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Exploring the Interplay between Conceptualizing and Realizing Inquiry—The Case of One Mathematics Teacher's Trajectory

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Abstract: Inquiry, an approach that departs from traditional mathematics teaching, empowers students through active participation and increased accountability in exploration, argumentation, evaluation, and communication of mathematical ideas. There is broad research consensus on the benefits of inquiry-based approaches to teaching and learning mathematics, including their potential to support equitable mathematics classrooms. While research has separately explored teachers' conceptions of inquiry and their efforts to enact the practice, little is known about the interplay between mathematics teachers' conceptions and enactment, and how it could be harnessed in professional development. In this study, we follow Alex, an experienced upper secondary mathematics teacher unfamiliar with inquiry, as he participates in a one-semester professional development course that draws on inquiry in multiple ways. His trajectory towards learning to teach through inquiry is revealed through patterns and shifts in his reflections and classroom actions. Our findings reveal significant developments in Alex's conception of inquiry and in how he realizes it in his classroom, identifying three paths that illuminate his inquiry trajectory: the teacher's role in inquiry interactions, a growing idea of inquiry, and orchestrating whole-class situations. In the interplay between enacting and reflecting, he moves from distributing authority separately between himself and 'the students' (as one unit) to fostering shared authority, a key aspect of empowerment, between himself and his students (as multiple voices) in both groupwork and whole-class episodes.

Keywords: inquiry-based mathematics teaching; conceptualization of inquiry; realization of inquiry; professional development; authority

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1. Introduction

Inquiry has received massive attention in educational research, mathematics curriculums, and professional development worldwide [1–3]. There is broad research consensus on the benefits of inquiry-based approaches to teaching and learning mathematics (e.g., [3,4]), including its potential to support equitable mathematics classrooms [2,5,6] Ernest [7] proposed the empowerment of students—with students experiencing a position of power through engagement in mathematics—as a goal for mathematics education, arguing that it is a step towards equity. Though it is a wide term, without a universal definition [8], inquiry is often referred to as an approach that departs from traditional mathematics teaching and empowers students through active participation and increased accountability in exploration, argumentation, evaluation, and communication of mathematical ideas.

In spite of the theoretical arguments and the policies supporting inquiry, this approach to teaching and learning is rare in day-to-day practice in science—where it originated—as well as in mathematics [1,9]. As researchers seek to understand the mechanisms at play in this phenomenon, several studies within science education suggest that teachers hold flawed ideas of what inquiry is (e.g., [9,10]), which seems to influence how inquiry is enacted (e.g., [11,12]). Summarized, "there seems to be confusion over what teaching science in inquiry really means and how that translates into classroom practice" ([13] p. 63). Research on mathematics teachers' ideas of what inquiry entails is scarce, and even less is known about how this might influence teachers' practices [14]. This article contributes to the field by exploring the interconnection between inquiry ideas and actions through a case study of an experienced upper secondary school mathematics teacher's path towards teaching through inquiry, drawing on his narratives and on observations of two lessons where he sets about bringing the approach to life.

What we do know from existing research is that it is in no way easy for teachers to shift towards inquiry-based approaches to mathematics teaching [14–16]. One way to support the teachers in this complex and challenging endeavor is through professional development (PD) courses. Based on a review of empirical research, not limited to the PD context, Stahnke et al. [17] suggested that teachers' knowledge and beliefs influence their in-the-moment decision-making and instructional practice. Recently, researchers have also looked at teacher development through the reflections teachers share after planning and teaching of lessons (e.g., [18,19]). Thus, PD courses should encourage teachers to experience and experiment with inquiry as well as to reflect on their views about mathematics teaching and inquiry [1,20,21]. Maaß et al. [14] stress that the ways in which teachers interpret inquiry-based approaches to mathematics after a PD course seem to be important in how they implement it in their own teaching, but that this possible connection is heavily understudied. Wee et al. [13] suggest that teachers participating in PD courses should also experiment with inquiry in their own teaching and, through this, develop their understanding of inquiry and how to teach through inquiry-based approaches.

Research thus emphasizes both reflecting on inquiry and enacting it as valued practices in PD. However, little is known about the interplay between these two practices, and about how this interplay contributes to teachers' professional development. We seek, therefore, to understand how inquiry is viewed and recontextualized by teachers over time, and further, how it is brought to life through their actions in the mathematics classroom. In doing so, we recognize that teachers' development occurs through iterative processes of experimentation and reflection, requiring a "careful study of the pathways teachers take as they grow as practitioners" ([22] p. 21). The findings of this study could provide insights into how teacher education institutions (in this case, PD) can support teachers' development. This is important, as inquiry "raises a quest for developmental work and professional development to support teachers in experimenting with and developing *their own* inquiry-based practice of mathematics teaching" ([21] p. 808, our emphasis).

A crucial argument for inquiry-based approaches in mathematics is that inquiry is a pedagogy supporting student empowerment [2,5,6]. Given the differences between the empowerment of students in traditional versus inquiry-based mathematics classrooms, power relationships in the classroom can significantly change when teachers transition towards inquiry-based approaches. In this study, we focus on the idea of shared authority [23]. More specifically, we are interested in the shift in authority relationships, i.e., who is in command, transitioning from the traditional classroom, where the teacher is the authority figure, to shared authority between teacher and students in inquiry-based approaches. We expect this lens to contribute to our understanding of teachers' pathways when learning to teach through inquiry.

The aim of this article is to provide a comprehensive picture of one teacher's trajectory, the pathways he takes as he engages in learning to teach through inquiry as he participates in a one-semester mathematics PD course, and how authority relationships are reflected in his trajectory. To our knowledge, few studies have taken this perspective. We will address this through the case of Alex, an in-service mathematics teacher with nearly 15 years of experience across subjects and grades, who still, in his own words, is unfamiliar with inquiry-based approaches. More specifically, we ask the following question:

> *What characterizes Alex's inquiry trajectory, interpreted through the interplay between his conceptualizations and realizations of inquiry?*

Conceptualization here refers to the teacher's formation of an idea of what inquiry entails, while realization addresses how it is brought to life in the mathematics classroom. Thereafter, based on our findings, we discuss the following question:

What are the connections between Alex's inquiry trajectory and the authority relationships in the mathematics classroom?

2. Theoretical Background

2.1. Inquiry in Mathematics Teaching

The following three facets form a natural basis for talking about and practicing inquiry as they are experienceable (can be observed and felt "in-action") and frequently highlighted across research: *the role of the students*, *the role of the teacher*, and *inquiry problems*. In Table 1, we have synthesized essential elements within the three facets based on a variety of frameworks and research on inquiry in mathematics (e.g., [2,16,21,24,25]).

Table 1. Theoretical framework for students' and teachers' roles in inquiry and inquiry problems.

Essential Elements of Inquiry in Teaching and Learning Mathematics.	
Students	Build on what they know to engage deeply with unfamiliar problems Collaboratively grapple with mathematical ideas Take on mathematical authority and responsibility
Teachers	Encourage and inquire into student reasoning Use student contributions to develop shared understandings and connections to formal mathematics Foster student empowerment through design, structure, and facilitation
Problems	Foster student engagement Are meaningful and relevant for students' daily lives Are related to mathematical ideas and concepts

Inquiry enables students to practice exploring the unknown through what is known, by connecting and building on their existing knowledge to develop what, for them, can be considered new insight and strategies [21,25,26]. As the social context can contribute to meaningful learning [16], e.g., enriching students' thinking [2], inquiry research often promotes collaboration. When grappling with unfamiliar problems together, students practice communicating, negotiating, and evaluating ideas [2,16,27] by actively engaging in each other's reasoning and working towards a shared understanding of the problem and its possible solutions [28]. Thus, we see students' active engagement in inquiry processes as exploration, argumentation, communication, and evaluation of mathematical ideas and relationships, where we consider all four elements as equally essential.

For teachers to promote students' inquiry-based learning, it is important to be curious about, encourage, and challenge students' explorations and argumentation through purposeful questioning and focusing actions that illuminate student thinking [2,16,29,30]. Such questions and actions "prompt students to explain their thinking and justify their solution strategies, with a focus on the reasoning the students utilized during the task as opposed to only the procedures used" ([25] p. 17). Argumentation is closely linked to the development of classroom norms [31], and an inquiry-oriented teacher is expected to promote collaboration and the sharing of ideas and argumentation between the students [2,25]. The structuring and orchestration of lessons is yet another highly important aspect of inquiry-based teaching [15,32]. Through the anticipation and monitoring of students' reasoning, teachers are better prepared to ask open and challenging questions with a learning purpose, not least to help students bridge their thinking to formal mathematics and the mathematical content they are grappling with. Connecting and sequencing students' ideas and arguments in a plenary discussion is one fruitful way to do so [2,15,25,32].

Teacher and student inquiry happens in interaction with a mathematical problem. In order to foster inquiry-based learning opportunities as described, the problems given to the students need to be related to mathematical ideas and concepts, as well as foster engagement and be perceived as meaningful and relevant by the students [16,21,24].

The three facets are artificially separated in Table 1; in reality, they are interwoven. For example, the way the teacher structures the inquiry activity—starting with selecting or designing problems that are cognitively challenging but still approachable through the students' existing knowledge [21,26]—lays a foundation for the students' active engagement in inquiry processes and for the teacher's further orchestration (cf. [15]) of the inquiry.

