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Alleviating the H_0 Tension in Scalar–Tensor and Bi-Scalar–Tensor Theories

Maria Petronikolou [1,2,*] and Emmanuel N. Saridakis [1,3,4]

[1] National Observatory of Athens, Lofos Nymfon, 11852 Athens, Greece; msaridak@noa.gr
[2] Department of Physics, National Technical University of Athens, Zografou Campus, 15773 Athens, Greece
[3] CAS Key Laboratory for Researches in Galaxies and Cosmology, Department of Astronomy, University of Science and Technology of China, Hefei 230026, China
[4] Departamento de Matemáticas, Universidad Católica del Norte, Avda. Angamos 0610, Casilla, Antofagasta 1280, Chile
* Correspondence: petronikoloumaria@mail.ntua.gr

Abstract: Herein, we investigate scalar–tensor and bi-scalar–tensor modified theories of gravity that can alleviate the H_0 tension. In the first class of theories, we show that by choosing particular models with a shift-symmetric friction term we are able to alleviate the tension by obtaining a smaller effective Newton's constant at intermediate times, a feature that cannot be easily obtained in modified gravity. In the second class of theories, which involve two extra propagating degrees of freedom, we show that the H_0 tension can be alleviated, and the mechanism behind this is the phantom behavior of the effective dark-energy equation-of-state parameter. Hence, scalar–tensor and bi-scalar–tensor theories have the ability to alleviate the H_0 tension with both known sufficient late-time mechanisms.

Keywords: H_0 tension; modified gravity; Horndeski; scalar-tensor; bi-scalar theories; generalized Galileon

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1. Introduction

Although the ΛCDM concordance paradigm of cosmology, which is based on general relativity alongside the cold dark matter sector and the cosmological constant, is very successful in describing the universe's evolution, it seems to exhibit possible disadvantages, at both the theoretical and phenomenological levels [1]. In the first category, one can find the cosmological constant problem, as well as the non-renormalizability of general relativity. In the second category one may find possible cosmological tensions.

In particular, a first tension is related to the present value of the Hubble parameter H_0, since the value estimated by the Planck collaboration was $H_0 = (67.27 \pm 0.60)$ km/s/Mpc [2], while the direct measurement of the 2019 SH0ES collaboration (R19) gave a value of $H_0 = (74.03 \pm 1.42)$ km/s/Mpc, representing a difference of about 4.4σ. Furthermore, there is the issue of the σ_8 related to matter clustering and the possible deviation of the cosmic microwave background (CMB) estimation [2] from the SDSS/BOSS measurement [3,4]. Although there has been much discussion as to whether these tensions are due to unknown systematics, it seems that at least the H_0 tension may indeed be a sign of new physics [5–44] (for a review see, [45]).

On the other hand, modified gravity refers to a very broad class of theories that aim to alleviate the non-renormalizability of general relativity, bypass the cosmological constant problem, and lead to improved cosmological evolution, at both the background and the perturbation levels [46,47]. In order to construct gravitational modifications, one can start from the Einstein–Hilbert action of general relativity and add extra terms in the Lagrangian, resulting in $f(R)$ gravity [48–57], Gauss–Bonnet and $f(G)$ gravity [58–61], cubic gravity [62], Lovelock gravity [63,64], etc. Alternatively, one can start form the equivalent torsional formulation of gravity and modify it suitably, resulting in $f(T)$ gravity [65–81],

$f(T, T_G)$ gravity [82,83], $f(T, B)$ gravity [84–88], etc. Additionally, a broad class of gravitational modifications are the scalar–torsion theories, which are constructed by one scalar field coupled to curvature terms. In particular, the most general four-dimensional scalar–tensor theory with one propagating scalar degree of freedom is Horndeski gravity [89] or equivalently generalized Galileon theory [90–101]. Finally, one may extend this framework beyond Horndeski theories [102–106], as well as to bi-scalar–tensor theories, in which one has two extra scalar fields [107,108].

The effect of modified gravity on late-time universe evolution is two-fold. The first aspect is that it induces new terms in the Friedmann equations, which can collectively be absorbed into an effective dark-energy sector. The second is that it typically leads to a modified Newton's constant. Hence, in every cosmology governed by a modified theory of gravity, one typically obtains Friedmann equations of the form [46]

$$H^2 = \frac{8\pi G_{eff}}{3}\left(\rho_m + \rho_{DE}^{eff}\right), \tag{1}$$

$$\dot{H} = -4\pi G_{eff}\left(\rho_m + \rho_{DE}^{eff} + p_m + p_{DE}^{\mathrm{eff}}\right), \tag{2}$$

where ρ_{DE}^{eff} and p_{DE}^{eff} are, respectively, the effective dark-energy density and pressure, and G_{eff} is the effective Newton's constant, all depending on the parameters of the theory. Hence, qualitatively, we deduce that in order to alleviate the H_0 tension in this framework, i.e., obtain a higher H_0 than standard lore predicts, we have two possible methods [109,110]: (i) one could try to obtain a smaller effective Newton's constant, since "weaker" gravity is reasonable to induce faster expansion, or (ii) one could try to obtain suitable modified-gravity-oriented extra terms in the effective dark-energy sector, which could lead to faster expansion, e.g., obtaining an effective dark-energy equation-of-state parameter $w_{DE} := p_{DE}/\rho_{DE}$ lying in the phantom regime.

