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Asymptotic Justification of Models of Plates Containing Inside Hard Thin Inclusions

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Abstract: An equilibrium problem of the Kirchhoff–Love plate containing a nonhomogeneous inclusion is considered. It is assumed that elastic properties of the inclusion depend on a small parameter characterizing the width of the inclusion ε as ε^N with $N < 1$. The passage to the limit as the parameter ε tends to zero is justified, and an asymptotic model of a plate containing a thin inhomogeneous hard inclusion is constructed. It is shown that there exists two types of thin inclusions: rigid inclusion ($N < -1$) and elastic inclusion ($N = -1$). The inhomogeneity disappears in the case of $N \in (-1, 1)$.

Keywords: Kirchhoff-Love plate; composite material; thin inclusion; asymptotic analysis

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1. Introduction

An equilibrium problem of a Kirchhoff–Love plate containing a nonhomogeneous inclusion is considered. It is assumed that the elastic properties of the inclusion depend on a small parameter characterizing width of the inclusion ε as ε^N with $N < 1$. The problem is formulated as a variational one; namely, as a minimization problem of the energy functional over a set of admissible deflections in the Sobolev space H^2. This implies that the deflections function is a solution of a boundary value problem for bi-harmonic operator (pure bending, see, e.g., [1–4]).

The aim of the present work is to justify passing to the limit as $\varepsilon \to 0$. To do this, we apply a method that was originally introduced in [5,6] for problems of gluing plates. The method is based on variational properties of the solution to the corresponding minimization problem and allows for finding a limit problem for any $N < 1$ simultaneously. It is shown that there exist two types of hard inclusions in dependence of N: thin rigid inclusion ($N < -1$) and thin elastic inclusion ($N = -1$). In case $N \in (-1, 1)$, the influence of the inhomogeneity disappears in the limit. We get limit problems in a variational form, which is convenient, for example, for numerical analysis by the finite element method.

Let us give a short survey of works that are close to the present investigation. Note that there are not so many works devoted to study of models of thin inclusions in plates. We mention [7–9], in which thin elastic inclusions in pates were studied. Papers [10–13] are devoted investigations of thin rigid inclusions. We refer to [14–21] for asymptotic analyses for different models of bonded structures in Elasticity. We indicate also paper [22], where a geometry-dependent state problem for a heterogeneous medium with defects is investigated in framework of anti-plane elasticity.

Finally, we mention paper [23], where the mechanical behavior of an anisotropic nonhomogeneous linearly elastic three-layer plate with soft adhesive, including the inertia forces, was studied, and the various limiting models in the dependence of the size and the stiffness of the adhesive was derived. The problem under consideration in the present paper is different from the mentioned paper because we consider the hard inhomogeneity lying strictly inside the plate and derive limiting problem depending on the size and stiffness of the inclusion. Wherein, the plate size does not vary and remains constant.

2. Statement of Problem

Let us fix a small parameter $\varepsilon \in (0,1)$ and consider an inhomogeneous rectangular plate $\Omega \subset \mathbb{R}^2$ with a thin rectangular inclusion $\Omega^\varepsilon_{inc} \subset \Omega$ of width $2\varepsilon d$, where d is diameter of Ω. Let us specify some notations:

$$\Omega = (-a_1, a_2) \times (-b_1, b_2),\ a_\alpha, b_\alpha > 0,\ \alpha = 1,2,$$

$$\Omega^\varepsilon_{inc} = (-\varepsilon d, \varepsilon d) \times (-c_1, c_2),\ \ 0 < c_\alpha < b_\alpha,\ \alpha = 1,2,$$

$$\Omega_\pm = \{(y_1, y_2) \in \Omega \ \mid \ \pm y_1 > 0\},$$

$$S = \partial\Omega_- \cap \partial\Omega_+,$$

$$S_{inc} = S \cap \Omega^\varepsilon_{inc},$$

$$\Omega^\varepsilon_{mat} = \Omega \setminus \overline{\Omega}^\varepsilon_{inc},\ \ \Omega^\varepsilon_\pm = \Omega^\varepsilon_{mat} \cap \Omega_\pm,$$

Note that, for all small enough $\varepsilon > 0$ a family of subdomains Ω^ε_{inc} lies strictly inside Ω. Besides, let us define the following notations:

$$\Omega^\varepsilon_{mid} = \{(y_1, y_2) \in \Omega \ \mid \ -\varepsilon d < y_1 < \varepsilon d,\ y_2 \in S\},$$

$$S^\varepsilon_\pm = \{(y_1, y_2) \in \Omega \ \mid \ y_1 = \pm\varepsilon d,\ y_2 \in S\},$$

We assume that S_{inc} is divided into three subsets $S_\alpha \subset S_{inc}$, where each S_α is an union of finite number of segments or empty set, $\alpha = 1,2,3$.