2.2. Teachers' Conceptualizations and Realizations of Inquiry

In this article, the term 'conceptualization' is used to capture a teacher's internal formation of an idea of what inquiry entails, revealed through the way they talk about inquiry, i.e., how one "describes and evaluates 'best practice' discursively" ([33] p. 315). We see conceptions as "personal constructs as they guide instructional decisions and impact the representation of the content (. . .) and that they are concept-centered and can be modified with additional information that adds to, challenges, or clarifies the conception" ([12] p. 1). Further, we view 'realizations' of inquiry as how inquiry is enacted in practice; more precisely, it refers to how "best practice" is brought to life in the mathematics classroom [33].

Even though teachers can *talk* about inquiry elements, this does not necessarily mean that these elements find their way into the classroom. Whitehead [34], first published in 1929, addressed the issue of ideas that are learnt but not used almost a hundred years ago, and many educational researchers have addressed inconsistencies between ideas and actions since (e.g., [33,35,36]). Much of the research on teachers' conceptualizations and realizations of inquiry, both in mathematics and science, has focused on comparisons and tensions between the two, and although there seems to be a strong connection between teachers' views about inquiry and how they enact inquiry in the classroom [9,14,17,32,37], researchers have also found misalignments between the two (e.g., [38,39]). Teachers might be able to conceptualize and plan inquiry practices but struggle to realize them [38], and they might be able to realize inquiry practices that they struggle to conceptualize [39]. There are also findings suggesting that even though teachers' conceptualizations and realizations are aligned, both are limited (e.g., [12,40]). Engeln et al. [16] found that even though most of the teachers in their survey study positioned themselves as having positive attitudes towards inquiry-based approaches, the vast majority reported using only certain inquiry elements in their practice, notably, mainly inquiry elements where the teachers remained in control. However, many of the above-mentioned studies rely on teachers' self-reported enactment of inquiry, which comes with a certain risk. For instance, Capps et al. [11] found that many teachers held flawed perceptions of what inquiry really was, which made them believe they were enacting inquiry practices in their classrooms when they probably were not.

Rather than focusing on alignments and misalignments between conceptualizations and realizations, this study focuses on how conceptualization and realization in conjunction, through cycles, can shape a teacher's trajectory for learning to teach mathematics through inquiry. There is a complex relationship between teachers' views, previous experiences, and practices [9]. The connection between views and classroom practices is non-linear and formed within a large system of connected factors [14,22]. In their study of four mathematics teachers' knowledge of inquiry and their inquiry practices as they engaged in a one-year PD program, through concept maps, interviews and observations, Chin et al. [41] found positive progress in both knowledge and practice among all four, but they were not causally related and there were no radical changes in practice over the course of the year. In contrast to most of the above-mentioned studies, they conclude that "our results taken as a whole indicate no obvious correlation between a person's knowledge of mathematics inquiry and her corresponding teaching practice" ([41] p. 859).

Many teachers conceptualize inquiry as an exploration process [9,13,42] more than processes of argumentation, evaluation, and communication of mathematical ideas. Kang et al. [42] studied 34 science teachers' conceptions of inquiry through a teaching scenario survey instrument. The teachers' conceptions were measured in terms of the characteristics they used to identify inquiry activities and compared to the five essential features of inquiry presented in *Inquiry and the National Science Education Standards* document [43]: (1) engaging in scientifically oriented questions, (2) giving priority to evidence, (3) formulating explanations based on evidence, (4) evaluating explanations in connection with scientific knowledge, and (5) communicating explanations. They found that the first three features were emphasized among the teachers, whereas the latter two were rarely used. The researchers claimed that the teachers' conceptualizations were thus limited to a traditional and narrow view of inquiry, due to the lack of connection between science content and inquiry teaching that lies in the evaluation and communication of explanations. Wee et al. [13] aimed to study how science teachers involved in a PD program focusing on inquiry-based activities changed their understanding of inquiry and inquiry teaching. They followed four teachers as they expressed their conceptualization of inquiry by drawing concept maps before and after they implemented inquiry-based teaching. They found that implementation did very little to improve the teachers' individual understanding of inquiry and their understanding of inquiry in the context of classroom instruction, particularly regarding the essential inquiry feature of communicating and justifying explanations, as also seen in the study of Kang et al. [42].

2.3. Inquiry and Shared Authority

Based on case studies, Ernest [7] hypothesized that empowerment is fostered by certain classroom experiences, such as mathematical risk-taking, experiencing success in genuine struggle, and collaborating. Inquiry radically shapes students' educational experience in this direction, as they formulate questions, mathematise, argue, prove, etc., and develop habits of mind with implications beyond school. It supports student empowerment not only because of the real-life relevance of the problems [6], but also because of the "vision of relationships between the different actors potentially involved within and outside the school system" ([21] p. 808). Given the differences between the empowerment of students in traditional versus inquiry-based mathematics teaching, we expect this lens to contribute to our understanding of Alex's trajectory. We limit our attention to one key aspect of student empowerment, a "shift in power relations so that the teacher listens to pupils in depth and allows them to make and express judgements and values their contributions" ([7] p. 13).

In the previous section, we argued that, despite the lack of a clear definition, inquiry-based approaches are recognizable by three interconnected facets: the problems, the role of the teacher, and the role of the students. While all three facets of inquiry are, in principle, interconnected, the teacher can more readily act on two of these: the task, and the role of the teacher; thus, these two can be perceived as more actionable by teachers, and therefore, more worthy of attention ([44] p. 12). Nevertheless, transitioning from traditional to inquiry-based teaching requires a shift in students' roles, too (e.g., getting to grips with different mathematical tasks, collaborating with their peers, taking ownership of mathematics). An obvious challenge for teachers is to identify and enact practices that enable students to meet the new expectations (e.g., working collaboratively [45]) and to cope with students' resistance to new practices (e.g., the use of challenging tasks or engaging with multiple solutions [46]). We are particularly interested here in the shift in authority relationships as Alex learns to teach through inquiry. Authority relationships differ significantly between traditional and inquiry-based approaches (see [6]), and the idea of *shared authority* is useful to capture this shift as we explore how a teacher learns to teach through inquiry. Following Amit and Fried [23], we understand an authority figure to be a person(s) whose statements and commands are accepted or obeyed without question (p. 147). Although inanimate objects such as textbooks and calculators can exert authority in mathematics [47], we limit our attention to people.

Empirical studies show that sharing authority is fraught with difficulties. For example, a case study of secondary mathematics teachers showed that sharing authority can be hindered by teachers' views of mathematics, with one teacher eliminating groupwork because of his conviction that his explanations were crucial to students' understanding [48]. More encouragingly, Ng et al. [49] showed that certain practices enacted by a teacher in whole-class discussions did help teacher in moving away from a pattern of positioning himself as the only authority. In contrast with these examples of the struggle to share authority, Arnesen and Rø [50] present a case where authority is shared between the teacher and the students, yet their analysis reveals a different set of challenges: considering the issue of shared authority in relation to the issue of supporting students' mathematical reasoning. Distinguishing between the potential of a teacher's moves to support shared authority and reasoning, respectively, they found that the teacher tended to prioritize shared authority at the expense of mathematical reasoning. Finally, tracking not one but two different agendas (connecting to children's mathematical thinking and connecting to children's funds of knowledge), Kinser-Traut and Turner [51] added a layer of complexity as they found that sharing authority is not a characteristic of a teacher's practice but is domain-specific. In our exploration of a teacher's conceptualization and realization of inquiry, the notion of authority enables us to capture the development of the teacher's scope of action on the student role.

3. Methods

3.1. Alex

Alex (pseudonym) was one of four teachers who volunteered to be observed and interviewed for a project looking at teachers' inquiry experiences and developments in relation to participating in a one-semester PD course offered by a Norwegian university to in-service lower secondary (grades 8–10) and upper secondary (grades 11–13) mathematics teachers. Elsewhere [52,53], we looked at all four teachers' conceptualizations of inquiry in terms of the teacher role, student role, and problems, both before and throughout their participation in the PD course. In this study, we focus only on Alex. Small-scale studies like this "are likely to remain a well-suited method for articulating the mechanisms of teacher learning" ([22] p. 24), which is in line with our aim.

Alex was chosen as our case because of his long teaching career combined with his self-reported newness to inquiry approaches to teaching mathematics. His teaching experience of nearly 15 years was mainly at the primary and lower secondary levels, but for the last year before entering the PD course, he had worked at an upper secondary school. Concurrently to participating in the PD course, Alex taught grade 12 mathematics for students who had chosen practical mathematics. (In Norway, upper secondary students choose between practical mathematics, social science mathematics, and natural science mathematics. Practical mathematics is considered as the least advanced option.) Many of the students knew each other and Alex from the previous year's grade 11 practical mathematics class, making their mathematics class a familiar environment for them. At the start of the PD course, Alex reported having a traditional, teacher-centered teaching approach; he described his typical lesson as an introduction by the teacher followed by individual work. However, he expressed being motivated to learn about inquiry and develop his practice accordingly, partly because of the centrality of the approach in the new Norwegian mathematics curriculum, and partly because he believed that inquiry could be fruitful for students at a range of mathematical attainment levels. Nevertheless, he voiced concerns about inquiry being an unfamiliar approach for his students not only in his lessons, but in their education in general. Accompanying this concern, Alex disclosed that inquiry was an unfamiliar approach for him as well, both from a student and teacher perspective.

While previous research points at in-service mathematics teachers' struggles with conceptualizing and realizing inquiry-based approaches (e.g., [38,39,41]), we want to contribute to a better understanding of the trajectory an experienced teacher—albeit a novice when it comes to inquiry—takes as he tries to develop inquiry-based approaches to teaching mathematics through repeated reflections and enactment.