In this work, we present two broad classes of modified gravity that can fulfill the above qualitative requirements in the correct quantitative way and alleviate the H_0 tension. The first class includes scalar–tensor theories [89], and the second includes bi-scalar–tensor theories [107,108]. Interestingly enough, we show that in the first class the mechanism behind the alleviation is the smaller G_{eff}, while in the second class it is the phantom dark energy. The rest of this paper is structured as follows: In Section 2 we briefly review scalar–tensor theories and present specific models that can alleviate the tension. In Section 3 we present bi-scalar–tensor theories and construct models alleviating the tension. Finally, in Section 4, we summarize the obtained results.

2. Scalar–Tensor Theories Alleviating the H_0 Tension

In this section, we briefly review scalar–tensor theories and then present particular models that can alleviate the H_0 tension. The most general Lagrangian with one extra scalar degree of freedom ϕ and curvature terms, giving rise to second-order field equations, is $\mathcal{L} = \sum_{i=2}^{5} \mathcal{L}_i$ [89,111,112], where

$$\mathcal{L}_2 = K(\phi, X), \tag{3}$$

$$\mathcal{L}_3 = -G_3(\phi, X)\Box\phi, \tag{4}$$

$$\mathcal{L}_4 = G_4(\phi, X)\, R + G_{4,X}\left[(\Box\phi)^2 - (\nabla_\mu\nabla_\nu\phi)\,(\nabla^\mu\nabla^\nu\phi)\right], \tag{5}$$

$$\begin{aligned}\mathcal{L}_5 = G_5(\phi, X)\, G_{\mu\nu}\,(\nabla^\mu\nabla^\nu\phi) - \frac{1}{6}\, G_{5,X}\big[&(\Box\phi)^3 - 3(\Box\phi)\,(\nabla_\mu\nabla_\nu\phi)\,(\nabla^\mu\nabla^\nu\phi) \\ &+ 2(\nabla^\mu\nabla_\alpha\phi)\,(\nabla^\alpha\nabla_\beta\phi)\,(\nabla^\beta\nabla_\mu\phi)\big].\end{aligned} \tag{6}$$

As usual, R is the Ricci scalar; $G_{\mu\nu}$ is the Einstein tensor; the functions K and G_i ($i = 3, 4, 5$) depend on ϕ and its kinetic energy $X = -\partial^\mu\phi\partial_\mu\phi/2$; and $G_{i,X} := \partial G_i/\partial X$,

$G_{i,\phi} := \partial G_i/\partial\phi$. Focusing on Friedmann–Robertson–Walker (FRW) geometry with the metric

$$ds^2 = -dt^2 + a^2(t)\,\delta_{ij}dx^i dx^j, \tag{7}$$

and adding the matter Lagrangian $\mathcal{L}_m$ corresponding to a perfect fluid with energy density ρ_m and pressure p_m, by performing variation we obtain the two generalized Friedmann equations:

$$\begin{aligned}
&2XK_{,X} - K + 6X\dot{\phi}HG_{3,X} - 2XG_{3,\phi} - 6H^2G_4 + 24H^2X(G_{4,X} + XG_{4,XX}) - 6H\dot{\phi}G_{4,\phi} \\
&-12HX\dot{\phi}G_{4,\phi X} + 2H^3X\dot{\phi}(5G_{5,X} + 2XG_{5,XX}) - 6H^2X(3G_{5,\phi} + 2XG_{5,\phi X}) = -\rho_m,
\end{aligned} \tag{8}$$

$$\begin{aligned}
&K - 2X(G_{3,\phi} + \ddot{\phi}G_{3,X}) + 2(3H^2 + 2\dot{H})G_4 - 8\dot{H}XG_{4,X} - 12H^2XG_{4,X} - 4H\dot{X}G_{4,X} \\
&-8HX\dot{X}G_{4,XX} + 2(\ddot{\phi} + 2H\dot{\phi})G_{4,\phi} + 4XG_{4,\phi\phi} + 4X(\ddot{\phi} - 2H\dot{\phi})G_{4,\phi X} \\
&-4H^2X^2\ddot{\phi}G_{5,XX} - 2X(2H^3\dot{\phi} + 2H\dot{H}\dot{\phi} + 3H^2\ddot{\phi})G_{5,X} + 4HX(\dot{X} - HX)G_{5,\phi X} \\
&+4HX\dot{\phi}G_{5,\phi\phi} + 2[2(\dot{H}X + H\dot{X}) + 3H^2X]G_{5,\phi} = -p_m,
\end{aligned} \tag{9}$$

