Education **2012**, *2*, 22-32; doi:10.3390/educ2010022

OPEN ACCESS
education
ISSN 2076-3344
www.mdpi.com/journal/education

Article

AI and Mathematical Education

Angel Garrido

Departamento de Matemáticas Fundamentales, Facultad de Ciencias de la UNED, Paseo Senda del Rey 9. 28040, Madrid, Spain; E-Mail: agarrido@mat.uned.es

Received: 23 November 2011; in revised form: 19 December 2011 / Accepted: 23 December 2011 / Published: 6 January 2012

Abstract: From ancient times, the history of human beings has developed by a succession of steps and sometimes jumps, until reaching the relative sophistication of the modern brain and culture. Researchers are attempting to create systems that mimic human thinking, understand speech, or beat the best human chess player. Understanding the mechanisms of intelligence, and creating intelligent artifacts are the twin goals of Artificial Intelligence (AI). Great mathematical minds have played a key role in AI in recent years; to name only a few: Janos Neumann (also known as John von Neumann), Konrad Zuse, Norbert Wiener, Claude E. Shannon, Alan M. Turing, Grigore Moisil, Lofti A. Zadeh, Ronald R. Yager, Michio Sugeno, Solomon Marcus, or Lászlo A. Barabási. Introducing the study of AI is not merely useful because of its capability for solving difficult problems, but also because of its mathematical nature. It prepares us to understand the current world, enabling us to act on the challenges of the future.

Keywords: mathematical education; mathematical logic; fuzzy logic; AI; computer science

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1. Introduction

The historical origin of Artificial Intelligence as a scientific field is usually believed to have been established at the Darmouth Conference (1956). In that year, John McCarthy coined the term, defined as "the science and engineering of making intelligent machines". However, this definition does not cover the full breadth of the field. AI is multi-connected with many other fields, such as Neuroscience or Philosophy; and we can trace its origins further back, to arcane origins, perhaps to Plato, Raymond Lully (Raimundo Lulio, in Spanish), G. W. Leibniz, Blaise Pascal, Charles Babbage, Leonardo Torres Quevedo, *etc.*, with their attempts to create thinking machines. A suggestive definition of the field would be to say that it is the area of Computer Science that focuses on creating machines that can engage in behaviors that humans consider intelligent.

The history of the field is long and fruitful; just recall topics as diverse and important as the famous Turing test, the Strong *vs*. Weak AI discussion, the Chinese Room argument, and so on. And modern AI has spanned more than fifty years now.

Frequently, AI requires Logic. However, Classical Logics show too many insufficiencies [7,8], necessitating introduction of more sophisticated tools, such as Fuzzy Logic, Modal Logic, Non-Monotonic Logic. Indeed, Mathematics can be thought of as a mere instance of First-Order Predicate Calculus, and therefore, a part of applied Monotonic Logic. The limitations of classical logic reasoning, and the clear advantages of Fuzzy Logic, are discussed later in this paper.

Among the things that AI needs to implement as representation are Categories, Objects, Properties, Relations, and so on. All of these are connected to Mathematics [1-6]; they are, as well, very good and illustrative examples, in the context of education. For instance, it is possible to show Fuzzy Sets together with the usual sets, also called Crisp, or Classical Sets, as a particular case; or to introduce concepts and strategies from Discrete Mathematics, such as the convenience of using Graph Theory tools in many fields.

The problems in AI can be classified in two general types: Search Problems and Representation Problems. Then, we have Logics, Rules, Frames, Nets, as interconnected models and tools. As it is easy to see, all of them are very mathematical topics.

The origin of ideas about thinking machines [2,7], and the mechanisms through which the human brain works; the possibility of mimicking its behavior if we create some computational structure similar to the neuron, or perhaps the neural system, its synapses or connections between neurons, to produce what is called a Neural Network… Rather than Science Fiction, or the plot of a film, these are real-world subjects of study, and they have been so for many years. And the interest in them has recently increased.

The central purpose of AI would be to create an admissible model for human knowledge. Its subject is, therefore, the "pure form". We try to emulate the reasoning of a human brain. Research directed to this goal can only happen in a succession of approximating steps, but attempts proceed always in this sense.