3.2. The PD Course

Mathematics PD courses can support teachers in developing more sophisticated ideas of inquiry (e.g., [14] and positively influence their enactment of inquiry (e.g., [20]). In this context, facilitating authentic inquiry experiences has been emphasized [1,21]. The PD course in this study, taking place in the Autumn of 2022, engaged its participating teachers in two cycles of experiencing, reflecting, designing, and realizing authentic inquiry (Figure 1). These aspects are advocated for the design of PD promoting inquiry [1,20]. Through the two cycles, the teachers visited and revisited inquiry in mathematics from both a learner and teacher perspective in both PD and school settings. The PD course included three five-day seminars at the university (one in August, one in October, and one in December), focusing mainly on single- and multivariable calculus complemented with sessions and reflections on mathematical pedagogy. Between seminars, the teachers worked at their respective schools as well as following asynchronous and synchronous remote lessons and exercises related to the mathematical curriculum in the PD course. The last seminar was dedicated to repetition and exams; thus, we focus on the first two seminars.

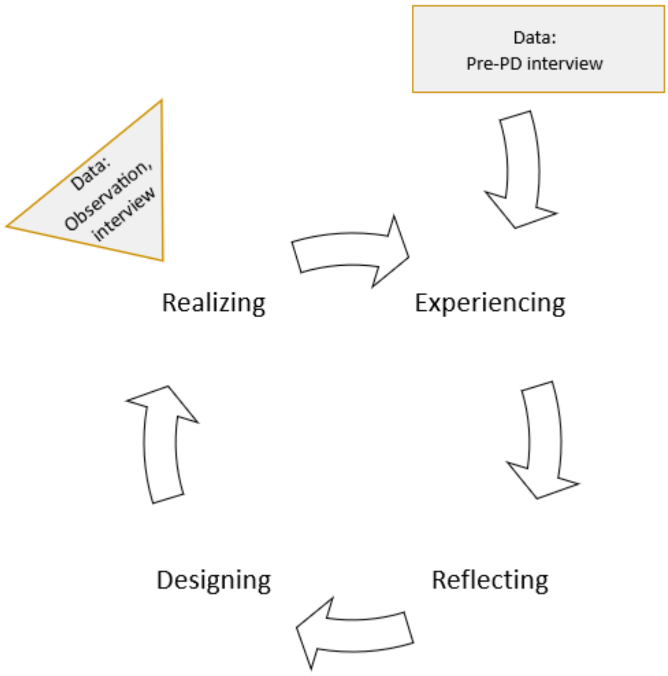


Figure 1. Model of the PD cycles and events of data collection in this study.

Experiencing. In each of the two seminars, the teachers worked in groups with an inquiry problem related to the mathematics curriculum in the course. The aim was for the teachers to participate in inquiry as learners of mathematics, to experience it in the ways that their students would (for example, by using their knowledge to collaboratively approach and grapple with an unfamiliar problem). The PD instructors modeled inquiry teaching, e.g., inquiring into teachers' reasoning, encouraging collaboration, and orchestrating whole-class summaries.

Reflecting. Subsequently, the instructors orchestrated sessions of reflecting on what inquiry in mathematics was. In the first seminar, the teachers discussed in their groups what they saw as essential for inquiry in mathematics, building on what they had just experienced and on their established perspectives of inquiry—a concept that had implicitly been circulating in the Norwegian educational curriculum for quite a while (see [14]) and in which the Norwegian term 'utforsking' is used in everyday language. This discussion was followed by a whole-class brainstorming session on inquiry. The collective identification of keywords could be seen as both soliciting and broadening teachers' perspectives. All contributions were encouraged and added to the blackboard without any validation by the instructor (Figure 2). (We include the picture of the board to give a sense of the number of keywords that were suggested. We are mindful that many readers will not be able to understand the Norwegian contents. However, it is impossible to offer a faithful translation by regarding the words (e.g., 'less structure', and 'understanding') isolated from the context, and an analysis of the class discussion is beyond the scope of this case study.) Subsequently, the first author synthesized the keywords into a mind map with three main categories: students, teachers, and problems (see Figure 3 for a translation to English). The mind map (in Norwegian) was given to the teachers, introducing the student role, teacher role, and inquiry problems as three facets of inquiry in mathematics that would guide future work in the course. By building on the teachers' perspectives and experiences, inquiry was portrayed in broad ways to allow for the teachers to develop their own variations [20].

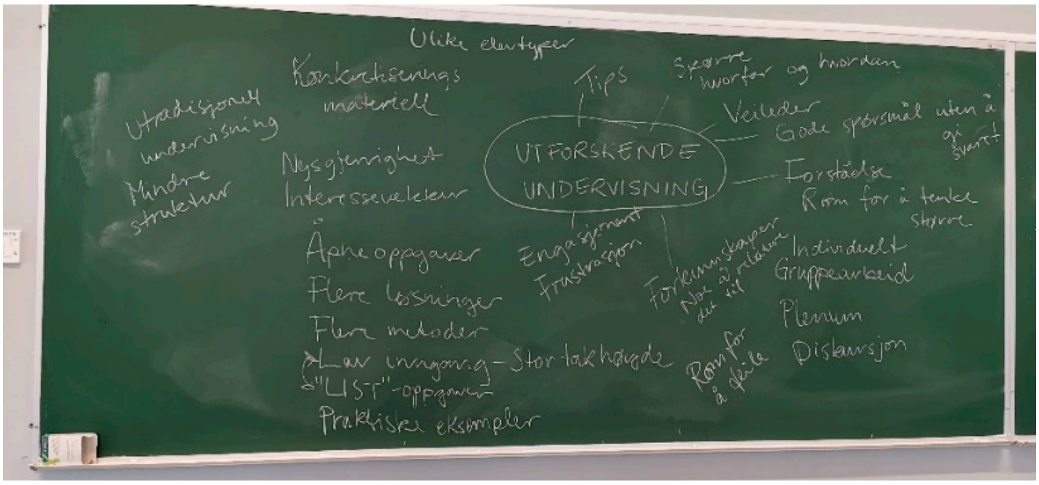


Figure 2. Picture of the blackboard at the end of brainstorming.

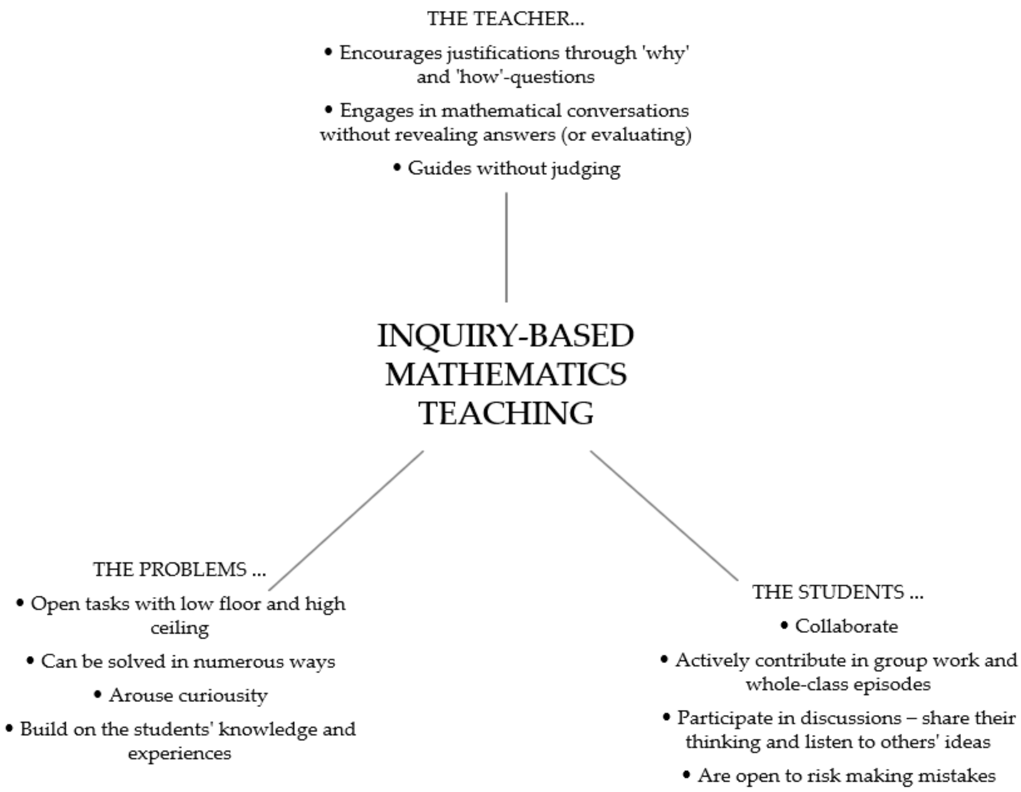


Figure 3. English translation of the synthesized mind map.

In the second seminar, the reflection session encouraged the teachers to reflect in their groups on the first cycle, and in particular, the design of an inquiry lesson and their realizations of inquiry by trying out this lesson in their own classrooms (see designing and realizing). This connected the PD course and the teachers' needs [1]. The teachers were encouraged to relate these reflections to the three facets (student, teacher, problem).

Designing. In both seminars, the teachers groupwise designed an inquiry lesson to try out in their own classrooms. In the first seminar, they were asked to design a problem for the lesson and agree on essential elements for realizing the lesson. The idea was to enable the teachers to design inquiry lessons that would be closely related to their own world, i.e., their practices [1]. Some support structures were offered, such as guiding questions (e.g., What makes a fruitful problem in your class? What specific aspects of the student and teacher roles do you want to focus on?), three versions of one example problem with varying levels of pre-decided strategies or procedures offered in the problem text [54,55], and the synthesized mind map in Figure 3. All these resources were given to the teachers in Norwegian. The problem Alex and his group designed was refined in collaboration with the first author, as the group only got around to making a first draft during the design session. The refined problem, subject to some modifications by Alex (e.g., adding some information) and translated to English by the authors, is shown in Figure 4.