with dots denoting derivatives with respect to t. Moreover, variation with respect to $\phi(t)$ gives

$$\frac{1}{a^3}\frac{d}{dt}(a^3J) = P_\phi, \tag{10}$$

with

$$\begin{aligned}
J :=\; &\dot{\phi}K_{,X} + 6HXG_{3,X} - 2\dot{\phi}G_{3,\phi} - 12HXG_{4,\phi X} + 6H^2\dot{\phi}(G_{4,X} + 2XG_{4,XX}) \\
&+2H^3X(3G_{5,X} + 2XG_{5,XX}) + 6H^2\dot{\phi}(G_{5,\phi} + XG_{5,\phi X}),
\end{aligned} \tag{11}$$

$$\begin{aligned}
P_\phi :=\; &K_{,\phi} - 2X(G_{3,\phi\phi} + \ddot{\phi}G_{3,\phi X}) + 6(2H^2 + \dot{H})G_{4,\phi} \\
&+6H(\dot{X} + 2HX)G_{4,\phi X} - 6H^2XG_{5,\phi\phi} + 2H^3X\dot{\phi}G_{5,\phi X}.
\end{aligned} \tag{12}$$

Lastly, as usual, we consider the matter conservation equation $\dot{\rho}_m + 3H(\rho_m + p_m) = 0$.

In the following, we present specific models of scalar–tensor theories that can alleviate the H_0 tension [113]. Since the Horndeski theory recovers ΛCDM cosmology for $G_4 = 1/(16\pi G)$, $K = -2\Lambda = const$, and $G_3 = G_5 = 0$, our strategy is to introduce deviations that are negligible at high redshifts, where the CMB structure is formed, but become significant at low redshifts, where local Hubble measurements take place.

We start by examining a subclass of Horndeski gravity that contains the G_5 term, which is called "non-minimal derivative coupling". In particular, we can consider models with $G_4 = 1/(16\pi G)$ and $G_3 = 0$, which is the case in ΛCDM cosmology, and impose a simple scalar-field potential and standard kinetic term, namely $K = -V(\phi) + X$. Additionally, since G_5 affects the friction terms of the scalar field [114–116], we make the G_5 term depend only on X, i.e., $G_5(\phi, X) = G_5(X)$. Inserting this into (8) and (9) gives the effective dark-energy density and pressure [113]:

$$\rho_{DE} = 2X - K + 2H^3X\dot{\phi}(5G_{5,X} + 2XG_{5,XX}), \tag{13}$$

$$p_{DE} = K - 2XG_{5,X}\left(2H^3\dot{\phi} + 2H\dot{H}\dot{\phi} + 3H^2\ddot{\phi}\right) - 4H^2X^2\ddot{\phi}G_{5,XX}, \tag{14}$$

and thus the dark-energy equation-of-state parameter becomes

$$w_{DE} \equiv \frac{p_{DE}}{\rho_{DE}}. \tag{15}$$

One can choose a suitable $G_5(X)$ in order for $H(z)$ to coincide with $H_{\Lambda\text{CDM}}(z) := H_0\sqrt{\Omega_{m0}(1+z)^3 + 1 - \Omega_{m0}}$ at $z = z_{\text{CMB}} \approx 1100$, namely $H(z \to z_{\text{CMB}}) \approx H_{\Lambda\text{CDM}}(z \to$

$z_{\rm CMB}$), but satisfy $H(z \to 0) > H_{\Lambda\rm CDM}(z \to 0)$. For simplicity, we focus on dust matter (i.e., $p_m = 0$), and without q loss of generality we consider $K = -V_0\phi + X$.