Initially, AI worked through idealizations of the real world. Its natural fields were, therefore, "formal worlds". Search procedures operated in the Space of States, which contains the set of all states (or nodes, in the case of representation by graphs), that we can obtain when we apply all the available operators. Many early AI programs used the same basic algorithm. To achieve a certain goal (winning a game or proving a theorem), they proceeded step by step towards it (by each time making a move or a deduction) as if searching through a maze and backtracking whenever they reached a dead end. This paradigm was called "reasoning as search".

2. Artificial Intelligence Techniques

The techniques for solving problems in AI are of two types:

- *Declarative:* it permits the description of the known aspects of the problem (it is the *Heuristic Treatment);* and
- Procedural*:* which itemizes the necessary paths to reach the solution to the problem (it is the *Algorithmic Treatment)*.

To pose problems is equivalent to constructing their solutions [3-6]. This requires: an *agent*, the system or program to execute; a *set of actions*, which allows us to reach such objectives; and a *procedure of election*, which allows us to select between different paths to reach the solution.

We may use a series of resources for approaching problems in AI, such as Logic, Rules, Associative Nets, Frames, and Scripts. The choice of methods must be based on the characteristics of the problem and our expectations about the type of solution. In many cases, we take two or more tools at a time, as is the case in Frame Systems with participation of Rules.

2.1. Rule Based Systems

Inference in Rule Based Systems (RBS) consists of establishing the certainty of a certain statement from the available information in a Base of Facts (BF) or Knowledge Base. We have two methods to concatenate rules: going forward and going backwards.

Rules are a clear improvement over Classical Logics [7], where the reasoning was monotonic, and inference is only possible without contradiction of pre-existing facts. The advantage of RBS is that we can delete or substitute facts from the Base of Facts, according to the new inferences obtained. All of them are provisional and modifiable. This makes this type of reasoning Non-Monotonic.

If there is more than one rule applicable at a certain point of the reasoning, which one should be executed first? The set of applicable Rules constitutes, at each step, the Conflict Set (which will be dynamic, obviously). The underlying decision problem is called Resolution of Conflicts, or Control of Reasoning.

2.2. Heuristics

Different strategies [7,8] exist to select the Rule to be executed next: Ordering of Rules, Control of Agendas, the Criterion of Actuality, and the Criterion of Specificity. The Criterion of Specificity leads to execution, first, of the more specific Rules, *i.e.*, those with more facts in their antecedent. So, between:

"R_1: if a, then b", and "R_2: if a and d then c"

we must select R_2, because it is more specific than R_1.

We also have *Mechanisms of Control in RBS.* One is the Mechanism of Refractarity, by which we forbid executing again a Rule, once utilized, if no new information exists which allows or recommends such a case. A second mechanism is Rule Sets, which allow us to activate or neutralize Blocks of Rules. A third is Meta-Rules, or Rules that treat (or substantiate) other Rules; such as Meta-Rules can collaborate in the Control of Reasoning, with the change or assignation of priorities to different Rules, according to the evolution of circumstances.

2.3. Networks

The more recent studies of networks deal with Bayesian Networks. They were introduced in the context of systems for medical diagnosis, where classical statistical techniques, such as the Bayes Rule, had been employed previously.

In the modeling of such problems, the following hypotheses are assumed: *Exclusivity, Exhaustivity* and *Conditional Independence* (CI). According to the *Exclusivity* assumption*,* two different diagnoses cannot be right at the same time. With *Exhaustivity,* we assume to have at our disposition all possible diagnoses. And by the *CI*, we assume that the found discoveries must be mutually independent to a certain diagnosis. The difficulty with these assumptions comes from their inadequacy to the real world. Bayesian Networks overcome these difficulties.

A *Bayesian Network* (BN) is represented as a pair *(G, D)*, where *G* is a directed, acyclic and connected graph, and *D* is a probability distribution, associated with random variables. Such distribution verifies the Property of Directional Separation, according to which the probability of a variable does not depend on its non-descendant nodes.

Inference in a BN consists of establishing on the net, for the known variables, their values, and for the unknown variables, their respective probabilities.

The objective of BNs in Medicine is to find the probability of success that we can give to certain diagnoses, knowing certain symptoms.

3. Problems in Artificial Intelligence

In AI, the problems can be classified according to their level [7]. In a first level, the problems concern decision, learning, perception, planning and reasoning. In a second level, the tasks of classification, representation and search. When we formulate a problem, we start from the statement, or the explanation of it, in natural language. Fundamentally, its treatment is based on the "level of knowledge", introduced by Newell, in 1981, as "abstract level of interpretation of systems, in AI". The "Rationality Principle" is also basic; according to which "if a system has the knowledge according to which one of its actions leads to one of its goals, then such action is carried out".