In our consideration, Ω is a composite plate, consisting of the elastic matrix Ω^ε_{mat} and the inhomogeneous inclusion $\Omega^\varepsilon_{inc} = \cup^3_{\alpha=1}\Omega^\varepsilon_\alpha$, where

$$\Omega^\varepsilon_\alpha = \{(y_1, y_2) \in \mathbb{R}^2 \ \mid \ -\varepsilon d < y_1 < \varepsilon d,\ y_2 \in S_\alpha\},\ \ \alpha = 1,2,3.$$

Moreover, in the sequel, we will use the following notations:

$$\Omega^\varepsilon_0 = \Omega^\varepsilon_{mid} \setminus \cup^3_{\alpha=1}\overline{\Omega}^\varepsilon_\alpha,$$

$$S_0 = S \setminus \overline{S}_{inc}.$$

Denote, by E_0, E^ε_α and k_0, k_α, Young's modules and Poisson's ratios of parts Ω_{mat} and $\Omega^\varepsilon_\alpha$ of the composite plate Ω, respectively, $\alpha = 1,2,3$. The compound character of the structure is expressed by the fact that E_0, k_0, and k_α are constants, while Young's modulus E^ε_α depends on ε, as follows:

$$E^\varepsilon_\alpha = \varepsilon^{N_\alpha} E_\alpha \ \text{ in } \ \Omega^\varepsilon_\alpha,\ \ \alpha = 1,2,3,$$

where N_1, N_2, N_3 are real numbers, such that

$$N_1 < -1,\ \ N_2 = -1,\ \ N_3 \in (-1, 1).$$

Parameters N_1 and N_2 correspond to hard inclusions in the plate Ω (see [6,24,25]). Moreover, put $N_0 = 0$.

Denote, by w, deflections of the composite plate Ω. Then the bending moments are defined by formulae (see, e.g., [26,27])

$$m_{ij}(w) = d^\varepsilon_{ijkl} w_{,kl},\ \ i,j = 1,2,\ w_{,kl} = \frac{\partial^2 w}{\partial y_k \partial y_l},$$

where the positive definite and symmetric tensor $\{d_{ijkl}\}$ is orthotropic with the following components:

$$d^{\varepsilon}_{iiii}(y) = D^{\varepsilon}(y),\ d^{\varepsilon}_{iijj}(y) = D^{\varepsilon}(y)k^{\varepsilon}(y),$$
$$d^{\varepsilon}_{ijij}(y) = d^{\varepsilon}_{ijji}(y) = D^{\varepsilon}(y)(1-k^{\varepsilon}(y))/2,\ i \neq j,\ i,j = 1,2, \tag{1}$$

$$D^{\varepsilon}(y) = \begin{cases} D_0 \ \text{ in } \ \Omega^{\varepsilon}_{mat}, \\ \varepsilon^{N_\alpha} D_\alpha \ \text{ in } \ \Omega^{\varepsilon}_{\alpha},\ \alpha = 1,2,3, \end{cases}$$

$$D_\alpha = \frac{E_\alpha h^3}{12(1-k^2_\alpha)},\ \alpha = 0,1,2,3,$$

$$k^{\varepsilon}(y) = \begin{cases} k_0 \text{ in } \Omega^{\varepsilon}_{mat}, \\ k^{\varepsilon}_{\alpha},\ \text{in } \Omega^{\varepsilon}_{\alpha},\ \alpha = 1,2,3, \end{cases}$$

h is a thickness of the plate Ω that is constant. Note paper [28], where it was shown non-standard behaviour in the asymptotic two-dimensional reduction from three-dimensional elasticity, when the thickness and size of inclusions depend on the same parameter.

The potential energy functional of the plate has the following representation (see [27]):

$$\Pi(w) = \frac{1}{2}\int\limits_{\Omega} d^{\varepsilon}_{ijkl} w_{,kl} w_{,ij}\, dy - \int\limits_{\Omega} f w\, dy,$$

where $f \in L_2(\Omega)$ is a bulk force acting on the plate Ω. Subsequently, the equilibrium problem of nonhomogeneous plate clamped on the external boundary $\partial\Omega$ can be formulated as the minimization problem: find a function $w_\varepsilon \in H^2_0(\Omega)$ such that

$$\Pi(w_\varepsilon) = \inf_{w \in H^2_0(\Omega)} \Pi(w). \tag{2}$$

Problem (2) is known to have a unique solution w_ε (see, e.g., [26,29]), which satisfies the variational equality:

$$\int\limits_{\Omega} d^{\varepsilon}_{ijkl} w_{\varepsilon,kl} w_{,ij}\, dy = \int\limits_{\Omega} f w\, dy \quad \forall w \in H^2_0(\Omega). \tag{3}$$

Moreover, the function w_ε is a unique solution the following boundary value problem:

$$(d^{\varepsilon}_{ijkl} w_{\varepsilon,kl})_{,ij} = f \ \text{ in } \ \Omega,$$

$$w_\varepsilon = \frac{\partial w_\varepsilon}{\partial \nu} = 0 \ \text{ on } \ \partial\Omega,$$

where ν is a unit normal vector $\partial\Omega$.