We start with the investigation of a model with

$$G_5(X) = \xi X^2. \tag{16}$$

In this case, (13) and (14) give

$$\rho_{DE} = \frac{\dot{\phi}^2}{2} + V_0\phi + 7\xi H^3 \dot{\phi}^5, \tag{17}$$

$$p_{DE} = \frac{\dot{\phi}^2}{2} - V_0\phi - \xi\dot{\phi}^4 \left(2H^3\dot{\phi} + 2H\dot{H}\dot{\phi} + 5H^2\ddot{\phi}\right). \tag{18}$$

As we mentioned above, we choose the model parameter V_0 and the initial conditions for the scalar field in order to obtain $H(z_{\rm CMB}) = H_{\Lambda\rm CDM}(z_{\rm CMB})$ and $\Omega_{m0} = 0.31$, in agreement with [2], and we handle ξ as the parameter that determines the late-time deviation from ΛCDM cosmology. We solve the Friedmann equation numerically, and in Figure 1 we depict $H(z)/(1+z)^{3/2}$ for different choices of ξ. As one can see, the model coincides with ΛCDM at high and intermediate redshifts, while at small redshifts it leads to higher values of H_0. In particular, H_0 depends on the model parameter ξ, and it can be around $H_0 \approx 74$ km/s/Mpc for $\xi = 1.3$ (we mention here that since ξ has dimensions of $[M]^{-9}$ and since $H_0 \approx 10^{-61}$ in Planck units, this gives $\xi^{1/9} \sim 10^{40}$ GeV^{-1}). Hence, one can see that the H_0 tension can be alleviated at 3σ if $1.2 < \xi < 1.7$.

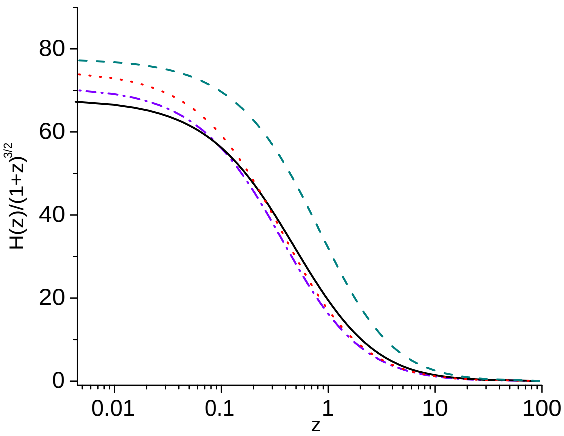

Figure 1. The normalized $H(z)/(1+z)^{3/2}$ in km/s/Mpc as a function of the redshift for ΛCDM cosmology (solid black line) and for scalar–tensor Model I with $V_0 = 0.06$ and $G_5(X) = \xi X^2$ for $\xi = 1.7$ (purple dashed and dotted line), $\xi = 1.4$ (red dotted line), and $\xi = 1.1$ (blue dashed line) in H_0 units. We imposed $\Omega_{m0} \approx 0.31$.

Let us now examine the mechanism behind the tension alleviation, following [113]. In the left-hand graph of Figure 2, we depict the effective dark-energy equation-of-state parameter w_{DE} given in (15). As we can see, it does not exhibit phantom behavior, i.e., it cannot be the cause of the increased H_0 [109,110]). On the other hand, we note that in scalar–tensor Horndeski gravity, one obtains an effective Newton's constant as follows [117,118]:

$$\frac{G_{eff}}{G} = \frac{1}{2}\left(G_4 - 2XG_{4,X} + XG_{5,\phi} - \dot{\phi}HXG_{5,X}\right)^{-1}. \tag{19}$$

In the right-hand graph of Figure 2, we depict the evolution of the normalized effective Newton's constant $\frac{G_{eff}}{G}$. As we can see, we obtain a decrease in the effective Newton's constant at intermediate redshifts, and as we mentioned in the introduction, this can lead to an increased H_0. Hence, we deduce that in the scenario at hand, the mechanism behind the tension alleviation is the decreased G_{eff}, i.e., a suitably weaker gravity.

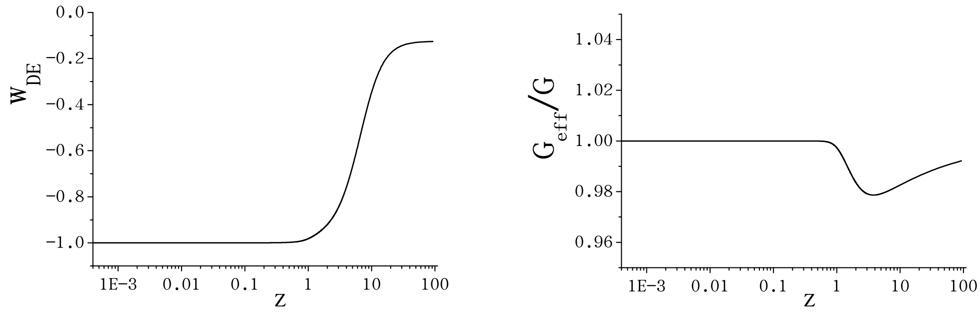

Figure 2. (**Left**): The effective dark-energy equation-of-state parameter w_{DE} given in (15) as a function of the redshift for Model I with $V_0 = 0.08$ and $\xi = 1.3$ in H_0 units. (**Right**): The corresponding normalized effective Newton's constant $\frac{G_{eff}}{G}$ given in (19) as a function of the redshift. The graphs are from [113].

