Supplementary Material for 'Identification in Parametric Models: The Minimum Hellinger Distance Criterion

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David Pacini[1]

Auxiliary Calculations

This Supplementary Material presents step-by-step calculations of the squared Hellinger distance in Examples 1–5.

Example 1. The squared Hellinger distance in this example is

$$\rho(\theta) = 1 - \int f_\theta^{1/2} f_{\theta_o}^{1/2} d\mu = 1 - \int_{-\infty}^{+\infty} \left[\frac{1}{\theta} 1(0 \le y \le \theta) \right]^{1/2} \left[\frac{1}{\theta_o} 1(0 \le y \le \theta_o) \right]^{1/2} dy$$

$$= 1 - \int_0^{\min(\theta,\theta_o)} \frac{1}{\theta^{1/2}\theta_o^{1/2}} dy = 1 - \frac{\min(\theta,\theta_o)}{\sqrt{\theta\theta_o}} = 1 - \frac{1}{2\sqrt{\theta\theta_o}} \left(\theta + \theta_o - |\theta - \theta_o| \right).$$

Since $\nabla \ln f_\theta = -\frac{d\ln\theta}{d\theta} = -\theta^{-1}$ for $0 < y < \theta$, the Fisher matrix is

$$\mathcal{I}(\theta) = \int (\nabla \ln f_\theta)^2 f_\theta d\mu - \left[\int \nabla \ln f_\theta d\mu \right]^2$$

$$= \int_{-\infty}^{+\infty} \theta^{-2}\theta^{-1} 1(0 \le y \le \theta) dy - \left[\int_{-\infty}^{+\infty} -\theta^{-1}\theta^{-1} 1(0 \le y \le \theta) dy \right]^2 = \theta^{-2} - \theta^{-2} = 0$$

[1]David.Pacini@bristol.ac.uk

Example 2. The squared Hellinger distance in this example is

$$\rho(\theta) = 1 - \int f_\theta^{1/2} f_{\theta_o}^{1/2} d\mu$$

$$= 1 - \int_{-\infty}^{+\infty} \left[\frac{m}{\theta} \exp\left(\frac{m}{m-1} - \frac{m}{\theta} y \right) 1\left(y \geq \frac{\theta}{m-1} \right) \right]^{1/2} \left[\frac{m}{\theta_o} \exp\left(\frac{m}{m-1} - \frac{m}{\theta_o} y \right) 1\left(y \geq \frac{\theta_o}{m-1} \right) \right]^{1/2} d\,$$

$$= 1 - \frac{m}{\sqrt{\theta\theta_o}} \int_{-\infty}^{+\infty} \left[\exp\left(\frac{m}{m-1} - \frac{m}{\theta} y \right) \right]^{1/2} \left[\exp\left(\frac{m}{m-1} - \frac{m}{\theta_o} y \right) \right]^{1/2} 1\left(y \geq \frac{\theta}{m-1} \right) 1\left(y \geq \frac{\theta_o}{m-1} \right)$$

$$= 1 - \frac{m}{\sqrt{\theta\theta_o}} \int_{\frac{\max(\theta,\theta_o)}{m-1}}^{\infty} \exp\left(\frac{m}{m-1} - \frac{(\theta+\theta_o)}{2\theta\theta_o} my \right) dy$$

$$= 1 - \frac{m}{\sqrt{\theta\theta_o}} \left[- \frac{2\theta\theta_o}{(\theta+\theta_o)m} \exp\left(\frac{m}{m-1} - \frac{(\theta+\theta_o)}{2\theta\theta_o} my \right) \right]_{y=\frac{\max(\theta,\theta_o)}{m-1}}^{y=\infty}$$

$$= 1 - \frac{m}{\sqrt{\theta\theta_o}} \frac{2\theta\theta_o}{(\theta+\theta_o)m} \left[\exp\left(\frac{m}{m-1} - \frac{(\theta+\theta_o)}{2\theta\theta_o} m \frac{\max(\theta,\theta_o)}{m-1} \right) \right]$$

$$= 1 - \frac{2\sqrt{\theta\theta_o}}{(\theta+\theta_o)} \exp\left(1 - \frac{(\theta+\theta_o)}{2\theta\theta_o} \max(\theta,\theta_o) \right)^{\frac{m}{m-1}}$$

$$= 1 - \frac{2\sqrt{\theta\theta_o}}{(\theta+\theta_o)} \exp\left(1 - \frac{(\theta+\theta_o)^2 + (\theta+\theta_o)|\theta-\theta_o|}{4\theta\theta_o} \right)^{\frac{m}{m-1}}$$

