $\mathcal{P}$. It follows from the consideration above that the image of this map is a Lagrangian submanifold of $\mathcal{P}^n$.

If action functional S is quadratic, the equations of motion are linear, and we can apply the constructions reported at the beginning of the appendix to quantize $\mathcal{P}$ and this Lagrangian submanifold. We obtain Hilbert space $\mathcal{H}$ and a vector (more precisely a ray) in $\mathcal{H}^n$.

If action functional S is conformally invariant, we can consider M an element of $\mathcal{P}(g, n)$. We obtain map $\sigma_{g,n} : \mathcal{P}(g, n) \to \mathcal{H}^n$ of Segal's approach to CFT. (In general, this map is defined up to a factor; this corresponds to CFT with a non-vanishing central charge.)

All our considerations can be applied to the case where the action functional is defined on commuting and anticommuting fields; then, we should work with symplectic superspaces and their Lagrangian submanifolds. This remark allows us to apply the above techniques to bosonic strings in flat 26-dimensional Minkowski space (in BRST formalism, all equations of motion are linear). In this case, we recover formulas of operator formalism of bosonic string theory [1].

Let us apply the same techniques in the formalism of L-functionals (see, for example, [4]). In this formalism, we assign to every vector Φ in the representation space of CCR (A1) or CAR (A3) a functional

$${}_{\Phi}(\alpha^*, \alpha) = \langle e^{-\alpha\hat{a}^*} e^{\alpha^*\hat{a}}\Phi, \Phi\rangle$$

or, more generally, to every density matrix K in this space a functional

$$L_K(\alpha^*, \alpha) = tr e^{-\alpha\hat{a}^*} e^{\alpha^*\hat{a}} K.$$

Here, $e^{-\alpha\hat{a}^*} = e^{-\alpha^k\hat{a}^*_k}$, where α^k are commuting parameters in the case of CCR and anti-commuting parameters in the case of CAR.

Nonlinear L-functional $L(\alpha^*, \alpha)$ corresponds to positive linear functional on Weyl algebra (a *-algebra with generators obeying CCR) or Clifford algebra (where CCR are replaced with CAR). For every element B of *-algebra $\mathcal{A}$, one can define two operators acting on the space of linear functionals on $\mathcal{A}$; one of them (denoted by the same symbol, B) transforms linear functional $\omega(A)$ into linear functional $\omega(AB)$, and the second one (denoted by the symbol $\tilde{B}$) transforms this functional into linear functional $\omega(B^*A)$. If functional $\omega(A)$ corresponds to vector Φ (i.e., $\omega(A) = \langle \Phi, A\Phi \rangle$) and $B\Phi = 0$, then $B\omega = 0$, and $\tilde{B}\omega = 0$. This remark allows us to write down the equations for functionals ω corresponding to vectors Φ that appear in operator formalism.

By representing linear functionals on Weyl or Clifford algebra as functionals $L(\alpha^*, \alpha)$, we can calculate operators on these functionals corresponding to generators $\hat{a}_k, \hat{a}_k^*$ (see [4]). Using this remark, we obtain equations for functionals $L(\alpha^*, \alpha)$ appearing in operator formalism.

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