3. Decomposition of the Problem and Coordinate Transformations

In the sequel, we will have deal with the problem (3). Let us rewrite it in an equivalent form. For this, we introduce the following set:

$$K_\varepsilon = \{v = (v_-, v_+, v_m) \in H^2(\Omega^{\varepsilon}_{-}) \times H^2(\Omega^{\varepsilon}_{+}) \times H^2(\Omega^{\varepsilon}_{m}) \ |$$
$$v_\pm = v_m,\ \ v_{\pm,1} = v_{m,1} \text{ a.e. on } S^{\varepsilon}_{\pm},$$
$$v_\pm = \frac{\partial v_\pm}{\partial \nu} = 0 \text{ a.e. on } \partial\Omega^{\varepsilon}_{\pm} \cap \partial\Omega\}.$$

Taking into account the (1), problem (3) can be reformulated, as follows: find a triplet $(w_{\varepsilon-}, w_{\varepsilon+}, w_{\varepsilon m}) \in K_\varepsilon$ satisfying a variational equality

$$\begin{aligned} &b_{\varepsilon-}(w_{\varepsilon-}, v_-) + b_{\varepsilon+}(w_{\varepsilon+}, v_+) + b_{\varepsilon m}(w_{\varepsilon m}, v_m) = \\ &= l_-(v_-) + l_+(v_+) + l_m(v_m) \ \ \forall (v_-, v_+, v_m) \in K_\varepsilon, \end{aligned} \tag{4}$$

where

$$b_{\varepsilon\pm}(u,v) = D_0 \int\limits_{\Omega^\varepsilon_\pm} (u_{,11}v_{,11} + u_{,22}v_{,22} + k_0(u_{,11}v_{,22} + u_{,22}v_{,11}) + 2(1-k_0)u_{,12}v_{,12})\, dy,$$

$$b_{\varepsilon m}(u,v) = \sum_{\alpha=0}^{3} D^\varepsilon_\alpha \int\limits_{\Omega^\varepsilon_\alpha} (u_{,11}v_{,11} + u_{,22}v_{,22} + k_\alpha(u_{,11}v_{,22} + u_{,22}v_{,11}) + 2(1-k_\alpha)u_{,12}v_{,12})\, dy.$$

$$l_{\varepsilon\pm}(u) = \int\limits_{\Omega^\varepsilon_\pm} fu\, dy, \ \ l_{\varepsilon m}(u) = \int\limits_{\Omega^\varepsilon_m} fu\, dy.$$

From the Calculus of Variations, it follows that problem (4) has a unique solution $(w_{\varepsilon-}, w_{\varepsilon+,\varepsilon m}) \in K_\varepsilon$ for all $\varepsilon > 0$ small enough (see, e.g., [2,26]). Herewith, $w_{\varepsilon\pm}$ and $w_{\varepsilon m}$ are restrictions of w_ε on subdomains $\Omega^\varepsilon_\pm$ and Ω^ε_m, respectively.

Next, we introduce coordinate transformations that map domains $\Omega^\varepsilon_\pm$ and Ω^ε_m onto domains independent of ε. For this, we consider two convex domains ω_1 and ω_2, such that

$$\overline{S} \subset \omega_1, \ \overline{\omega}_1 \subset \omega_2, \ \partial\omega_2 \cap \{y_1 = -a_1\} = \emptyset, \ \partial\omega_2 \cap \{y_1 = a_2\} = \emptyset,$$

and a smooth cut-off function θ, such that

$$\theta = 1 \text{ in } \overline{\omega}_1, \ 0 < \theta < 1 \text{ in } \omega_2, \ \theta = 0 \text{ in } \mathbb{R}^2 \setminus \overline{\omega}_2.$$

Let us introduce the following notations:

$$\Omega_m = \{(z_1, z_2) \in \mathbb{R}^2 \ | \ -d < z_1 < d, \ z_2 \in S\},$$

$$S_\pm = \{(z_1, z_2) \in \mathbb{R}^2 \ | \ z_1 = \pm d, \ z_2 \in S\},$$

$$\Omega_\alpha = \{(z_1, z_2) \in \mathbb{R}^2 \ | \ -d < z_1 < d, \ z_2 \in S_\alpha\}, \ \alpha = 0,1,2,3,$$

$$S^\pm_\alpha = \{(z_1, z_2) \in \mathbb{R}^2_z \ | \ z_1 = \pm d, \ z_2 \in S_\alpha\}, \ \alpha = 0,1,2,3.$$

and define coordinate transformations in the domains $\Omega_\pm$ and Ω_m as follows:

$$y_1 = x_1 \pm \varepsilon d\theta(x_1, x_2), \quad y_2 = x_2, \ \ (x_1, x_2) \in \Omega_\pm, \ \ (y_1, y_2) \in \Omega^\varepsilon_\pm, \tag{5}$$

$$y_1 = \varepsilon z_1, \quad y_2 = z_2, \ \ (z_1, z_2) \in \Omega_m, \ \ (y_1, y_2) \in \Omega^\varepsilon_m. \tag{6}$$