Since $\nabla \ln f_\theta = -m/\theta$ for $y > \theta/(m-1)$, the Fisher matrix is

$$\mathcal{I}(\theta) = \int (\nabla \ln f_\theta)^2 f_{\theta_o} d\mu - \left[\int \nabla \ln f_\theta f_{\theta_o} d\mu \right]^2$$

$$= \int_{y \geq \theta/(m-1)}^{+\infty} \left(-\frac{m}{\theta} \right)^2 f_{\theta_o}(y) dy - \left[\int_{y \geq \theta/(m-1)}^{+\infty} -\frac{m}{\theta} f_{\theta_o} d\mu \right]^2 = \left(\frac{m}{\theta} \right)^2 - \left(\frac{m}{\theta} \right)^2 = 0$$

Example 3. The squared Hellinger distance in this example is

$$\rho(\theta) = 1 - \int f_\theta^{1/2} f_{\theta_o}^{1/2} d\mu$$

$$= 1 - \int_{-\infty}^{+\infty} \left[(\sqrt{2\pi})^{-1} \exp\left[-(y - \theta^2)^2/2 \right] \right]^{1/2} \left[(\sqrt{2\pi})^{-1} \exp\left[-(y - \theta_o^2)^2/2 \right] \right]^{1/2} dy$$

$$= 1 - (\sqrt{2\pi})^{-1} \int_{-\infty}^{+\infty} \exp\left[-(y - \theta^2)^2/2 \right]^{1/2} \exp\left[-(y - \theta_o^2)^2/2 \right]^{1/2} dy$$

$$= 1 - (\sqrt{2\pi})^{-1} \int_{-\infty}^{+\infty} \exp\left[-(y - \theta^2)^2/4 \right] \exp\left[-(y - \theta_o^2)^2/4 \right] dy$$

$$= 1 - (\sqrt{2\pi})^{-1} \int_{-\infty}^{+\infty} \exp\left(\frac{-(y - \theta^2)^2 - (y - \theta_o^2)^2}{4} \right) dy$$

$$= 1 - (\sqrt{2\pi})^{-1} \int_{-\infty}^{+\infty} \exp\left(\frac{-(y^2 + \theta^4 - 2\theta^2 y) - (y^2 + \theta_o^4 - 2\theta_o^2 y)}{4} \right) dy$$

$$= 1 - (\sqrt{2\pi})^{-1} \int_{-\infty}^{+\infty} \exp\left(\frac{-(y^2 + \theta^4 - 2\theta^2 y) - (y^2 + \theta_o^4 - 2\theta_o^2 y)}{4} \right) dy$$

$$= 1 - (\sqrt{2\pi})^{-1} \int_{-\infty}^{+\infty} \exp\left(-\frac{1}{2} y^2 + \frac{1}{2}(\theta^2 + \theta_o^2) y - \frac{(\theta^4 + \theta_o^4)}{4} \right) dy$$

Set $a = -\frac{1}{2}$, $b = \frac{1}{2}(\theta^2 + \theta_o^2)$, and $c = -\frac{(\theta^4 + \theta_o^4)}{4}$. By completing the square

$$\rho(\theta) = 1 - (\sqrt{2\pi})^{-1} \int_{-\infty}^{+\infty} \exp\left(a(y + d)^2 + g \right) dy$$

$$= 1 - \exp(g)(\sqrt{2\pi})^{-1} \int_{-\infty}^{+\infty} \exp\left(a(y + d)^2 \right) dy.$$

where $d = \frac{b}{2a}$ and $g = c - \frac{b^2}{4a}$. Developing

$$g = -\frac{(\theta^4 + \theta_o^4)}{4} - \frac{\left(\frac{1}{2}(\theta^2 + \theta_o^2) \right)^2}{4\left(-\frac{1}{2} \right)} = -\frac{(\theta^2 - \theta_o^2)^2}{8}, \quad d = \frac{\frac{1}{2}(\theta^2 + \theta_o^2)}{2\left(-\frac{1}{2} \right)} = -\frac{1}{2}(\theta^2 + \theta_o^2)$$