Finally, let us discuss the perturbative behavior of the model at hand. As one can show [111,119,120], in order for the Horndeski/generalized Galileon theory to be free from Laplacian instabilities associated with the scalar field propagation speed, one should have

$$c_S^2 \equiv \frac{3(2w_1^2 w_2 H - w_2^2 w_4 + 4w_1 w_2 \dot{w}_1 - 2w_1^2 \dot{w}_2)}{w_1(4w_1 w_3 + 9w_2^2)} \geq 0, \tag{20}$$

while in order to avoid perturbative ghosts, one should have

$$Q_S \equiv \frac{w_1(4w_1 w_3 + 9w_2^2)}{3w_2^2} > 0. \tag{21}$$

Additionally, the light speed in these theories is [111]

$$c_T^2 \equiv \frac{w_4}{w_1} \geq 0, \tag{22}$$

which at late times should be very close to 1, in agreement with LIGO/Virgo bounds [121]. By studying c_S^2, Q_S, and c_T^2, one can show that the scenario at hand is viable [113], although with a certain amount of fine-tuning required.

We can examine other models that can lead to similar behavior, namely a smaller effective Newton's constant due to the friction term $G_5(X)$ that can result in a higher H_0. For instance, a model with $G_5(X) = \lambda X^4$ also leads to $H_0 \approx 74$ km/s/Mpc for $\lambda = 1$ in H_0 units (since λ has dimensions of $[M]^{-17}$, we acquire $\lambda^{1/17} \sim 10^{30}$ GeV^{-1}), and the tension can be alleviated at 3σ if $0.5 < \lambda < 1.2$ in H_0 units. On the other hand, one can see that models with odd powers of X do not solve the tension, since the last term in (19) changes signs, and this does not guarantee that G_{eff}/G will remain smaller than 1.

Finally, we can consider combinations of monomial forms, such as

$$G_5(X) = \xi X^2 + \lambda X^4, \tag{23}$$

in which case we have more freedom to obtain the desired decreased G_{eff}/G. We elaborate the equations numerically, and in Figure 3 we present the normalized Hubble constant evolution. As one can observe, the H_0 tension is alleviated due to the decreased effective Newtons' constant.

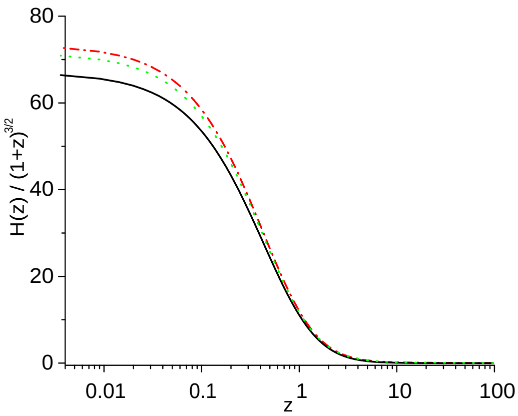

Figure 3. The normalized $H(z)/(1+z)^{3/2}$ in km/s/Mpc as a function of the redshift for ΛCDM cosmology (solid black line) and for the combined scalar–tensor Model (23) with $V_0 = 0.08$ and with $\xi = 1.5$, $\lambda = 0.001$ (green dotted line) and $\xi = 1.3$, $\lambda = 0.002$ (red dashed and dotted line) in H_0 units. We imposed $\Omega_{m0} \approx 0.31$.

We close this section by mentioning that although in modified theories of gravity in general one acquires an effective Newton's constant in a different manner to standard theories, a viable $G_{eff}/G < 1$ is not easily obtained. For instance, in $f(R)$ gravity, where [56]

$$\frac{G_{eff}}{G} = \frac{1}{f_R}\frac{1+4\frac{k^2}{a^2}\frac{f_{RR}}{f_R}}{1+3\frac{k^2}{a^2}\frac{f_{RR}}{f_R}}, \tag{24}$$

with k being the wave number, one can see that under the viability conditions $f_{,R} > 0$ for $R \geq R_0$ (with R_0 representing the present value of the Ricci scalar) and $f_{,RR} > 0$ for $R \geq R_0$ [48], as well as $0 < \frac{Rf_{,RR}}{f_{,R}}(r) < 1$ at $r = -\frac{Rf_{,R}}{f} = -2$ [48], $G_{eff}/G < 1$ cannot be obtained. On the other hand, this is indeed possible in $f(T)$ gravity, where $G_{eff} = G\left(1+\frac{\partial f(T)}{\partial T}\right)^{-1}$ [122]. However, scalar–tensor theories may present such behavior quite easily.