It is not difficult to show that for all sufficiently small coordinate transformations (5) and (6) map bijectively the domains $\Omega_\pm$ and Ω_m onto $\Omega^\varepsilon_\pm$ and Ω^ε_m, respectively, (see, e.g., [30,31]). Note that the subdomain $\Omega^\varepsilon_\alpha$ is mapped into subdomains Ω_α, $\alpha = 0,1,2,3$.

Denote, by $\Phi^\pm_\varepsilon(x)$ and $J^\pm_\varepsilon$, Jacobian matrices and Jacobians of transformations (5), respectively,

$$\Phi^\pm_\varepsilon(x_1, x_2) = \begin{pmatrix} 1 \pm \varepsilon d\theta_{,1}(x_1, x_2) & \pm\varepsilon d\theta_{,2}(x_1, x_2)) \\ 0 & 1 \end{pmatrix},$$

$$J^\pm_\varepsilon(x_1, x_2) = \det \Phi^\pm_\varepsilon(x_1, x_2) = 1 \pm \varepsilon d\theta_{,1}(x_1, x_2).$$

Coordinate transformations (5) and (6) establish one-to-one correspondences between spaces $H^2(\Omega_\pm)$, $H^2(\Omega_m)$ and $H^2(\Omega^\varepsilon_\pm)$, $H^2(\Omega^\varepsilon_m)$, respectively. Moreover, the set K^ε is transformed into a set K_ε,

$$K_\varepsilon = \{v = (v_-, v_+, v_m) \in H^{2,0}(\Omega_-) \times H^{2,0}(\Omega_+) \times H^2(\Omega_m) \ | \\ v_\pm|_S = v_m|_{S_\pm}, \ v_{\pm,1}|_S = \frac{1}{\varepsilon} v_{m,1}|_{S_\pm}\},$$

where

$$H^{2,0}(\Omega_\pm) = \{v_\pm \in H^2(\Omega_\pm) \ | \ v_\pm = \frac{\partial v_\pm}{\partial \nu} = 0 \text{ a.e. on } \partial\Omega^\varepsilon_\pm \cap \partial\Omega\}.$$

Hereinafter, we assume that, for any functions $v_\pm(x)$, $x \in \Omega_\pm$, and $v_m(z)$, $z \in \Omega_m$, equality $v_\pm|_S = v_m|_{S_\pm}$ means that

$$v_\pm(0, x_2) = v_m(\pm d, z_2), \ \ x_2 = z_2 \in S.$$

Introduce the following notations:

$$w^\varepsilon_\pm(x_1, x_2) = w_{\varepsilon\pm}(x_1 \pm \varepsilon d\theta(x_1, x_2), x_2), \ \ (x_1, x_2) \in \Omega_\pm,$$

$$w^\varepsilon_m(z_1, z_2) = w_{\varepsilon m}(\varepsilon z_1, z_2), \ \ (z_1, z_2) \in \Omega_m.$$

Becase of the smoothness of coordinate transformations (5), we have asymptotic expansions for the transformations of the second-order derivatives for (5) (see, e.g., [30–33])

$$w_{\varepsilon\pm,ij} = w^\varepsilon_{\pm,ij} + \varepsilon P^\pm_{ij}(\varepsilon, w^\varepsilon_\pm), \tag{7}$$

with

$$|P^\pm_{ij}(\varepsilon, w^\varepsilon_\pm)| \le C(|w^\varepsilon_{\pm,k}| + |w^\varepsilon_{\pm,kl}|), \ \ i, j, k, l = 1, 2.$$

Besides, we have for (6)

$$w_{\varepsilon m,11}(y_1, y_2) = \frac{w^\varepsilon_{m,11}(z_1, z_2)}{\varepsilon^2}, \ w_{\varepsilon m,12}(y_1, y_2) = \frac{w^\varepsilon_{m,12}(z_1, z_2)}{\varepsilon}, \ w_{\varepsilon m,22}(y_1, y_2) = w^\varepsilon_{m,22}(z_1, z_2).$$