The problems in AI can be finally classified in two principal types, Search and Representation Problems [8].

3.1. Representation

For representation, we need concepts such as Trees and Graphs; Structure of Facts, for instance: stacks, queues and lists; and the knowledge about the Complexity of Algorithms is also crucial.

3.2. Search

In the search process, we have two options: without information of the domain (Blind Search); and with information of the domain (Heuristic Search). In the first case, we can choose, according to the type of problem, between *Search in extent,* and *Search in depth.*

There are other methods, derived from the previous method, such as *Searching in Progressive Depth* and *Bidirectional Searching,* both with names allusive enough of their nature.

A different method, in this case not derived, is the *General Search in Graphs.* In such a procedure, the possibility of immediate translation to a matrix expression is obvious, through representing graphs as incidence matrices.

Blind Search, or search without information of the domain, appears with the initial attempts to solve game problems by idealizations of the real world, and trying to obtain automatic proofs.

The searching process could be conducted in state spaces. Such a searching procedure has applicability on problems provided with some characteristics, when we can associate a state to each different situation of the domain. There is then a series of initial states: there are some operators, which allow us to take steps between the successive states and there is a final state. In such processes, the correspondence between State and Node of the graph, and between *arc* (edge or link of the graph) and *operator* is clear.

Searching in extent. We advance in the graph through levels. So, we obtain the lesser cost solution, if it exists. In Search in Depth, the search is expanded, one link at a time, from the root node. If we reach a blind alley in our walk through the graph, we backtrack until the nearest node and, from this, we take one ramification in the graph. It is usual to establish an exploration limit, or depth limit, fixing the maximal length of the path, from the root.

Heuristic Search, *i.e.*, when we are searching with knowledge of the domain. At first sight, it may be naively thought that the computer can explore every path. But this is too optimist. Such exploration would be very difficult, because of the phenomenon of "combinatorial explosion" of branching. Spatial and temporal complexity can advise us against its realization. For this, we need to select the more promising trajectories. In this way, we cannot obtain the best solution (optima), but an efficient approach to it.

Now, we introduce one new mathematical tool, the so-called *heuristic evaluation function, f.* By such a function, we assign a value, $f(n)$, to each node n. So, $f(n)$ gives us the estimation of the real distance (unknown), from the current node, n, to the final node, m.

There are also strategies designed for the treatment of *Searching problems with two adversaries*. In this case, the general purpose is to select the necessary steps to win the game. Chess is perhaps the most commonly considered game in this context; in fact, it was at the origin of these methods. We must now assume alternative moves. In each move, the ideal would be when the player knows his possibilities and realizes the more unfavorable move for her adversary. But it is impossible to control this completely, in general, because of the "combinatory explosion". So, we need to develop a depth-search-tree, with limited depth, applying the Principle of Rationality. To estimate the goal, we introduce a more sophisticated function, the aforementioned heuristic evaluation function, which would measure, for each node, the possibilities of the players.

4. Artificial Intelligence in the Classroom

All of these problems, their methods of solution or approximation tools, may be implemented in the classroom [3-6]. For instance mathematical games such as Chess or Go can be introduced to improve the logical capacities of the students, in any scholar level. In elemental education, it may be an interesting stimulus to promote hidden potentialities; and in higher levels, it may contribute to strengthen the logical and deductive skills of the beginner researchers.

Tools that are in use currently in Mathematics and AI, as Graph Theory, may be used to introduce classical and very exciting questions, such as the Halting Problem, or the Traveling Salesman Problem (TSP), or an open question for the 21st century, P *vs* NP.

The study of Mathematics can be supported by the introduction of games such as those mentioned above: Chess, Checkers, Stratego or Sudoku. Not only that: our students can be introduced to more subtle analyses; an example would be the Prisoner's Dilemma. Students would have a large quantity of information readily available on the Web, so researching for the rules, tricks and hints to master games like Chess or Go can be an interesting activity in itself. Further motivation for the students may come in the reading of papers, or Web articles, on the history of the games; this would be illustrative and motivating for the students.