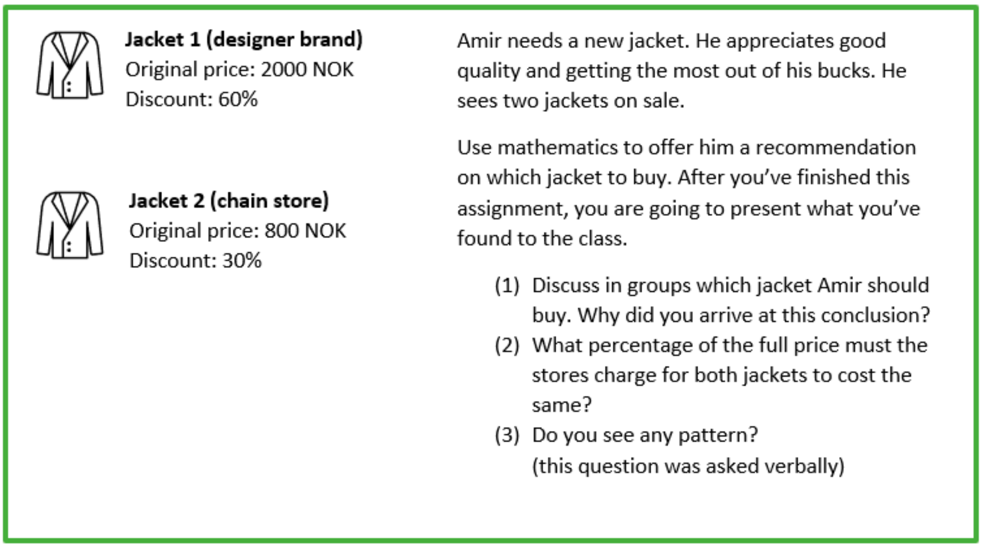
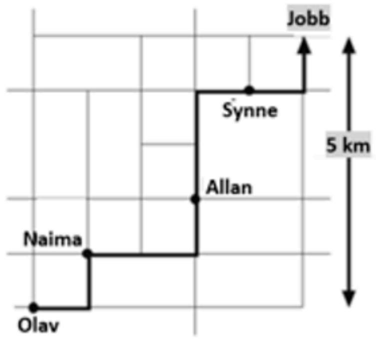


Figure 4. The problem for the first lesson.

The course instructors and the first author found that the first design session was too complex, with the teachers spending most of their time designing their problem and bringing little attention to what the students should do when interacting with the problem or on how they as teachers could facilitate the inquiry. Therefore, extra emphasis was put on the student and teacher roles in inquiry in the design session in the second seminar. A problem that could easily be adapted [20] and used in the teachers' classrooms was given to the teachers (see Figure 5 for a translation to English). In the same groups as before, the teachers tried to solve the problem in different ways, guided by the following questions: How might your student try to solve this problem? What ideas might they bring? And how would you as a teacher support them as they grapple with their ideas? Subsequently, all groups shared some of their solutions on the blackboard. The groups were then asked to design a lesson using this problem. Some guiding structures were the questions above and encouragements to discuss how they would use the students' ideas. Both the problem and the other resources were given to the teachers in Norwegian.



The figure above shows where each person lives, and the route Olav drives.

Olav has gotten his driver's license and drives to his summer job. Every day, he picks up Naima, Allan and Synne along the way. They also ride with him back to their homes.

The four of them decide to split the driving expenses, which are 1000 NOK.

Discuss in groups: How much should each of them pay? Remember to justify your suggestions.

Preferably come up with more than one suggestion. Provide arguments supporting what you think is the best way to split the expenses.

Figure 5. The problem for the second lesson, adapted from "Sharing Petrol Costs" from Bowland Maths, https://www.bowlandmaths.org.uk/materials/pd/online/pd_05/resources.html (accessed on 7 October 2022).

Because of this adjustment to the design phase, we did not obtain data on Alex's trajectory in terms of how his reflections on inquiry problems would be realized into a concrete problem for the second lesson. Consequently, our data regarding this facet was not as rich as for the other two facets (students and teachers). In this study, we therefore focus on the student and teacher facets.

Realizing. After each of the two seminars, the teachers tried out their designed inquiry lesson individually in their own mathematics classroom. The idea was for them to experiment and bring to life what they, inspired by their conceptualizations of inquiry and the experiences and reflections they had participated in in the PD course so far, saw as essential elements and actions for inquiry in mathematics. While they experienced inquiry as learners in the PD course, they now experienced it from a teacher's perspective. This type of personal experience with teaching through inquiry is key to teachers' professional development [1]. In the first cycle, they were also asked to hand in an individual reflection note on what they had focused on in their realization of the inquiry lesson, how it went, and what they had learnt from it.

3.3. Data Collection

Data were collected in all four phases of the cycles (experiencing, reflecting, designing, realizing) through recorded observations and interviews. Observations of teachers' pedagogical actions reveal the 'what' and 'how' of their teaching, but other perspectives are needed to encompass the 'why' [56]. To explore the qualitative aspects of, and ultimately understand, Alex's trajectory, we thus needed to combine and cross-validate observations from his realizations with reflections available through interviews [57]. We draw on data from three events: A pre-PD interview approximately two months before Alex started in the PD course, and observations of his realization of both the first and second lesson followed by interviews with Alex about his realizations.

The observations involved placing an audio recorder on Alex to record all interactions with the students and whole-class episodes. In addition, the first author was present at the back of the classroom, taking notes without interfering in the lesson. The aim of the pre-PD interview, conducted digitally and video-recorded, was to become familiar with Alex's current teaching practice and his conception of inquiry in mathematics before his participation in the PD course—a baseline. The two interviews after Alex's lessons were conducted in his office and video-recorded. The topics were (i) Alex's reflections on his realization of the lesson; (ii) his reflections on what inquiry in mathematics entails; and (iii) the connections between (i) and (ii). This gave us insight into his reflections on events that stood out for him from the lessons, in relation to his conceptualization of the student and teacher roles, and his ideas on new and refined inquiry elements to feed into his future realizations. All three interviews were semi-structured, allowing for Alex's conceptions and reflections to form the conversation. The interviews were conducted by the first author, whom we will sometimes refer to as the "researcher".

3.4. Analysis of Data

While previous research has often used scoreboards and schemes to score, or check the boxes of, inquiry elements present in a lesson or interview (e.g., [12,41,58]), our research looked at the presence (or absence) of inquiry elements together with the quality of the enacted inquiry elements. We achieved this through a thematic analysis [59] of the recorded data. Thematic analysis is especially fruitful when looking for patterns, and tracing developments, in and between observations and interviews [59]. This flexible analytical process provided us with space to explore the concurrent developments and patterns in Alex's conceptualizations and realizations of inquiry and how they together formed his trajectory.

The first author repeatedly listened to the recordings to identify critical events [60], which then were transcribed. As our focus is on Alex's trajectory in terms of how he conceptualized and realized inquiry, critical events included how he facilitated inquiry throughout the lessons and on interview extracts where he reflected on essential inquiry elements. Two of the authors coded the transcripts through iterative processes, moving between transcripts, coding, and recoding, supported by theoretically founded codes and supplementary codes emerging from the data. Table 2 shows the coding guide for the interviews. The details for the problem facet have been omitted, due to the changes in the PD design explained above.

Table 2. Coding guide for interviews. Codes that were inductively produced are marked with a *.

Inquiry Facet	Category (Codes in Parentheses)
Students	*Collaborative and communicative processes* (argue and challenge; build on ideas; discuss; evaluate; explain; share and listen to ideas; shared understanding) *Student thinking* (connect existing knowledge; explore; find strategies and solutions; see that there are multiple strategies and solutions *; use knowledge in new situations) *Authority and accountability* (actively engage; responsibility and ownership)
Teachers	*Interactions with student reasoning* (ask for justifications; challenge student thinking; direct *; encourage new solution or path; few prescriptions; foster collaboration; guide and support; inquire into student thinking; purposeful questioning *) *Brokering* (bridge student ideas and formal mathematics; connect students' thinking with each other) *Structure and planning* (anticipate student thinking; plan activity; select and sequence; summarize)

For the observations, we looked at how the teacher brought to life or tried to facilitate inquiry (e.g., the interview code 'argue' would have the observation code 'teacher encouraging student argumentation'). Any disagreements in coding were discussed and resolved. An utterance or observation could be assigned multiple codes if it addressed multiple inquiry elements. In addition to the assigned codes, the absence of some codes was equally interesting to us. From the coded utterances, we looked for patterns within an interview or observation, and based on the patterns, we wrote narratives of the respective events. These narratives guide the result section. From patterns across the narratives, we identified three main paths for Alex's trajectory. We let these paths emerge through the analysis before summarizing them in the discussion (Table 3 in Section 5.1) and discussing them to address the first research question.

We recognize the inherent asymmetry in the two facets of inquiry in Table 2; while both are equally accessible in teacher's conceptualizations, they may be seen as within the teacher's scope of action or not, depending on the actor they feature. This then has implications for the realization of inquiry. To capture such nuances that can explain what features of inquiry the teacher might choose to experiment with, we turn to the concept of shared authority. Inspired by Wagner and Herbel-Eisenmann [47], we characterize the authorities and their relationships in the narratives by interpreting who decides what happens during mathematics lessons and who decides what is true in mathematics.