In summary, as one can see, the above sub-class of scalar–tensor gravity can alleviate the H_0 tension due to the effect of the kinetic-energy-dependent G_5 term on decreasing G_{eff}.

3. Bi-Scalar–Tensor Theories Alleviating the H_0 Tension

In this section, we present another class of modified gravity that can lead to the alleviation of the H_0 tension, namely bi-scalar theories of gravity. These theories are determined by the action [107,108]

$$S = \int d^4\sqrt{-g}\, f\Big(R, (\nabla R)^2, \Box R\Big), \tag{25}$$

with $(\nabla R)^2 = g^{\mu\nu}\nabla_\mu R \nabla_\nu R$. In this work, we focus on models with $f(R, (\nabla R)^2, \Box R) = \mathcal{K}((R, (\nabla R)^2) + \mathcal{G}(R, (\nabla R)^2)\Box R$. We can rewrite the above action by transforming the Lagrangian using double Lagrange multipliers, in which case one can clearly see that they correspond to bi-scalar–tensor theories of gravity. Hence, introducing the scalar fields ϕ and χ through $g_{\mu\nu} = \frac{1}{2}e^{-\sqrt{\frac{2}{3}}\chi}\hat{g}_{\mu\nu}$ and $\varphi := f_\beta$, with $\beta := \Box R$, we obtain [123]

$$\begin{aligned} S = \int d^4x\sqrt{-\hat{g}}\Big[&\frac{1}{2}\hat{R} - \frac{1}{2}\hat{g}^{\mu\nu}\nabla_\mu\chi\nabla_\nu\chi - \frac{1}{\sqrt{6}}e^{-\sqrt{\frac{2}{3}}\chi}\hat{g}^{\mu\nu}\mathcal{G}\nabla_\mu\chi\nabla_\nu\phi + \frac{1}{4}e^{-2\sqrt{\frac{2}{3}}\chi}\mathcal{K} \\ &+\frac{1}{2}e^{-\sqrt{\frac{2}{3}}\chi}\mathcal{G}\hat{\Box}\phi - \frac{1}{4}e^{-\sqrt{\frac{2}{3}}\chi}\phi\Big]. \end{aligned} \tag{26}$$

Thus, varying the above action in terms of the metric, we extract the Friedmann equations as follows [107,108]:

$$H^2 = \frac{1}{3}(\rho_{DE} + \rho_m) \tag{27}$$
$$2\dot{H} + 3H^2 = -(p_{DE} + p_m), \tag{28}$$

where we define an effective dark-energy sector with energy density and pressure given by

$$\begin{aligned}\rho_{DE} := &\frac{1}{2}\dot{\chi}^2 - \frac{1}{4}e^{-2\sqrt{\frac{2}{3}}\chi}\mathcal{K} - \frac{2}{3}\dot{\phi}^2\left[\dot{\phi}\left(\sqrt{6}\dot{\chi} - 9H\right) - 3\ddot{\phi}\right]\mathcal{G}_B \\ &+\frac{1}{2}e^{-\sqrt{\frac{2}{3}}\chi}\left[\dot{B}\dot{\phi}\mathcal{G}_B + \frac{\phi}{2} + \dot{\phi}^2(\mathcal{G}_\phi - 2\mathcal{K}_B)\right],\end{aligned} \tag{29}$$

$$p_{DE} := \frac{1}{2}\dot{\chi}^2 + \frac{1}{4}e^{-2\sqrt{\frac{2}{3}}\chi}\mathcal{K} + \frac{1}{2}e^{-\sqrt{\frac{2}{3}}\chi}\left(\dot{B}\dot{\phi}\mathcal{G}_B + \dot{\phi}^2\mathcal{G}_\phi - \frac{\phi}{2}\right), \tag{30}$$

Here, $\mathcal{K} = \mathcal{K}(\phi, B)$; $\mathcal{G} = \mathcal{G}(\phi, B)$;