All of these techniques have been implemented [5,6] in the classroom with secondary level students, and it has been shown that their interest has increased simultaneously in Mathematics and Information Technology. With higher education students, at the undergraduate university level of studies of Mathematics, Physics, and Computer Science, a very positive reaction has been obtained, with an improvement in both their interest for the subjects and their results.

I consider that this approach places Computer Science in the role Physics and its problems have played in the past, as a support of mathematical reasoning [8,9]; although Physics should not disappear from the picture, being a necessary aid.

I propose showing such Methods through the parallel study of Mathematics and Computer Science foundations. Other Computer Science subfields could be carriers of this method too, but perhaps AI is the current better choice, given its characteristics, which practically coincide with many mathematical techniques and objectives.

The benefits of such an innovative educative method must consist in a more progressive adaptation of Mathematical Education to modern times [4,7], with the final purpose of producing adaptive and creative minds, capable of solving new problems and challenges.

5. AI and Educational Challenges

The interest of the study of Artificial Intelligence is based on our increasing need for creating new manners of thinking and interpreting the mechanisms of the brain and of human reasoning. Inside this new and powerful science, stimulated by the constant advances of the Sciences of Computation and of the rapidity, efficiency and efficiency of the computational processes, the so-called Non-classic Logics are adequately placed. Among them, the one that finds more applications is Fuzzy Logic, and the reason for this is that it simulates the way in which we reason as human beings. Instead of the black-and-white simplified version of reality of Boolean binary classical logics, we are used to grades of truth, shades of grey: instead of the absolute truth or falsehood, yes or no, our way of thinking admits for gradation.

It is, therefore, crucial for the formation in Mathematics of the new generations of students that these innovative concepts become part of the curricula, as part of the subjects related to Mathematics, such as Physics, Chemistry, or Engineering.

We say that a language is formal when its syntax is precisely given. Mathematical Logic is the study of the formal languages. Usually, it is called Classical Logic, being dychotomic, or bi-valuated, only either True or False.

The modern study of Fuzzy Logic [7] and partial contradictions had its origins early in the 20th century, when the great English philosopher and mathematician Bertrand Russell found the Ancient Greek paradox at the core of modern set theory and logic.

According to the old riddle, a Cretan asserts that all Cretans lie. So, is he lying? If he lies, then he tells the truth and does not lie. If he does not lie, then he tells the truth and so lies. Both cases lead to a contradiction because the statement is both True and False.

The set of all sets is a set, and so it is a member of itself. Yet the set of all apples is not a member of itself, because its members are apples and not sets. Perceiving the underlying contradiction, Russell then asked: "Is the set of all sets that are not members of themselves a member of itself?" An essential question, because: If it is, it isn't; if it isn't, it is.

Faced with such a conundrum, classical logic surrenders. But fuzzy logic says that the answer is half true and half false, a 50–50 divide. Fifty percent of the Cretan's statements are true, and 50 percent are false. The Cretan lies 50 percent of the time and does not lie the other half. When membership is less than total, a bivalent system might simplify the problem by rounding it down to zero or up to 100 percent. Yet 50 percent does not round up or down.

In the 1920s, independently of Russell, the Polish mathematical logician Lukasiewicz worked out the principles of Multivalued Logic (MVL, by acronym), in which statements can take fractional truth values between the1's and 0's of Boolean logic (which is a binary logic).

The physicist and philosopher Max Black, in 1937, applied MVL to lists, or sets of objects, and in so doing drew the first fuzzy set curves. Following Russell's lead, Black called the sets "vague" (equivalent term to "fuzzy").

Three decades later, Lofti A. Zadeh published his very famous paper called "Fuzzy Sets". Zadeh applied Lukasiewicz's logic to every object in a set and worked out a complete algebra for fuzzy sets. Even so, fuzzy sets were not put to use until the middle of the 1970s, when Mamdani designed a fuzzy controller for a steam engine. Since then, the term "fuzzy logic" has come to mean any mathematical or computer system that reasons with fuzzy sets.

The Fuzzy Logic is a successful generalization of the Mathematical or Classical Logic [7,8]. It deals with the problem of ambiguity in Logic. Because Classical Logics show many insufficiencies for the problems of AI, a more flexible tool is needed, allowing for a gradation of certainty, indicating different degrees of membership to a set, or fulfillment of a property or relationship, and so on.