After applying coordinate transformations (5) and (6) to (4), we get that the triplet $(w^\varepsilon_-, w^\varepsilon_+, uw^\varepsilon_m) \in K_\varepsilon$ is a unique solution to the following variational equality:

$$b^\varepsilon_-(w^\varepsilon_-, v_-) + b^\varepsilon_+(w^\varepsilon_+, v_+) + b^\varepsilon_m(w^\varepsilon_m, v_m) = l^\varepsilon_-(v_-) + l^\varepsilon_+(v_+) + l^\varepsilon_m(v_m) \ \forall (v_-, v_+, v_m) \in K_\varepsilon, \tag{8}$$

where, taking into account (7) and (1),

$$b^\varepsilon_\pm(u, v) = b_\pm(u, v) + r_\pm(\varepsilon, u, v),$$

$$b_\pm(u, v) = D_0 \int\limits_{\Omega_\pm} (u_{,11}v_{,11} + u_{,22}v_{,22} + k_\pm(u_{,11}v_{,22} + u_{,22}v_{,11}) + 2(1 - k_\pm)u_{,12}v_{,12})\, dx,$$

$$|r_\pm(\varepsilon, u, v)| \le c_\pm(\varepsilon) \left(\|u\|^2_{H^2(\Omega_\pm)} + \|v\|^2_{H^2(\Omega_\pm)}\right), \ 0 \le c_\pm(\varepsilon) = o(1) \text{ as } \varepsilon \to 0, \tag{9}$$

$$b_m^\varepsilon(u,v) =$$
$$= D_0 \int\limits_{\Omega_m} \left(\frac{u_{,11}v_{,11}}{\varepsilon^3} + \varepsilon u_{,22}v_{,22} + \frac{k_m}{\varepsilon}(u_{,11}v_{,22} + v_{,22}w_{,11}) + \frac{2(1-k_m)}{\varepsilon} u_{,12}v_{,12} \right) dz +$$
$$+ D_1 \int\limits_{\Omega_m} \left(\frac{u_{,11}v_{,11}}{\varepsilon^{3-N_1}} + \frac{u_{,22}v_{,22}}{\varepsilon^{-N_1-1}} + \frac{k_m}{\varepsilon^{1-N_1}}(u_{,11}v_{,22} + v_{,22}w_{,11}) + \frac{2(1-k_m)}{\varepsilon^{1-N_1}} u_{,12}v_{,12} \right) dz +$$
$$+ D_2 \int\limits_{\Omega_m} \left(\frac{u_{,11}v_{,11}}{\varepsilon^4} + u_{,22}v_{,22} + \frac{k_m}{\varepsilon^2}(u_{,11}v_{,22} + v_{,22}w_{,11}) + \frac{2(1-k_m)}{\varepsilon^2} u_{,12}v_{,12} \right) dz +$$
$$+ D_3 \int\limits_{\Omega_m} \left(\frac{u_{,11}v_{,11}}{\varepsilon^{3-N_3}} + \varepsilon^{N_3+1} u_{,22}v_{,22} + \frac{k_m}{\varepsilon^{1-N_3}}(u_{,11}v_{,22} + v_{,22}w_{,11}) + \frac{2(1-k_m)}{\varepsilon^{1-N_3}} u_{,12}v_{,12} \right) dz,$$

$$l_\pm^\varepsilon(v) = \int\limits_{\Omega_\pm} f(x_1 \pm d\theta(x_1,x_2), x_2)(1 \pm d\theta_{,1}(x_1,x_2)v\, dx,$$

$$l_m^\varepsilon(v) = \varepsilon \int\limits_{\Omega_m} f(\varepsilon z_1, z_2) v\, dz,$$

$$|l_\pm^\varepsilon(v)| \le C\|v\|_{L_2(\Omega_\pm)}, \tag{10}$$

$$|l_m^\varepsilon(v)| \le C\varepsilon\|v\|_{L_2(\Omega_m)}. \tag{11}$$

4. Limit Problem

To justify passing to the limit as $\varepsilon \to 0$, we need some auxiliary lemma proved in [5,6].

Lemma 1 (Poincare-typé inequalities). *For any triplet $(v_-, v_+, v_m) \in K_\varepsilon$ and $\varepsilon \in (0,1)$, the inequalities*

$$\|v_m\|^2_{L_2(\Omega_m)} \le C\left(\|v_{m,11}\|^2_{L_2(\Omega_m)} + \|v_\pm\|^2_{H^{2,0}(\Omega_\pm)}\right),$$
$$\|v_{m,1}\|^2_{L_2(\Omega_m)} \le C\left(\|v_{m,11}\|^2_{L_2(\Omega_m)} + \varepsilon^2\|v_{\pm,1}\|^2_{L_2(S)}\right)$$

hold, where a constant $C > 0$ does not depend on (v_-, v_+, v_m) and $\varepsilon > 0$.