4. Results

All excerpts in the result section are translated from Norwegian by the authors.

4.1. Pre-PD Interview

The main idea of inquiry, to Alex, was unstructured tasks where the students had to use and combine their existing knowledge. The tasks should have a low entrance point, so that "all students can accomplish something", but also include multiple solution paths. When elaborating on his conceptualization of inquiry, Alex continued to focus on the dynamics between the (individual) student and the tasks. He shared his ambitions for the students to discover what, for them, would be new ways to solve the problems (as opposed to just using formulas and strategies they already knew), to see how mathematics is connected across topics, and to become aware that "there are more ways to Rome" (sic). Alex did not reflect unprobed on what his role as a teacher in inquiry was, but was encouraged to share his views (Box 1):

Box 1. Excerpt 1.

Researcher: What do you think is the teacher's role in inquiry then?
Alex: Ehm, to have the toolbox. And then the students can get tools as they need them.
Researcher: Could you elaborate on that?
Alex: Yes, so (. . .) imagine you're at the woodwork room (. . .) [The students] come to the teacher and ask: "Do you have anything so that I can do such and such?". "Yes. What about this hand plane, it will make it smooth". (. . .) That you at least point them in the right direction of what they can look for to accomplish the task they are doing.

This conceptualization of the teacher role in mathematical inquiry portrayed the teacher role as steering the students in "the right direction". Similar views were also shared when Alex spoke about the students' role, where he talked about himself illustrating multiple ways to solve problems and the students choosing the one they liked the best.

4.2. Lesson 1

4.2.1. Observation Lesson 1

Throughout the lesson, Alex engaged in interactions with groups where he began making efforts to encourage students to explore (by finding multiple strategies and solutions to the problems) and communicate their mathematical ideas (mainly by telling Alex their solutions in groupwork and presenting their calculations and solutions in a plenary session). However, most of the student–teacher interactions remained on superficial levels. The excerpt in Box 2 illustrates the superficial level of his interactions and how he declined invitations from the students to be involved in their inquiry, which was another characteristic of Alex's interactions with the students throughout the lessons:

Box 2. Excerpt 2.

Alex: What have you found out?
Student 1: Well, I have found some arguments.
Alex: You have?
Student 1: Yes, do you wanna hear?
Alex: I want you to write them down, so that you can present them for the class later. Maybe even make something visual, maybe a PowerPoint or-
Student 2: Is this what we are gonna present?
Alex: Yes, and you need some arguments too.

In this excerpt, Alex emphasized the importance of supporting arguments but shut down the students' invitation for him to hear their arguments. Thus, he declined the opportunity to inquire into their reasoning. This happened both in group settings (where he declined two groups' efforts to share their arguments with him) and in whole-class episodes (where he quickly moved on from, or shut down, student suggestions to patterns). While the students inquired on the problem in their groups, there were also several episodes where students asked Alex questions or expressed confusion, only for Alex to walk away without reacting to their questions or confusion. At one point, a student reached out to Alex, telling him that his evasive responses did not help her, only to be shut down by Alex telling her that "the point is that I'm not giving you any concrete answers".

The few examples of Alex attempting to encourage students to support their ideas with arguments were almost exclusively general encouragements like the one in Box 2, and thus not related to specific aspects of their ideas. The superficial nature of Alex's interactions with the students was also reflected in how quickly he moved on from their inquiry. Most of his initial questions when approaching a group encouraged the students to explain their results, inviting them to communicate their solutions to him. However, once the students had responded to the question, Alex seemed to initiate a new process. At the start of the groupwork, Alex's go-to-response was "can you make a visual representation of this?" Later in the lesson, the go-to-response changed to "can you find another way to solve it?". Hence, the students' inquiry processes were not followed up or further elaborated on, resulting in few discussions between students or between students and Alex when he was present in a group.

Alex's efforts to support his students' inquiry processes when their progression paused seemed to include several instances of simplifying the problem and directing student processes. This happened both in groupwork and whole-class episodes. In Excerpt 3 (Box 3), we enter a whole-class episode in the lesson. All groups had, in random order, presented their answers to the first and second part of the problem, and Alex now wanted to make a table with the groups' answers to the second part (percentages paid for each jacket if they cost the same):

Box 3. Excerpt 3.

Alex: Now we're gonna make a table. (. . .) Did you arrive at any percentages where it [sic] was the same?

Student: We found that. . . Well, we didn't really find any percentages that were equal, but where the price was the same.

Alex: Yes, this was complicated. The jackets cost the same if. . .? If you have a 70% discount on the expensive jacket [jacket 1] and 25% discount on the cheap one [jacket 2]. Then what do you pay?

Student: What?

Alex: What do you pay? If you have 70% discount, then how much do you pay?

Students: 30%

Alex: Yes, right. So, if I pay 30% of the expensive jacket and 75% of the cheap jacket, then they cost. . .?

Students: 400.

As a student from one of the groups commented the imprecise question from Alex, he responded by reducing the problem to a standard task and directing the students to an answer. This type of interaction was typically seen where students' progression had paused and especially where the students expressed doubt.

Alex instructed the students to discuss in their groups as he launched the problem, but this focus was not maintained throughout the lesson. When Alex interacted with the students it was mainly in the form of short one-to-one interactions between himself and individual students, and he did not encourage collaborative actions.

4.2.2. Interview Proceeding Lesson 1

When reflecting on the lesson, Alex explained how he tried to distance himself from the inquiry processes (Box 4):

Box 4. Excerpt 4.

Alex: My thought was to give them minimum information. Minimum steering, like, I should steer them as little as possible. That they should try to find methods themselves. (. . .) Well, that was my thoughts on my- to hold back on the information, 'cause it's so easy to give too much information and direct them into one trajectory.

(. . .)

Researcher: How did you plan to hold back? Were there any questions you wanted to ask or-

Alex: To be encouraging and say "Yes! Now you're onto something, could you do more?". Like, that kind of opening questions. "Is there anything else you could do?".

Through such reflections, Alex seemed to conceptualize inquiry as an approach where the teacher provides little direction, and the students take on the responsibility of "finding methods". An essential part of the teacher's role for Alex seemed to be to encourage students to explore new strategies. When asked further about his interactions with the students, Alex expressed that he was unsure of what would really be purposeful questions to support the students' inquiry. He could not identify any questions he asked during the lesson that he felt were more fruitful than others but agreed, after some gentle reminding from the researcher, that "the combination of all those 'why' and 'what' and 'how did you find this', that's what makes the students progress". He concluded that he still had to figure out the "right" questions.

Alex took from the lesson that teacher preparation is essential for a fruitful inquiry process, stating that "if the problem is well thought through, you'll manage to guide them during the activity". To be better prepared for inquiry lessons, he had to "be aware of all possible methods". Thus, for Alex, anticipating different ways in which the students might approach a problem came forth as an important part of inquiry that he wanted to enact. Similarly, he addressed the need to improve his orchestration of whole-class episodes, as shown in Excerpt 5 (Box 5) about the various strategies and solutions chosen by the groups:

Box 5. Excerpt 5.

Researcher: Did you reflect on how to highlight this? That there were so many ways to find an answer?

Alex: I did, and that's an area where I can improve. I mean, I can be better at picking up what they do. And demonstrate that "alright, you guys used this method, and you guys used this, and you used this". (. . .) If I had been a little more focused, I could have picked up on all those different methods and weighted them against each other.

Researcher: So, what you're saying here is that helping the students to see connections between methods is a typical teacher task [in inquiry]?

Alex: Mhm.

Here, Alex shared a view of whole-class inquiry consisting of students presenting their methods and solutions, and the teacher making connections between them. His focus was on his own contribution to the whole-class episode.

As for collaborative aspects in his lesson, Alex disclosed that the class had not worked much with establishing what was expected of them when working together, but that he felt that active participation was implicit for collaboration. Alex emphasized that he had arranged the students in groups of three because he felt this was a good group size for everyone to be able to engage, and that he had purposefully assigned students to the different groups.

4.3. Lesson 2

4.3.1. Observation Lesson 2

The following excerpt (Box 6) illustrates how Alex's interactions with the groups included numerous efforts to inquire into their reasoning.

Box 6. Excerpt 6.

Alex: What have you found out?
Student 1: Ehm. . . Is it- each square is one kilometer?
RAlex: What-why do you think that?
Student 1: Because, like 1, 2, 3, 4, 5 [counts squares]. And it's longer if he's picking them all up.
Alex: Mhm. . .

Through opening questions such as "what are your thoughts?" and "what have you found out 'til now?" Alex showed interest in the students' inquiry as well as their results. We also noticed several episodes of both general encouragements to find arguments ("and then you should argue for your solution") and requests for argumentation that was connected to a concrete student idea (as illustrated in Box 6). Alex's inquiry into student thinking, together with some direct encouragement to engage in collaborative reflections (e.g., "what do you think, [name]?" and "does anyone have another idea?"), shows how collaboration and communication were promoted in this lesson. As before, he constantly encouraged his students to explore new methods and answers to the problem throughout the lesson. There were very few episodes where Alex directed the students' thinking; rather, he made several efforts to support student inquiry by trying to activate their previous knowledge. However, as illustrated in Excerpt 7 (Box 7), this was oftentimes met with confusion:

Box 7. Excerpt 7.

Alex: What have you found out 'til now?
Student 1: Ehm. . . We don't really know.
[Students talking about not wanting to charge their friends to ride with them]
Alex: Well, yes, but have you seen anything like this before? What is it that you are wondering about?
Student 1: It's. . . The whole thing doesn't make sense.
Alex: What doesn't make sense?
Student 1: Like, where have we seen this before?