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and then replacing

$$\rho(\theta) = 1 - \exp\left(-\frac{(\theta^2 - \theta_o^2)^2}{8}\right)(\sqrt{2\pi})^{-1}\int_{-\infty}^{+\infty}\exp\left(-\frac{1}{2}\left(y - \frac{1}{2}(\theta^2 + \theta_o^2)\right)^2\right)dy$$

$$\rho(\theta) = 1 - \exp\left(-\frac{(\theta^2 - \theta_o^2)^2}{8}\right),$$

where for solving the integral one can use the change of variables $x = y - b$ and $z = x/\sqrt{2}$

$$\int_{-\infty}^{+\infty}\exp\left(-\frac{1}{2}\left(y - b\right)^2\right)dy = \int_{-\infty}^{+\infty}\exp(ax^2)dx = \sqrt{2}\int_{-\infty}^{+\infty}\exp(-z^2)dx = \sqrt{2}\sqrt{\pi}$$

and the Gaussian integral.

Since $\nabla\ln f_\theta = (y - \theta^2)2\theta$, the Fisher information is

$$\mathcal{I}(\theta) = \int(\nabla\ln f_\theta)^2 f_{\theta_o}d\mu - \left[\int\nabla\ln f_\theta f_{\theta_o}d\mu\right]^2$$

$$= \int_{-\infty}^{+\infty}4\theta^2(y - \theta^2)^2 f_{\theta_o}(y)dy - \left[\int_{-\infty}^{+\infty}(y - \theta^2)2\theta f_{\theta_o}(y)dy\right]^2 = 4\theta^2$$

Example 4. For a step-by-step calculation of the squared Hellinger distance between multivariate normal distributions see e.g., Pardo (2005, p. 51). To verify that $\arg\min_\theta\rho(\theta)$ is not a singleton, start by noticing that minimizing $\rho(\theta)$ is equivalent to maximizing

$$\mu, \Omega \mapsto \alpha(\mu, \Omega) := -\frac{1}{4}\ln[\det(\Omega)] + \frac{1}{2}\ln\left[\frac{1}{4}\det(\Omega + \Omega_o)\right] + \frac{1}{4}(\mu - \mu_o)'(\Omega + \Omega_o)^{-1}(\mu - \mu_o)$$

The first order condition for this maximization problem is

$$\frac{\partial\alpha(\mu, \Omega)}{\partial\mu} = \frac{1}{2}(\Omega + \Omega_o)^{-1}(\mu - \mu_o) = 0$$

$$\frac{\partial\alpha(\mu, \Omega)}{\partial\Omega} = -\frac{adj(\Omega)}{4\det(\Omega)} + \frac{adj(\Omega + \Omega_o)}{2\det(\Omega + \Omega_o)} + \frac{1}{4}(\mu - \mu_o)'\frac{\partial(\Omega + \Omega_o)^{-1}}{\partial\Omega}(\mu - \mu_o) = 0.$$

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where we have used Jacobi's formula to express the derivative of the determinant of a matrix in terms of the adjugate of the matrix. Re-arrangeing the first equation in the last display yields $\mu = \mu_o$. Replacing in the second equation, one obtains

$$-\frac{adj(\Omega)}{2\det(\Omega)} + \frac{adj(\Omega + \Omega_o)}{\det(\Omega + \Omega_o)} = 0.$$

Since, for any matrix A, $adj(A) = \det(A)A^{-1}$, one has

$$(\Omega + \Omega_o)^{-1} - \frac{1}{2}\Omega^{-1} = 0 \text{ and } \Omega(\Omega + \Omega_o)^{-1} - \frac{1}{2}I_2 = 0,$$

which shows that (μ_o, Ω_o) is not the only critical point of the function $\mu, \Omega \mapsto \alpha(\mu, \Omega)$.