Our main result is the following theorem.

Theorem 1. *Let $w^\varepsilon = (w_-^\varepsilon, w_+^\varepsilon, w_m^\varepsilon)$ be a solution to (8); let $w_0 \in K_0$ be a solution to the following variational equality:*

$$b(w_0, w) + 4d(1-k_2)D_m \int\limits_{S_2} \frac{\partial(w_{0,1}|_{S_2})}{\partial z_2} \frac{\partial(w_{,1}|_{S_2})}{\partial z_2}\, dz_2 = l(w)\ \forall w \in K_0, \tag{12}$$

where

$$K_0 = \{w \in H_0^2(\Omega)\ |\ w = \alpha x_2 + \beta \text{ a.e. on } S_1,\ \alpha, \beta \in \mathbb{R};\ w_{,1} \in H^1(S_2)\}.$$

Denote, by $w_\pm$, a restriction of w to subdomain $\Omega_\pm$ and, moreover, put

$$w_m(z_1, z_2) = w_0(z_1, 0) \quad \textit{for} \quad (z_1, z_2) \in \Omega_m.$$

Then, the following convergences

$$w_\pm^\varepsilon \rightharpoonup w_\pm \quad \textit{weakly in} \quad H^2(\Omega_\pm),$$

$$w_m^\varepsilon \rightharpoonup w_m \quad \textit{weakly in} \quad L_2(\Omega_m),$$

take place as $\varepsilon \to 0$.

Proof. Let us substitute $(w_-^\varepsilon, w_+^\varepsilon, w_m^\varepsilon)$ in (8) as a test function. Taking into account Lemma, (9)–(11), we obtain an estimate

$$\|w_-^\varepsilon\|^2_{H^{2,0}(\Omega_-)} + \|w_+^\varepsilon\|^2_{H^{2,0}(\Omega_+)} + \\ + \left\|\frac{w^\varepsilon_{m_0,11}}{\varepsilon^{\frac{3}{2}}}\right\|^2_{L_2(\Omega_0)} + \left\|\frac{w^\varepsilon_{m_0,12}}{\varepsilon^{\frac{1}{2}}}\right\|^2_{L_2(\Omega_0)} + \|\varepsilon^{\frac{1}{2}} w^\varepsilon_{m_0,22}\|^2_{L_2(\Omega_0)} + \\ + \left\|\frac{w^\varepsilon_{m_1,11}}{\varepsilon^{\frac{3-N_1}{2}}}\right\|^2_{L_2(\Omega_1)} + \left\|\frac{w^\varepsilon_{m_1,12}}{\varepsilon^{\frac{1-N_1}{2}}}\right\|^2_{L_2(\Omega_1)} + \left\|\frac{w^\varepsilon_{m_1,22}}{\varepsilon^{\frac{-N_1-1}{2}}}\right\|^2_{L_2(\Omega_1)} + \\ + \left\|\frac{w^\varepsilon_{m_2,11}}{\varepsilon^{2}}\right\|^2_{L_2(\Omega_2)} + \left\|\frac{w^\varepsilon_{m_2,12}}{\varepsilon}\right\|^2_{L_2(\Omega_2)} + \left\|w^\varepsilon_{m_2,22}\right\|^2_{L_2(\Omega_2)} + \\ + \left\|\frac{w^\varepsilon_{m_3,11}}{\varepsilon^{\frac{3-N_3}{2}}}\right\|^2_{L_2(\Omega_3)} + \left\|\frac{w^\varepsilon_{m_3,12}}{\varepsilon^{\frac{1-N_3}{2}}}\right\|^2_{L_2(\Omega_3)} + \|\varepsilon^{\frac{N_3+1}{2}} w^\varepsilon_{m_3,22}\|^2_{L_2(\Omega_3)} \le C \quad (13)$$

with a constant C independent of ε. Here, by $w^\varepsilon_{m_\alpha}$, denote a restriction of w^ε to Ω_α, $\alpha = 0,1,2,3$. Moreover, from (13), Lemma, and definition of the set K_ε, we additionally have

$$\|w^\varepsilon_m\|_{L_2(\Omega_m)} \le C, \;\; \|w^\varepsilon_{m,1}\|_{L_2(\Omega_m)} \le C\varepsilon. \quad (14)$$