$B = 2e^{\sqrt{\frac{2}{3}}\chi}g^{\mu\nu}\nabla_\mu\phi\nabla_\nu\phi$; and, for simplicity, we set the Planck mass to 1. Moreover, varying the action with respect to the scalar fields, we obtain their evolution equation as follows [107,108]:

$$\begin{aligned}&\ddot{\chi} + 3H\dot{\chi} - \frac{1}{3}\dot{\phi}^2\left[\dot{\phi}\left(3\sqrt{6}H - 2\dot{\chi}\right) + \sqrt{6}\ddot{\phi}\right]\mathcal{G}_B + \frac{1}{\sqrt{6}}e^{-2\sqrt{\frac{2}{3}}\chi}\mathcal{K} \\ &+\frac{1}{2\sqrt{6}}e^{-\sqrt{\frac{2}{3}}\chi}\left[2\dot{B}\dot{\phi}\mathcal{G}_B - \phi + 2\dot{\phi}^2(\mathcal{K}_B + \mathcal{G}_\phi)\right] = 0,\end{aligned} \tag{31}$$

with

$$\begin{aligned}&\frac{1}{3}e^{-\sqrt{\frac{2}{3}}\chi}\left[\dot{\phi}\left(-9H + \sqrt{6}\dot{\chi}\right) - 3\ddot{\phi}\right]\mathcal{K}_B + \frac{1}{6}\dot{B}\left\{3e^{-\sqrt{\frac{2}{3}}\chi}\dot{B} + 4\dot{\phi}\left[\dot{\phi}\left(9H - \sqrt{6}\dot{\chi}\right) + 3\ddot{\phi}\right]\right\}\mathcal{G}_{BB} \\ &+\frac{1}{3}e^{-\sqrt{\frac{2}{3}}\chi}\left[\dot{\phi}\left(9H - \sqrt{6}\dot{\chi}\right) + 3\ddot{\phi}\right]\mathcal{G}_\phi + \left\{e^{-\sqrt{\frac{2}{3}}\chi}\dot{B}\dot{\phi} + \frac{2}{3}\dot{\phi}^2\left[\dot{\phi}\left(9H - \sqrt{6}\dot{\chi}\right) + 3\ddot{\phi}\right]\right\}\mathcal{G}_{B\phi} \\ &+\left[\frac{4}{3}\dot{\phi}\left(9H - 2\sqrt{6}\dot{\chi}\right)\ddot{\phi} - \frac{1}{\sqrt{6}}e^{-\sqrt{\frac{2}{3}}\chi}\dot{B}\dot{\chi} + \dot{\phi}^2\left(18H^2 + 6\dot{H} - 3\sqrt{6}H\dot{\chi} - \frac{2}{3}\dot{\chi}^2 - \sqrt{6}\ddot{\chi}\right)\right]\mathcal{G}_B \\ &-e^{-\sqrt{\frac{2}{3}}\chi}\dot{\phi}^2\mathcal{K}_{B\phi} + \frac{1}{2}e^{-\sqrt{\frac{2}{3}}\chi}\dot{\phi}^2\mathcal{G}_{\phi\phi} - e^{-\sqrt{\frac{2}{3}}\chi}\dot{B}\dot{\phi}\mathcal{K}_{BB} - \frac{1}{4}e^{-2\sqrt{\frac{2}{3}}\chi}\mathcal{K}_\phi + \frac{1}{4}e^{-\sqrt{\frac{2}{3}}\chi} = 0,\end{aligned} \tag{32}$$

where $\mathcal{G}_{B\phi} = \mathcal{G}_{\phi B} \equiv \frac{\partial^2\mathcal{G}}{\partial B\partial\phi}$, etc. Finally, one can define the effective dark-energy equation-of-state parameter as $w_{DE} := p_{DE}/\rho_{DE}$.

Let us now extract specific models that coincide with ΛCDM cosmology at CMB redshifts while deviating from it at low redshifts, giving rise to a higher H_0. The first model that we can examine is Model I, with

$$\mathcal{K}(\phi, B) = \frac{1}{2}\phi - \frac{\zeta}{2}B \quad \text{and} \quad \mathcal{G}(\phi, B) = 0. \tag{33}$$

In this case, (29) and (30) give

$$\rho_{DE} = \frac{1}{2}\dot{\chi}^2 - \frac{1}{8}e^{-2\sqrt{\frac{2}{3}}\chi}\phi + \frac{1}{4}e^{-\sqrt{\frac{2}{3}}\chi}\left(\phi + \zeta\dot{\phi}^2\right), \tag{34}$$

$$p_{DE} = \frac{1}{2}\dot{\chi}^2 + \frac{1}{8}e^{-2\sqrt{\frac{2}{3}}\chi}\phi - \frac{1}{4}e^{-\sqrt{\frac{2}{3}}\chi}\left(\phi - \zeta\dot{\phi}^2\right). \tag{35}$$

We solve the cosmological equations numerically, and in the left-hand graph of Figure 4 we depict the normalized combination $H(z)/(1+z)^{3/2}$ as a function of the redshift for

ΛCDM cosmology and for Model I with different values of ζ. We find that H_0 depends on the model parameter ζ, as expected, and for $\zeta = -10$, it is around $H_0 \approx 74$ km/s/Mpc, which is consistent with its direct measurement (note that in natural units this corresponds to a typical value of $\zeta^{1/4} \sim -10^{-19}$ GeV^{-1}).