Alex frequently asked the students if they had seen anything like the problem before or if they knew any information that they could use to solve their queries. Often, such questions were used as follow-up questions after inquiring into their ideas. Box 7 illustrates how Alex's inquiries into student reasoning at times were restricted to one question before abandoning the students' ideas to rather ask them if they had "seen anything like this before", and how this seldom resulted in any progress.

The orchestration of whole-class episodes took a different route than in the first lesson. Alex invited groups to share their ideas in plenary, in a sequenced order, and he built on ideas to progress the collective reasoning. In the following excerpt (Box 8), the class had agreed that each side on the squares in the problem figure equaled one kilometer:

Box 8. Excerpt 8.

Alex: And the question then becomes, how can I use that information? That's some of the information that we have. How can I use that to split the cost of driving to and from work? What are your thoughts on that? [Directs the question to a specific group]
Student: Well, we were a little unsure of that. So, our first solution was just to make it simple, though that's not fair to all four. But we just made that [suggestion]. We took 1000 NOK and divided it by 4.
Alex: Mhm. Why?
Student: Cause then everyone pays the same.
Alex: Yes, well, that's one method. Very good. Did you have any suggestions? [directs the question to another group]
[Different groups share their ideas on Alex's request]

As the excerpt shows, Alex sometimes challenged students' contributions by asking for arguments or mathematical calculations behind the suggestions. The students were also encouraged to build on others' ideas when Alex asked the groups to use the agreed information to find new solutions. However, only Alex questioned their suggestions; unless they were sharing an idea, the students were left as spectators.

4.3.2. Interview Proceeding Lesson 2

When reflecting on the lesson, Alex disclosed that he felt much more prepared this time (than for Lesson 1) in terms of purposeful questions, inspired by George Polya's work [61], which he had become familiar with when preparing for the second PD seminar. He also noticed how anticipating student reasoning to the problem together with his group in the PD seminar effected his ability to inquire into his students' reasoning, as he was familiar with the different ways that they might attack the problem or the challenges they could encounter. In Box 9, the importance of purposeful questioning was highlighted as Alex reflected on what he saw as essential for the inquiry-based teacher role:

Box 9. Excerpt 9.

Researcher: (. . .) So, what about your role in inquiry then? As a teacher. What's your job?
Alex: It is to ask the right questions. I get that now, more than- or, I've known it before too, but I haven't known what those questions were.
Researcher: So, you see a development there?
Alex: I see a development there. That now as I have sort of a script for it- Of course, I'll add something to it, I'll make my own formulations, but having something, a small script on what to ask the students. That was very nice.

It is not clear what Alex meant when he talked about "having a script", but it is natural to believe that he was referring to questions he discussed with his group in the PD seminars and questions from Polya's work (e.g., "have you seen anything like this before?"). His orchestration of the lesson, and his questions in particular, made him feel that his students were actively engaged: "I got more answers than I thought I would. And more students engaged when they got such open questions. Like, not the direct 'what did you find?', but [questions] about how far they had come, what was on the table at the moment, what was their thinking like. So, it was more including for the whole group". However, monitoring and summarizing the student inquiry after the questions had been posed was challenging to Alex, especially "walking around in the classroom and thinking quick enough to keep up with their thinking, (. . .) to not just stand there scratching my head and wonder like 'uhm, is this right?' but really (. . .) listen to what they're actually saying". In other words, moving from evaluating the correctness of their solutions to inquiring into their arguments and explanations was a big leap for Alex. He also disclosed that he was not entirely content with the processes his questions led to. He stated that "it was kind of fixed, like, it didn't proceed from those first questions". However, this was "partly because the students' progression didn't deepen". Therefore, Alex had "stopped the questioning because I didn't get anything more from it. We didn't make it to the next phases". He believed that encouraging the students to explore multiple methods and solutions and inviting them to share their suggestions with the class, on the other hand, had helped the students develop ownership of their ideas.

In the final excerpt (Box 10), Alex reflected on how his conception of inquiry in whole-class episodes had developed:

Box 10. Excerpt 10.

Alex: Well, I guess that it has become clearer to me that it's more of a conversation than me steering it. [The students] should contribute much more. (. . .)

Researcher: Ok. So, it's the students' ideas that are brought up in the plenary part, do I understand you correct then? [Alex confirms]. What do you do then, like, why are you there?

Alex: I'm a moderator, a summarizer, or. . . Yes.

Researcher: So, your job is kind of to decide who-

Alex: Well, yes, you control it a little bit, maybe, but the students should contribute more into that summary than normally maybe. Or, not maybe, more than normally.

5. Discussion

5.1. Alex's Inquiry Trajectory in Three Paths

We identify three main paths in Alex's inquiry trajectory throughout the semester: *the teacher role in inquiry interactions*, *a growing idea of inquiry*, and *orchestrating whole-class situations*. These three paths, summarized in Table 3, inform the rest of this section before we turn our attention to how shared authority is reflected in Alex's trajectory.

Table 3. Three main paths in Alex's inquiry trajectory.

	Pre-PD	**Observation Lesson 1**	**Interview Lesson 1**	**Observation Lesson 2**	**Interview Lesson 2**
The teacher's role	The "woodwork teacher" handing out tools	Removing himself from the inquiry (or directing it)	Acknowledging the importance of teacher questioning and preparation	Polya-inspired inquiry into students' work	Purposeful questioning as an essential element of inquiry in mathematics
Growing idea of inquiry	Inquiry in mathematics equals (individual) discovery	Finding multiple solutions and strategies Touching upon communication and argumentation	"How and why"—supplementing exploration with argumentation	Students exploring, explaining, and sharing ideas	Exploration and argumentation. Students as active communicators
Orchestrating whole-class situations	Whole-class situations not mentioned	Show-and-tell Teacher directing students towards right answers	The importance of good teacher summaries to connect student ideas	Selecting and sequencing students' contributions to display multiple solutions and strategies	Whole-class summaries as dialogues between students and teacher

5.1.1. The Teacher Role in Inquiry Interactions—From the "Woodwork Teacher" to the Curious Questioner

Our results illuminate how Alex's focus moves from directing the students (pre-PD), through distancing himself from their mathematical activity (Lesson 1), to exposing and inquiring into their ideas (Lesson 2). From an empowerment perspective, this is a major shift in how students are positioned [7] as contributors to the mathematical activity, their contributions being not only valuable for their own learning, but also non-trivial from the vantage point of the teacher.

Only when specifically asked to does Alex reflect on his own role as the inquiry-based teacher in the pre-PD interview, and when he does, it is in very traditional ways (Box 1). Alex's analogy of "the woodwork teacher" has few similarities with research characterizations of the teacher role in inquiry settings. Still, we see him feeding this perspective into his lesson when he seemingly believes the students' progress is at risk (Box 3). Data from his first lesson illuminate, however, how Alex mainly distances himself from the students' inquiry processes either by shutting down their efforts to share their inquiry with him (Box 2), not following up on his novice efforts to ask the students what they've found out (Box 2), or by simply leaving them with their questions unanswered. Providing the students with intellectual space to ponder on mathematical problems speaks to their mathematical empowerment [6] and is essential for inquiry [2,21]; however, students' intellectual space must be accompanied by elements of structure where the teacher supports the students in their struggles and promotes and elicits their inquiry [25]. We see that when Alex distances himself from the inquiry, his students become confused about what is expected of them, trying to infer the expectation (e.g., argue or present, as shown in Box 2) and even to communicate their confusion to Alex. His distancing is in great conflict with his view before entering the PD-course, and we speculate that he is facing a conflict between the transmission-based "woodwork teacher" and the guiding teacher he later emphasizes. If so, this is a conflict he resolves via two extremes: directing the students when they do not offer ideas and distancing himself from their ideas when they present some.

Alex relates his detached teacher role to giving students space to choose their own solution paths (Box 4), a signal that his reflections on the teacher role have changed drastically since he started the PD course. He also identifies anticipating student reasoning and preparing purposeful questions, identified as questions about 'what', 'how', and 'why', as essential elements for inquiry that concern him and that he feeds into his next lesson. His open questions in Lesson 2, inspired by the four steps of problem solving [61], are intended to illuminate students' inquiry processes and activate their previous knowledge, aspects highly emphasized in inquiry research (e.g., [2,25]). Alex's interactions with his students in this lesson suggest that he is now valuing and trying to inquire into their reasoning, positioning the students as key sources of the mathematical ideas in focus. However, our findings also reveal how he is only in the early stages of adapting to the inquiring teacher role, leaning on the 'what', 'how', and 'why' and Polya-inspired questions as a script (Box 9). The questioning techniques are not a natural part of his repertoire yet and he stumbles when trying to adapt them (Box 7), not knowing when to use the questions, how to continue his inquiry into the students' reasoning from them, or how to keep up with the students' ideas.

Changing from transmission-based teaching practices to student-active approaches like inquiry is in no way a simple task as it requires teachers to rethink their own role in the classroom [14]. However, looking at Alex's path, he gradually broadens his conceptualization of the teacher role and his realizations—though still wobbly—become more refined and focused on uncovering the students' inquiry. This becomes visible in the way he reflects on his actions, talks about changes he wants to make, and plans his next steps.