Estimates (13) and (14) entail the existence of functions $w_\pm \in H^{2,0}(\Omega_\pm)$, $w_m \in L_2(\Omega_m)$, $p_\alpha, q_\alpha, r_\alpha \in L_2(\Omega_\alpha)$, $\alpha = 0,1,2,3$, such that for some subsequence $\{\varepsilon_n\}_{n=1}^\infty$ still denoted by ε, the following convergences:

$$\begin{aligned} w^\varepsilon_\pm &\rightharpoonup w_\pm \quad \text{weakly in} \quad H^2(\Omega_\pm), \\ w^\varepsilon_m &\rightharpoonup w_m \quad \text{weakly in} \quad L_2(\Omega_\alpha), \\ \varepsilon^{\frac{N_\alpha-3}{2}} w^\varepsilon_{m,11} &\rightharpoonup p_\alpha \quad \text{weakly in} \quad L_2(\Omega_\alpha), \\ \varepsilon^{\frac{N_\alpha-1}{2}} w^\varepsilon_{m,12} &\rightharpoonup q_\alpha \quad \text{weakly in} \quad L_2(\Omega_\alpha), \\ \varepsilon^{\frac{N_\alpha+1}{2}} w^\varepsilon_{m,22} &\rightharpoonup r_\alpha \quad \text{weakly in} \quad L_2(\Omega_\alpha) \end{aligned} \quad (15)$$

hold as $\varepsilon \to 0$, with $r_2 = w_{m,22}$. Moreover, from (13) and (14), it follows that

$$w^\varepsilon_{m,1} \to w_{m,1} = 0 \quad \text{strongly in} \quad L_2(\Omega_m), \quad (16)$$

$$w^\varepsilon_{m,11} \to w_{m,11} = 0 \quad \text{strongly in} \quad L_2(\Omega_m), \quad (17)$$

$$w^\varepsilon_{m,22} \to w_{m,22} = 0 \quad \text{strongly in} \quad L_2(\Omega_1), \quad (18)$$

and there exists $u \in L_2(\Omega_{m_2})$ such that

$$\frac{w^\varepsilon}{\varepsilon} \rightharpoonup u \quad \text{weakly in} \quad L_2(\Omega_2).$$

From definition of the set K_ε, after passing to the limit as $\varepsilon \to 0$, we obtain

$$w_m|_{S_\pm} = w_\pm|_S. \quad (19)$$

Because $w_{m,1} = 0$ in Ω_m (see (16)), w_m does not depend on z_2. Therefore, taking into account (17), we conclude that there exists a function $\beta(z_2) \in L_2(\Omega_m)$ such that

$$w_m(z_1, z_2) = \beta(z_2), \ \ (z_1, z_2) \in \Omega_m.$$

Condition (18) means that the function w_m is affine in the domain Ω_m with respect to z_2, i.e., there exists $\delta, \gamma \in \mathbb{R}$, such that

$$w_m(z_1, z_2) = \delta z_2 + \gamma \ \text{ in } \ \Omega_1. \tag{20}$$

Because of (19), we have

$$w_-|_S = w_+|_S. \tag{21}$$

Now, let us show that $w_\pm$ satisfy the following equality:

$$w_{+,1} = w_{-,1} \quad \text{on} \quad S. \tag{22}$$

Indeed, from the relation

$$\int_{-d}^{d} w^\varepsilon_{m,11}(z_1, z_2) dz_1 = w^\varepsilon_{m,1}(d, z_2) - w^\varepsilon_{m,1}(-d, z_2),$$

it follows that

$$\int_a^b |w^\varepsilon_{m,1}(d, z_2) - w^\varepsilon_{m,1}(-d, z_2)|^2 \, dz_2 \leq 2d \|w^\varepsilon_{m,11}\|^2_{L_2(\Omega_m)}.$$

Due to estimate (13) and the equalities $w^\varepsilon_{m,1}(\pm d, z_2) = \varepsilon w^\varepsilon_\pm(0, z_2)$ for $z_2 \in (a, b)$ (see the definition of the set K_ε), we obtain

$$\|w^\varepsilon_{+,1} - w^\varepsilon_{-,1}\|_{L_2(S)} \leq \frac{2d}{\varepsilon} \|w^\varepsilon_{m,11}\|_{L_2(\Omega_m)} \to 0$$

as $\varepsilon \to 0$. From (15) (the first line) and the compactness of trace operator, it follows

$$w^\varepsilon_{\pm,1} \to w_{\pm,1} \ \text{ strongly in } \ L_2(S)$$

as $\varepsilon \to 0$, and (22) holds.

At last, using the same arguments as in [6], we can prove additionally that

$$w_{\pm,1}|_{S_2} \in H^1(S_2) \tag{23}$$

and, moreover,

$$p_2 = -k_m w_{m,22} \ \text{ in } \ \Omega_2,$$

$$q_2 = \frac{\partial (w_{-,1}|_{S_2})}{\partial z_2} \ \text{ in } \ \Omega_2,$$

$$u = w_{-,1}|_{S_2} \ \text{ in } \ \Omega_2.$$

Now, let us define a function

$$w_0(x) = \begin{cases} w_-(x) & x \in \Omega_-, \\ w_+(x) & x \in \Omega_+. \end{cases} \tag{24}$$

Conditions (19)–(23) imply that the function w_0 belongs to the set K_0.