With the introduction of concepts and methods of Fuzzy Logic, the ideas of sets, relations and so on, are modified in the sense of covering adequately the indetermination or imprecision of the real world. We define the "world" as a complete and coherent description of how things are or how they could have been. In the problems related with this "real world", which is only one of the "possible worlds", the Monotonic Logic seldom works. Such type of Logic is classical in formal worlds, such as Mathematics. However, it is necessary to provide our investigations with a mathematical construct that can express all the "grey tones", not the classical representation of real world as either black or white, either all or nothing, but as in the common and natural reasoning, through progressive gradation.

Let A and B be two fuzzy sets, not necessarily on the same universe of discourse. The implication between them is the relation R: A→B such that $A \rightarrow B \equiv A * B$, where * is an outer matrix product using the logical operator AND. To each Fuzzy Predicate, we can associate a Fuzzy Set, defined by such property, that is, composed of the elements of the universe of discourse such that totally or partially verify such conditions. So, we can prove that the class of fuzzy sets, with the operations union (∪), intersection (∩), and *c* (with *c* the complement set operation), does not constitute a Boolean Algebra, because neither the Contradiction Law nor the Middle Excluded Principle hold. Turning to the first mentioned definitions, both proofs can be expressed easily, by counter-examples, in an algebraic or geometric way.

Fuzzy Rules are linguistic IF-THEN constructions that have the general form "IF A THEN B", where A, B are propositions containing linguistic variables. A is called the premise, or antecedent, and B is the consequent (or action) of the fuzzy rule. In effect, the use of linguistic variables and fuzzy IF-THEN rules exploits the tolerance for imprecision and uncertainty [7].

In this respect, Fuzzy Logic imitates the ability of the human mind to summarize data and focus on decision-relevant information. For these reasons, it is a very interesting way to advance, as an interesting access point to many new fields of Mathematics.

6. New Educational Paradigms

As for the games and mathematical riddles, there are many very good texts that contain them. We all have in mind some of the main ones, in particular on this topic and in general on mathematical divulgation, are the works of George Pólya, [3-6]. Also, some other very interesting authors must be cited, for instance Richard Courant [1], Raymond Smullyan, Martin Gardner, Miguel de Guzmán, Marco Livio, Ian Stewart, Piergiorgio Oddifredi, Marcus Du Sautoy, *etc*.

Another very important aspect, which is in the habit of instigating many disagreeable "surprises", resides in the skill to adapt the material to the audience. This is an art and it is often learned only with time [9], a part of the more or less "innate" qualities in every person. However, it is possible to promote and learn. What surely must not be done is to try to teach pupils to adapt to the subject in the form that it is studied at higher education faculties, but rather the other way around. The level and manner of introducing techniques have to be adapted to the pupil. Memory is certainly not the only quality a student must cultivate, and therefore, they must not be simply "handed a book" to be learned.

The rejection and the difficulties that this incorrect route has caused to many teachers tends to the infinite. The audience must be in the center of our planning, their knowledge, skills and possibilities have to be born in mind, instead of trying to organize an imitation of the way the old subject was presented to us at university. It is a good exercise to take a written topic to a higher, or more difficult level, or to choose an article from a scientific magazine and to try to turn it into something that the pupils could understand. Even more, it could motivate them, predispose them in favor, and not against, the study of Mathematics in the future.

7. New Tools on Mathematical Education

We may denominate Information and Communication Technologies (ICTs) the set of technologies which allow us to acquire, produce, store, treat, communicate, and register information, by voice, images and data, all of them translated to optical, acoustical, or electromagnetic signals.

New technologies permit the development of new didactic materials through different supports: Internet, digital discs, *etc*., encapsulated into computing support, or traveling through the Web.

Amongst the immense range of Internet applications, the World Wide Web will be the space with more possibilities. Web pages allow us to publish any digital element (text, photo, video, *etc*.), and personal creations of an educational type without commercial editors. For these reasons we have seen an exponential increase in its use recently.

Furthermore, the didactic material available on the Web can be accessed globally, independently of place and time. Not only the quantity of available material on the Web is increasing rapidly, also its access is more free, thanks to the so-called Free Software, really free for any user.