5.1.2. A Growing Idea of Inquiry—Inquiry Is More Than Exploration

Many teachers conceptualize inquiry mainly as exploration (e.g., [9,13,42], and this seems to also be the case for Alex in his pre-PD reflections. Inquiry, to Alex, is about discovering strategies, methods, and solutions. This perspective offers students some opportunities for epistemological empowerment [7] compared to traditional teaching (e.g., by strengthening their sense of autonomy), but not as much as in inquiry (e.g., by limiting their ownership of mathematics). In particular, no attention is paid to heavier analytical processes such as negotiation, argumentation, and evaluation, nor to collaboration, despite these aspects being highly valued in research efforts to conceptualize inquiry [2,21,25]. Alex maintains his explorative focus throughout both lessons, with frequent encouragement to find new representations, strategies, and solutions, which is also an expressed aim for him (Box 4). However, separating this from other research (e.g., [13,42]), we witness a growing awareness of the more argumentative and communicative aspects of inquiry. Alex takes steps towards sharing students' explorations in his lessons (in plenary in lesson L and both in plenary and via teacher–student interactions in Lesson 2) and gradually encourages the students to justify their suggestions (through general encouragements in Lesson 1 and more direct requests in Lesson 2). From our point of view, these aspects grow from Alex's experiences in the classroom and gradually receive more attention in the way he talks about inquiry, for example, in how the questions he asked in Lesson 2 made more students communicate their thinking, and his focus in Lesson 1 on how argumentation (the 'what', 'how', and 'why') is important for students' progress. Unlike his shutting down of students' efforts to share their arguments in Lesson 1 (Box 2), in Lesson 2, Alex specifically asks for them (Box 6, Box 8).

Argumentation is closely related to the development of classroom norms [31], which takes time and continued support; hence, Alex's early awareness and clumsy efforts to encourage argumentation can be seen as the first steps towards establishing expectations of argumentation in his inquiry classroom. Nevertheless, adding to a massive body of science research and in line with the few studies in mathematics, evaluative features [2,16,27] never become a part of Alex's path towards learning to teach through inquiry. Mathematical ideas are explored, shared, and sometimes supported with a statement addressing the *why*, but not assessed, as illustrated by the students' passive role in the whole-class episodes unless they are sharing ideas.

While previous research has focused on (science) teachers' conceptions and realization of explorative, communicative, evaluative, and argumentative processes of inquiry, often referring to the NCR [43] elements (e.g., [13,42]), our study also sheds light on Alex's reflections and actions towards the collaborative features of inquiry. Our findings suggest that while Alex talks about collaboration in positive terms, he struggles to feed this into his actions in the lessons, making it an idea that is talked about but not realized [33,34]. The students are placed in groups, but very little attention is paid to evaluations and negotiations—aspects of inquiry that are emphasized in research (e.g., [2,16,27]). We see the same regarding attempts to work towards shared understandings (as highlighted in, e.g., [28]). This is a question of limiting students' opportunities for epistemological empowerment through validation of ideas, and also a question of differential access. Especially in Lesson 1, we see Alex address individual students, engaging in one-to-one interactions, raising the question of equity—which students are genuine participants in the mathematical activity, and which are merely present? ([44] p. 33). We see some efforts in Lesson 2 to engage students by orienting them towards each other in the groups (e.g., by asking them if they agree with their group), as well as in the teacher–student interactions (e.g., asking if anyone on the group has other suggestions), which might suggest that Alex's ideas of the value of collaboration are developing and starting to take form in his realizations. Collaborative approaches such as inquiry strongly contrast with the traditional transmission teaching [14,44] that has guided Alex's practice up until his participation in the PD course, which might bring some context to the situation. We conclude that the individual focus in Alex's conceptualization of inquiry as individual exploration remains throughout the course, but is gradually complemented by an increased awareness of, and small experiments on, broadening this perspective and making small shifts in his actions.

5.1.3. Orchestrating Whole-Class Situations

Promoting and orchestrating episodes of whole-class discussions in inquiry can be challenging for teachers [15,32], and Alex is no exception. From an empowerment perspective, whole-class discussions are considered key practices for equity and the polar opposite of the individualized mathematical activity of traditional teaching ([44] p. 31). Before he starts in the PD course, whole-class discussion has no place in Alex's conceptualization of inquiry, but in his first lesson, he already invites the students to share their solutions with the class. Like so many before him, he realizes this as a sequence of unstructured, randomly selected show-and-tell with no connections made between student contributions. Structuring and orchestrating discussions is essential for inquiry-based teaching [15,30], to bridge students' ideas with each other and with formal mathematics [25,32]. Reflecting on this lesson, he recognizes the importance of preparing a summary to "pick up what they do" and compare the student ideas that have been presented (Box 5). Alex sees this as the teacher's job, a perception which allows him to remain in control of the situation (cf., [16]). Making summaries can be seen as a systematization of mathematical student contributions [32], but should then follow episodes of students comparing, contrasting, and challenging their own and each other's contributions [32]—Alex's reflections do not address student participation in whole-class episodes. His realization of the whole-class discussion in Lesson 2 illustrates that he now grapples with important elements for such events, such as selecting and sequencing [15] and trying to act as a broker [25] by using student contributions to develop a shared understanding of the problem [25,28] and encourage the groups to build on this in their further inquiry processes (as seen in Box 8). At this point, the brokering seems to be restricted to building shared understanding rather than connecting classroom mathematics to formal mathematics [25,32]. In his realizations, the students are left as a passive audience if they are not sharing an idea themselves; they are not invited to react to others' contributions (as we have reflected on in the previous section).

Alex, after the second lesson, shares with the researcher that he now sees whole-class episodes as a dialogue between the students and himself (Box 10). Looking at this path of Alex's trajectory, he gradually moves towards orchestrating whole-class discussions as they are portrayed in the research [2,15,25,32]. Menezes et al. [32] portray the teacher's role in whole-class discussions as "to coordinate the interaction among different students, orchestrating the discussion, promoting the mathematical quality of the presented explanations and argumentations" (p. 307). Though the interactions remain between individual students and Alex, his actions and reflections illustrate a growing awareness of dialogic structures and the importance of explanations and arguments being shared with the whole class. As we leave him, he welcomes students' ideas as important contributions, orchestrating ordered routes of eliciting student inquiry to build shared understanding and illuminate the various ways in which a mathematical problem can be approached.

5.2. Alex's Inquiry Trajectory in Light of Shared Authority

The analysis of two facets of inquiry (the teacher role and the student role) in our data enabled us to identify three paths of learning to teach through inquiry, paths we expect to be specific to Alex rather than shared by all PD participants. Our position is that Alex has the power to shape his paths; the enactment of inquiry practices is key to learning from professional development [1], but from a situated perspective, his enactment will be influenced by his responsibilities as a teacher and by the pedagogical reasoning underpinning his actions [56]. In particular, we assume that, during his two realizations of inquiry, Alex selected what aspects of inquiry he tried to learn about, valuing, as teachers tend to, actionable knowledge ([44] p. 13) and considering a smooth lesson to be a successful lesson (p. 15). In other words, we believe that Alex shaped his learning by concentrating on aspects of inquiry that allowed him to act (e.g., realizing the teacher role is more immediately actionable for him than realizing the student role), and decided on what aspects needed work based on key events that disrupted the smooth running of the lesson. We return with an interpretative stance to the narratives of Alex's paths, looking to understand what goals he might have set, and what implications these have for how Alex might attempt to realize inquiry. We do so through the lens of authority [23]. We argued that inquiry classrooms differ from traditional classrooms in terms of who is positioned as an authority figure [6], making this lens suitable to capture change for Alex, an experienced teacher in traditional approaches, but a novice to inquiry. We found that Alex chose specific aspects for his experimentation, aspects that reflected his concerns:

- Prior to PD, he believed that during inquiry, authority should be distributed between himself and the students (as a group) in separate agentic spaces. He was concerned with whether the students would play their roles, but he did not appear to see this problem as actionable and had no clear goals for his learning.
- In Lesson 1, he developed actionable ways of following up on his concern: reminding students of their roles and keeping quiet. After push-back from students, Alex revised his agenda to foster—through questioning—shared authority during groupwork and distributed authority during whole-class episodes.
- In Lesson 2, he experienced partial success in sharing authority during groupwork, through his more responsive questioning. Perhaps encouraged by this or overwhelmed by the burden on him in the distributed authority of whole-class discussion, he revised his agenda to sharing authority both in groupwork and in whole-class episodes.

We elaborate on these interpretations here, to characterize the three data collection points.

Pre-PD. Pre-PD, Alex envisaged authority in inquiry as distributed between the teacher and the students; they have authority while working on the task but relinquish it to him when they ask for help. The students were seen as a uniform mass, no distinctions were made between them beyond what is implicit in the aspiration that all may "accomplish something", and no reference was made to the interactions between students. This reflects his background of traditional teaching where authority is firmly in the hands of the teacher [6]. Alex reduced the teacher role at this time to manifesting intellectual authority when called upon (by identifying "the right direction" and offering the right help), a typical interaction pattern in classrooms where the teacher is the dominant authority [50]. However, he anticipated difficulties stemming from students not playing their part. He appeared to perceive this concern outside his agentic space, as he did not connect it to any action he might attempt.

Lesson 1. Two moves dominated Alex's actions during the first lesson, and both appeared connected to his concern prior to course start. First, he communicated explicitly and repeatedly that students have authority during groupwork and during the whole-class presentation. Secondly, he stopped himself from acting as an authority: he listened to the students who were stuck but provided no support, or he told students what to do (communicated solutions, produced visual representations, etc.) but failed to manifest intellectual authority (e.g., he listened to their solutions, and then, moved on without validating, challenging, extending, etc.). Alex's pre-PD concern was clear in his vocalization of the students' reimagined roles in inquiry [21], and perhaps also his search for an actionable way [44] to foster the students' roles. While keeping quiet does not appear on the surface as acting, in fact, it is a clear departure from the traditional teacher role [6] and it is not a trivial change [48].