In order to proceed with a problem defining the function w_0, we take arbitrary function $v \in C^2(\Omega) \cap K_0$ and define three functions v_-, v_+, v_m by

$$v_- = v|_{\Omega_-}, \ \ v_+ = v|_{\Omega_+},$$

$$v_m(z_1, z_2) = v(0, z_2), \ \ (z_1, z_2) \in \Omega_m.$$

Subsequently, for these functions, we consider a triplet $(v_- + \varepsilon\psi_-, v_+ + \varepsilon\psi_+, v_m + \varepsilon\psi_m) \in K_\varepsilon$, where $\psi_m(z_1, z_2) = v_{,1}(0, z_2)z_1$ for $(z_1, z_2) \in \Omega_m$, and $\psi_\pm \in H^{2,0}(\Omega_\pm)$ is arbitrary extensions of ψ_m in domains $\Omega_\pm$, such that

$$\psi_\pm|_S = \psi_m|_{S_m^\pm}, \quad \psi_{\pm,1} = 0 \ \text{ on } \ S,$$

and substitute it in (8). Since $v_{m,11} = 0$ and $\psi_{m,11} = 0$ in Ω_m, weak convergences in (15) and Formulas (23) allows for us to pass to the limit as $\varepsilon \to 0$ and obtain the following relation:

$$b_-(w_-, v_-) + b_+(w_+, v_+) + 4d(1-k_2)D_2 \int\limits_{S_2} \frac{\partial(w_{-,1}|_{S_2})}{\partial z_2} \frac{\partial(v_{-,1}|_{S_2})}{\partial z_2}\, dz_2 =$$

$$= l_-(v_-) + l_+(v_+) \ \ \forall v \in C^2(\Omega) \cap K_0.$$

Taking into account (24) and the fact that $C^2(\Omega) \cap K_0$ is dense in K_0, we obtain (12). □

Assuming that the solution w_0 to variational problem (12) has additional regularity, by applying the generalized Green formula (see, e.g., [2,26]), we deduce differential equations and boundary conditions for the functions w_0:

$$D_0 \Delta^2 w_0 = f \ \text{ in } \ \Omega \setminus (\overline{S}_1 \cup \overline{S}_2),$$

$$w_0 = \frac{\partial w_0}{\partial \nu} = 0 \ \text{ on } \ \partial\Omega,$$

$$w_0 = \delta_0 x_2 + \beta_0 \ \text{ on } \ S_1, \ \ \delta_0, \beta_0 \in \mathbb{R},$$

$$[m^1(w_0)] = 0 \ \text{ on } \ S_1,$$

$$\int\limits_{S_1} [t^1(w_0)]\, dx_2 = 0, \ \ \int\limits_{S_1} [t^1(w_0)] x_2\, dx_2 = 0,$$

$$[t^2(w_0)] = 0 \ \text{ on } \ S_2,$$

$$p = w_{0,1} \ \text{ on } \ S_2,$$

$$4dD_2(1-k_2)p_{,22} = [m^2(w_0)] \ \text{ on } \ S_2,$$

$$p_{,2} = 0 \ \text{ at } \ \partial S_2,$$

where $m^\alpha(w_0)$ and $t^\alpha(w_0)$ are bending moments and transverse forces, respectively, defined by

$$m^\alpha(w_0) = D_\alpha\Big(k_\alpha \Delta w_0 + (1-k_\alpha)\frac{\partial^2 w_0}{\partial \nu^2}\Big),$$

$$t^\alpha(w_0) = D_\alpha \frac{\partial}{\partial \nu}\Big(\Delta w_0 + (1-k_\alpha)\frac{\partial^2 w_0}{\partial \tau^2}\Big),$$

$\nu = (1, 0)$ and $\tau = (-1, 0)$ are an unit normal vector and an unit tangent vector, respectively, $\alpha = 1, 2$.

The mechanical interpretation of boundary conditions can be found in [6], see also [10,34,35].

5. Concluding Remarks

We proposed a method of asymptotic derivation of plate models containing hard thin inclusions lying strictly inside the plate. The method is based on the variational properties of the solution of the equilibrium problem and allows for one to simultaneously construct all possible cases of hard thin inclusions. It is shown that there exist two type of thin inclusions in the Kirchhoff–Love plate, namely, the rigid inclusion S_1 for $N < -1$ and the elastic inclusion S_2 for $N = -1$. The inhomogeneity disappears in the case of $N \in (-1, 1)$. The last means that we have no any peculiarity along the set S_3.

In the conclusion, we note that the proposed method does not allow considering the case of the exponent $N \geq 1$ simultaneously with the case of the exponent $N < 1$, because, for the first case, we need to use other type of test functions (see [6]), which cannot be substituted in variational equality for the second case of the exponent.

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