The use of ICTs as part of the learning methods is advantageous with respect to the very old resources of more classic educational schools. Such advantages are: *instructional flexibility*, so facilitating a different and more adequate pace of learning; *complementarity of codes*, allowing the student to obtain best information from different sensorial channels, without or almost without noise; *improved motivation*, because many investigations show that the use of diverse ICTs instills a better motivation for the students, which implies a greater implication on the process of learning; collaborative and Cooperative Activities, so producing an improving verbal interaction and participation in the school tasks, and potentiating social relations.

Such set of ideas is very tempting indeed, according to the aforementioned advantages. Nevertheless the use of new ICTs inside the classrooms has potential disadvantages, which must be considered to plan for its reasonable use:

- *Pseudo-information.* It is totally necessary to provide our students with tools that permit them to select true relevant information from spam and bad, biased, information and also to capacitate them to distinguish the real world from its manipulated image.
- *The saturation of information* has to be mitigated with appropriate tools to analyze, understand and integrate knowledge.
- *Technological dependencies.* The use of ICTs is indeed interesting, but we must recognize their limitations; their use has to be supported, but not so much as transforming them into a new and totemic idol.

Through problem solving we attempt to evaluate the degree of acquisition of certain mathematical competencies with ICTs and without ICTs. Such educational competencies are: resolution of Problems; use of Resources and Tools; to communicate; to interpret; to represent; to think, and Reasoning and Argumentation.

In the Mathematical area, we may distinguish between two categories: Attitudes towards Mathematics, and Mathematical Attitudes.

With "Attitudes towards Mathematics" we refer to the valuation and high regard to such scientific discipline and language, and the increasing interest to learn this useful tool, where it dominates the affective component over the cognitive component; so stating in terms of interest, satisfaction, curiosity, *etc*. The "Mathematical Attitudes" possess in fact a very cognitive character, and refer to the manner to use general capacities, such as flexibility of thought, mental opening, critical spirit, objectivity, tenacity, accuracy, creativity, decision, and so on, which are very essential to any work related to Mathematics, either of a theoretical or applied nature.

We must avoid the very extended, but erroneous because of its artificiality, division of Mathematics between Pure Mathematics (also called Fundamentals), and Applied Mathematics. Such territories are, in any case, evolving with time, and that is a totally subjective boundary. Because we can only observe two types of Mathematics: the good and the bad Mathematics, independent of when they are to be applied—just now, or in the more distant future—and also by Fuzzy Theory interpretation, with all the intermediate degrees of "goodness".

8. Conclusions

We have briefly covered some of the aspects of how Fuzzy Logic mimics the ability of the human mind to summarize data and focus on decision-relevant information. For these reasons, the introduction to the study of Fuzzy Logic is an interesting access point for a number of new fields for Mathematical Education.

Studying Artificial Intelligence is interesting not only because of its potential to tackle many open problems, both inside the field and in application to others scientific areas, and even the study of the humanities, but also because it is a new and strongly creative branch of Mathematics, and it prepares us to understand the current world, enabling us to act on the challenges of the future.

References

1. Courant, R. *What Is Mathematics? An Elementary Approach to Ideas and Methods*; Oxford University Press: New York, NY, USA, 1996.
2. Mc Corduck, P. *Machines Who Think*, 2nd ed.; A. K. Peters Ltd.: Natick, MA, USA, 2004.
3. Polya, G. *How to Solve It: A New Aspect of Mathematical Method*; Princeton University Press: Princeton, NJ, USA, 2009.
4. Polya, G. *Mathematics and Plausible Reasoning*; Princeton University Press: Princeton, NJ, USA, 1990.
5. Polya, G. *Mathematical Discovery: On Understanding, Learning and Teaching Problem Solving*; Wiley: Hoboken, NJ, USA, 1981.
6. Polya, G.; Kirkpatrick, H. *The Stanford Mathematics Problem Book: With Hints and Solutions*; Dover Publications: Mineola, NY, USA, 2009.
7. Russell, S.; Norvig, P. *Artificial Intelligence: A Modern Approach*, 2nd ed.; Prentice Hall: Upper Saddle River, NJ, USA, 2003.
8. Searle, J. Minds, brains, and programs. *Behav. Brain Sci.* **1980**, *3*, 417–457.
9. Taylor, H.; Taylor, L. *George Pólya: Master of Discovery*; Book Surge Publishing: Charleston, SC, USA, 2006.

10. Turing, A. Computing machinery and intelligence. *Mind* **1950**, *LIX (236)*, 433–460.