Let us now study the mechanism behind the H_0 alleviation. In the right-hand graph of Figure 4, we present the evolution of $w_{DE}(z)$. One can observe that most of the time it lies in the phantom regime, which, as we discussed in the introduction, is a way in which one can obtain tension alleviation. Hence, contrary to the case of single-scalar–tensor theories discussed in the previous section, where a decreased G_{eff} is the cause of tension alleviation, in the present bi-scalar theories it is the phantom behavior that leads to a higher H_0.

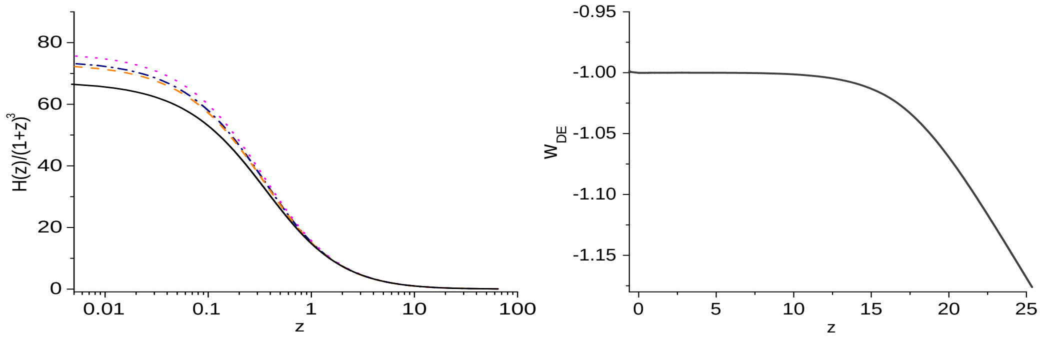

Figure 4. (**Left**): The normalized $H(z)/(1+z)^3$ in km/s/Mpc as a function of the redshift for ΛCDM cosmology (solid black line) and for bi-scalar–tensor Model I with $\zeta = -8$ (orange dashed line), $\zeta = -10$ (blue dashed and dotted line), and $\zeta = -12$ (magenta dotted line) in Planck units. We imposed $\Omega_{m0} \approx 0.31$. (**Right**): The corresponding effective dark-energy equation-of-state parameter w_{DE} as a function of the redshift for $\zeta = -10$ in Planck units.

We can proceed to the investigation of other models within the examined class. For instance, we can examine Model II, characterized by

$$\mathcal{K}(\phi, B) = \frac{1}{2}\phi \quad \text{and} \quad \mathcal{G}(\phi, B) = \xi B. \tag{36}$$

Repeating the same steps as in the previous model, we find that the present Hubble value H_0 depends on the model parameter ξ. In particular, for $\xi = -10$, it is around $H_0 \approx 74$ km/s/Mpc (in natural units, $\xi \sim -10$ corresponds to a typical value of $\xi^{1/8} \sim -10^{-19}$ GeV^{-1}). Similarly to the previous case, the mechanism behind the alleviation is the phantom behavior. Hence, we conclude that bi-scalar–tensor theories are very efficient in alleviating the H_0 tension.

4. Conclusions

We investigated scalar–tensor and bi-scalar–tensor modified theories of gravity that can alleviate the H_0 tension. In general, gravitational modifications affect the late-time evolution of the universe through the new terms they introduce into the Friedmann equations, namely in the effective dark-energy sector, as well as through the effective Newton's constant they induce. If these effects lead to weaker gravity (a smaller G_{eff}) at suitable redshifts, or to more repulsive effective dark energy (for instance, exhibiting phantom behavior), then they can cause faster expansion compared to the ΛCDM paradigm and thus lead to an increased H_0 value.

As a first class of models, we examined scalar–tensor theories, namely Horndeski/generalized Galileon gravity. Choosing particular models with a shift-symmetric G_5 friction term, we were able to alleviate the tension by obtaining a smaller effective Newton's constant at intermediate times, a feature that cannot be easily obtained in modified gravity theories. Additionally, we showed that the models at hand were free from perturbative instabilities, and they could have a gravitational-wave speed equal to the speed of light, though with a certain amount of fine-tuning necessary.

As a second class, we examined bi-scalar–tensor theories, namely theories involving two extra propagating degrees of freedom. Choosing particular models, we showed that the H_0 tension can be alleviated, and the mechanism behind this is the phantom behavior of the effective dark-energy equation-of-state parameter.

In summary, scalar–tensor theories with one or two scalar fields have the ability to alleviate the H_0 tension with both sufficient mechanisms. Such capabilities may be added to the other known phenomenological advantages of these theories and act as an additional indication that they could be good candidates for the description of nature.

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