These interaction patterns led to confrontation over what constitutes help in mathematics. The students openly expressed dissatisfaction, and, recognizing that he and the students shared the authority to define what was acceptable, Alex even explained his response ("the point is that I am not giving you any concrete answers"), hinting at constraints from his plan "to hold back the information", or from an external authority (the teaching approach itself, the researcher observing the lesson, or the mathematics teacher educators). Perhaps the confrontation, disrupting the smoothness of the lesson ([44] p. 15), prompted Alex to revise his agenda, seeking better ways of interacting with the students. This labeling of interactions as 'questioning' is consistent with his intention of not giving away the answers ("holding back" to respect student authority), but his agentic space had expanded to sharing authority between students and the teacher during groupwork, a notoriously difficult task [48]. Furthermore, he acknowledged that there are more than two authorities ('the students' and the teacher) in the classroom. Moves such as inviting students to produce their own solutions do contribute to positioning them as intellectual authorities individually, but not collectively [49]. During groupwork, Alex's moves led to the production of many different and potentially unexpected mathematical ideas. This challenged not only him (his questioning during groupwork), but also the students during the whole-class episodes. Following this revised interpretation, in the next iteration, he might attempt to distribute the authority during whole-class episodes between students, by giving them the authority to bring their own mathematical thinking, and the teacher, who has the authority to integrate these into a whole. Although his planned actions would fundamentally change the nature of the whole-class episodes, from show-and-tell to a phase where isolated ideas become integrated [2,15,25,32], it still positions the teacher as the main authority [50] and it is not clear that it would trigger development in Alex's trajectory for sharing authority between himself and the students. The awareness of students as a diverse group, however, is relevant for fostering shared authority among students.

Lesson 2. Following on from his concerns emerging from Lesson 1, Alex attempted to reconcile the competing demands of the teacher and students by sharing intellectual authority. He sought ways to say something helpful, without limiting the students' scope of action. At times, he was successful at relating, in some way, to the specific ideas brought up by the students, and at other times, he simply voiced a generic question without making a reasoned choice. A student's frustration ("it doesn't even make sense") with one of the new questions Alex was trying out ("have you seen anything like this before?") leads us to note that, although not conducive to inquiry, the episode created a space for the student to exert shared intellectual authority and disrupted the smoothness of the interaction, creating a need for improvement [44].

During Lesson 2, Alex refined his idea of students as authorities for knowledge production and validation. He not only invited students to take on specific roles (produce a solution during groupwork and present it during whole-class episodes), but by orienting students towards each other (to listen to alternative methods and build on them) through discursive moves that have shown promise in fostering shared authority among students [49]. The experience disrupted his assumptions on students' potential as authorities ("I got more answers than I thought I would"). Perhaps interpreting the smoothness of the experience ([44] p. 15) as a sign of partial success for his questioning, his agenda for the next inquiry-based lesson is to continue fostering shared authority, not only during groupwork but also during whole-class discussion. This goal is actionable [44] for him, through improved questioning. It is unclear what prompted Alex to reconsider authority relationships during whole-class discussion. There were no observable disruptions in the smoothness of the lessons that can be linked to this. We hypothesize that Alex's revised goal could be, in part, explained by the demands it places on Alex to remain the sole authority in connecting diverse student contributions. An alternative is that the situation is similar to that in the study of Kinser-Traut and Turner [51], where successes in sharing authority when connecting to students' mathematical thinking provided reinforcement of the teacher's conviction that it was the right thing to do.

6. Final Reflections

The study sheds light on how a teacher's conceptions of inquiry in mathematics throughout a PD course can influence how inquiry is brought to life in their classroom, a heavily understudied connection [14]. Moreover, it also highlights the opposite relationship, namely, how realizing inquiry lessons—and reflecting on them afterwards—can influence how inquiry is conceptualized. As researchers have sought to understand the mechanisms at play in mathematics teachers' efforts to conceptualize and realize inquiry, limitations in both conceptions and enactment have been addressed (e.g., [38,39]), as have alignments between the two (e.g., [32]). Separating from previous research, we focused on the trajectory shaped by the interplay between the conceptualization and realization of inquiry, identifying three paths that illuminate Alex's inquiry trajectory: the teacher's role in inquiry interactions, a growing idea of inquiry, and orchestrating whole-class situations. In this interplay, we glimpsed issues of student empowerment such as equitable participation and opportunities to validate and create mathematical knowledge, noticing particular tensions and synergies. Exploring their full complexity is beyond the scope of this paper. Our focus remained on the key issue of authority in mathematics. This lens allowed us to understand Alex's trajectory as driven by his own agenda for learning, with goals materializing through the interplay of his conceptualization and realization of inquiry, specifically by identifying actionable knowledge (the conceptualization of inquiry) and a lack of smoothness incidents (the realization of inquiry) [44]. He moved from distributing authority separately between himself and 'the students' (as one unit) to fostering shared authority between himself and his students (as multiple voices) in both groupwork and whole-class episodes.

In summary, our findings contrast those suggesting that realizing inquiry only has a minimal effect on teachers' conceptions of the approach, or that teachers' development in inquiry knowledge and practices after PD is small (e.g., [13,41]). As we have discussed, Alex lets his experiences, conceptions, and concerns feed into his lessons, and from the lessons, he reflects on and identifies new focus areas and concerns and refines his conceptualizations of inquiry in mathematics. This interplay forms his trajectory towards learning to teach mathematics through inquiry, which, for Alex, includes significant concurrent developments in both how he conceptualizes inquiry and how he realizes it. Ozel and Luft [12] argue that conceptions can be modified if challenged, clarified, or added to, and we suggest that trying to bring inquiry to life in the classroom can do just that—challenge, clarify, and add to the teacher's conception of inquiry. These modifications can again feed into enactment. However, our findings also suggest that this is a complex process; for example, Alex engaged in several unsophisticated and unreasoned efforts to implement what he saw as purposeful questions to inquire into the students' thinking but that, maybe due to his clumsy timing, ended in confusion. Further practice and refinements are needed to continue Alex's trajectory in learning to teach through inquiry. This argues for more research that does not simply count if or how many times an inquiry element is implemented into teaching but also analyzes its quality.

In the case of Alex, the analysis of shared authority provides a different lens for interpreting his trajectory—the goals he sets for himself and how he informs the revision of these goals based on his experimentation. This lens allowed us to explore potential mechanisms for the formulation of teacher goals, such as discerning between what is actionable and what is not, re-evaluating following disruptions in the smoothness of the lesson, reducing the load on the teacher, or persevering with what runs smoothly. However, Alex is a special case; he starts the course with a traditional teaching approach. Shared authority will most likely fail to capture changes in teachers who work collaboratively, and it does not account for the rich details of teaching through inquiry.

We acknowledge that Alex's trajectory was probably affected by the communities he was part of, for example, the group he and the other participants in the larger project formed, and the activities they participated in during the PD course. These influences on Alex's trajectory have not been thoroughly investigated in this work, and we welcome research that binds together PD experiences and the trajectories formed by the individual teacher. Though we cannot draw any conclusions on causality, we can pinpoint some connections between the PD course design and Alex's trajectory. PD should be relevant to teachers' daily practices [1]. Ensor [33] stresses that teachers should learn from activities rather than imitate them (and though he focuses on teacher education, we see this as also relevant for PD), and Hayward et al. [20] advocate for portraying inquiry in broad ways rather than through specific techniques. In the course in this study, the teachers were provided with creative and intellectual space to learn from their inquiry experiences both as students and teachers, and to, individually and collectively, refine their conceptions of the approach and choose what elements they wanted to emphasize in their efforts to bring inquiry into their classrooms. Though this was not our focus, we see that Alex, through the course, is provided several opportunities to, with the support from his peers and instructors, renavigate his inquiry compass. His paths are shaped by choices in the design of the PD course: the choice in mathematics education to build on teachers' pre-existing conceptualizations of inquiry (Figure 2), but structure these into three facets (Figure 3);

the choice to engage teachers in task design in the first seminar (Figure 4), and to work with a chosen task in the second (Figure 5); the choice to include compulsory classroom experimentation with inquiry between seminars, etc. Swan and Swain ([62] p. 175) stress that PD should support teachers' development by involving them in collaborative iterations where they "analyse, test and refine classroom activities that exemplify research based principles". As mathematical classroom activities consider how the students interact with tasks, each other, and the teacher [63], the iterations of experience, design, realization, and reflection in the PD course can be seen as supporting aspects that enable teachers to gradually refine their conceptualizations and realizations of inquiry in mathematics.

In this research, Alex volunteered to participate and was motivated to develop his teaching in an inquiry-based direction. He is in no way representative of any larger group of mathematics teachers; this trajectory is *his* trajectory. However, he provides empirical evidence not only that teachers' conceptions and enactments of inquiry can develop in a PD setting, but also *how* they can develop and on their possible paths and interplay. We encourage more research, to obtain a more nuanced understanding of the different trajectories teachers might take in similar settings, as teachers, despite having similar backgrounds, might have different paths of development in mathematics inquiry knowledge and teaching [41]. There is also evidence that teachers might only slowly, if at all, feed their PD learning and experiences into their teaching practice (e.g., [64]). This not only underpins the need for more and broader research, but also raises another question: Will the trajectory Alex has started on continue after his participation in the PD course? Artigue et al. [8] stress that this is an important question for inquiry-based mathematics education, as isolated activities and situations are interesting in themselves, but not enough for sustainable professional development.

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