Example 5. The squared Hellinger distance in this example is

$$\rho(\theta) = 1 - \int f_\theta^{1/2} f_{\theta_o}^{1/2} d\mu$$

$$= 1 - \int_{-\infty}^{+\infty} \left[\frac{1}{2}\exp(-|y - \theta|)\right]^{1/2} \left[\frac{1}{2}\exp(-|y - \theta_o|)\right]^{1/2} dy$$

$$= 1 - \frac{1}{2}\int_{-\infty}^{+\infty} \exp\left(-\frac{1}{2}|y - \theta| - \frac{1}{2}|y - \theta_o|\right) dy$$

Using the change of variable $z = y - \theta$ and defining $\delta = \theta - \theta_o$

$$\rho(\theta) = 1 - \frac{1}{2}\int_{-\infty}^{+\infty} \exp\left(-\frac{1}{2}|z| - \frac{1}{2}|z + \delta|\right) dz = 1 - \frac{1}{2}\int_{-\infty}^{+\infty} \exp\left(-\frac{1}{2}[|z| + |z + \delta|]\right) dz$$

$$= 1 - \frac{1}{2}\int_{\max(0,-\delta)}^{+\infty} \exp\left(-\frac{1}{2}(2z + \delta)\right) dz$$

$$- \frac{1(\delta < 0)}{2}\int_0^{-\delta} \exp\left(-\frac{1}{2}(-\delta)\right) dz - \frac{1(\delta > 0)}{2}\int_{-\delta}^0 \exp\left(-\frac{1}{2}(\delta)\right) dz$$

$$- \frac{1}{2}\int_{-\infty}^{\min(0,-\delta)} \exp\left(-\frac{1}{2}[-(2z + \delta)]\right) dz$$

Solving each of the fourth integrals, we obtain:

$$\int_{\max(0,-\delta)}^{+\infty} \exp\left(-\frac{1}{2}(2z+\delta)\right)dz = \left[-\exp\left(-z-\frac{\delta}{2}\right)\right]_{\max(0,-\delta)}^{+\infty}$$

$$= -\left[-\exp\left(-\max(0,-\delta)-\frac{\delta}{2}\right)\right] = \exp\left(-\frac{|\delta|}{2}\right)$$

$$1(\delta<0)\int_0^{-\delta} \exp\left(-\frac{1}{2}(-\delta)\right)dz = 1(\delta<0)\exp\left(\frac{\delta}{2}\right)(-\delta)$$

$$1(\delta>0)\int_{-\delta}^0 \exp\left(-\frac{1}{2}(\delta)\right)dz = 1(\delta>0)\exp\left(-\frac{\delta}{2}\right)\delta$$

$$\int_{-\infty}^{\min(0,-\delta)} \exp\left(-\frac{1}{2}[-(2z+\delta)]\right)dz = \left[\exp\left(z+\frac{\delta}{2}\right)\right]_{-\infty}^{\min(0,-\delta)}$$

$$= \exp\left(\min(0,-\delta)+\frac{\delta}{2}\right) = \exp\left(-\frac{|\delta|}{2}\right)$$

Replacing back in the squared Hellinger distance

$$\rho(\theta) = 1 - \frac{1}{2}\exp\left(-\frac{|\delta|}{2}\right) - \frac{1(\delta<0)}{2}\exp\left(\frac{\delta}{2}\right)(-\delta) - \frac{1(\delta>0)}{2}\exp\left(-\frac{\delta}{2}\right)\delta - \frac{1}{2}\exp\left(-\frac{|\delta|}{2}\right)$$

$$= 1 - \exp\left(-\frac{|\delta|}{2}\right) - \frac{\delta}{2}\left[1(\delta>0)\exp\left(-\frac{\delta}{2}\right) - 1(\delta<0)\exp\left(\frac{\delta}{2}\right)\right]$$

To calculate the Fisher matrix, one has $\ln f_\theta(y) = \ln(1/2) - |y - \theta|$ and then

$$
\begin{aligned}
\mathcal{I}(\theta) &= \int (\nabla \ln f_\theta)^2 f_{\theta_o} d\mu - \left[\int \nabla \ln f_\theta f_{\theta_o} d\mu \right]^2 \\
&= \int_{-\infty}^0 (-1)^2 f_{\theta_o}(y) dy + \int_0^{+\infty} (1)^2 f_{\theta_o}(y) dy - \left[\int_{-\infty}^0 (-1) f_{\theta_o}(y) dy + - \int_0^{+\infty} (1) f_{\theta_o}(y) dy \right]^2 \\
&= \int_{-\infty}^\infty f_{\theta_o}(y) dy = \int_{-\infty}^0 \frac{1}{2} \exp[(y - \theta)] dy + \int_0^{+\infty} \frac{1}{2} \exp[-(y - \theta)] dy = 1
\end{aligned}
$$