

Article

On the Solution of Generalized Banach Space Valued Equations

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Abstract: We develop a class of Steffensen-like schemes for approximating solution of Banach space valued equations. The sequences generated by these schemes are, converging to the solution under certain hypotheses that are weaker than in earlier studies. Hence, extending the region of applicability of these schemes without additional hypotheses. Benefits include: more choices for initial points; the computation of fewer iterates to reach a certain accuracy in the error distances, and a more precise knowledge of the solution. Technique is applicable on other schemes our due to its generality.

Keywords: Steffensen-like schemes; convergence order; Banach space; fixed point

MSC: 65G99, 65H10



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1. Introduction

A plethora of problems from optimal control, variational inequations, mathematical programming and other disciplines can be converted to finding a solution x_* of the generalized equation

$$0 \in F(x) + Q(x), \quad (1)$$

with $F : \mathbb{B}_1 \rightarrow \mathbb{B}_2$ being a continuous operators, Q a set valued operator mapping $\mathbb{B}_1$ into subsets of $\mathbb{B}_2$ that have closed graph, and $\mathbb{B}_1, \mathbb{B}_2$ are standing for Banach spaces.

The Steffensen-like scheme (SLS)

$$0 \in F(x_n) + F[q_1(x_n), q_2(x_n)](x_{n+1} - x_n) + Q(x_{n+1}), \quad (2)$$

has been developed (see [1]) to produce a sequence approximating x_* , and $F[\cdot, \cdot]$ is a divided difference of order one.

Notice that $G := 0$, then SLS reduces to the classical Steffensen scheme (SS)

$$x_{n+1} = x_n - F[q_1(x_n), q_2(x_n)]^{-1}F(x_n), \quad (3)$$

and if F is differentiable (Fréchet) with $q_j(x) = x$, then SLS becomes Newton's scheme (NS) for solving Equation (1), which is given by

$$0 \in F(x_n) + F'(x_n)(x_{n+1} - x_n) + Q(x_n). \quad (4)$$

Special choices of mapping q_j have been studied, such as:

$$\begin{aligned} q_1(x) &= x \\ q_j(x) &= x_n - F'(x_n)^{-1}F(x_n) \quad (\text{Newton's scheme}); \quad q_1(x_n) = x_n, \\ q_2(x_{n-1}) &= x_n - F[x_{n-1}, x_{n-2}]^{-1}F(x_{n-1}) \quad (\text{secant scheme}); \\ q_1(x_{n-1}) &= x_n + a_n(x_{n-1} - x_n), \text{ and } q_2(x_n) = x_n; \\ q_1(x) &= x \text{ and } q_2(x) = x - F(x). \end{aligned}$$

Dontchev [2] showed convergence under the continuity due to Aubin $(F + Q)^{-1}$ together with continuity by Lipschitz for NS. Argyros [3] showed convergence of NS when F is $i \geq 2$ differentiable.

Convergence of NS under Hölder like conditions can be found in [4]. For

$$q_1(x_n) = \alpha x_n + (1 - \alpha)x_{n-1} \text{ and } q_2(x_n) = x_n, \quad \alpha \in [0, 1),$$

scheme (2) was studied in [5]. Later a local convergence result was prescribed in [1] who generalized earlier results [6–8].

Based on the aforementioned, there is a need to find unifying convergence criteria under weaker conditions. Moreover, we are concerned with optimization considerations. These ideas and concerns constitute our motivation for this paper. In particular, we extend the region of applicability of SLS by finding more initial point x_0 ; providing fewer iterates for achieving a certain accuracy and we give a better information related to the where about x_* . These are obtained without additional hypothesis by developing the center-Lipchitz idea. This technique may be utilized to extend the applicability of other schemes with the same benefits. Another novelty of our idea is that all the benefits are obtained without additional to the previous works hypotheses.

In order to create a self-contained study, we present a background and auxiliary definitions and results in Section 2. The convergence analysis local is given for various cases in Sections 3 and 4 with examples. A concluding Section 7 completes this study.

2. Background and Auxiliary Results

We present material necessary for only understanding our results. A more detailed inside in the topics developed can be found in [2,9–11]. A distance from $x \in \mathbb{B}_1$ to a region $\mathbb{S}$ in $(\mathbb{B}_3, \|\cdot\|)$ (Banach space) is given by

$$DIST(x, \mathbb{S}) := \inf \|x - y\|, \quad (y \in \mathbb{S}).$$

If $\mathbb{A} \subseteq \mathbb{B}_3$, then the excess E is defined as

$$E(\mathbb{A}, \mathbb{S}) := \sup DIST(x, \mathbb{S}), \quad (x \in \mathbb{A}).$$

If $T : \mathbb{B}_1 \rightarrow \mathbb{B}_2$ stands for a set-valued operator, then

$$GPHT = \{(u, v) \in \mathbb{B}_1 \times \mathbb{B}_2, v \in T(u)\}$$

to be the graph of T and $T^{-1}(v) = \{u \in \mathbb{B}_1, v \in T(u)\}$ is the inverse operator of T . Let $L(\mathbb{B}_1, \mathbb{B}_2)$ stand for the space of continuous linear operators mapping $\mathbb{B}_1$ into $\mathbb{B}_2$. Then, notation $U[x, r]$ is used to denote a ball (closed) of center $x \in \mathbb{B}_1$ and radius $r > 0$.

Definition 1 ([2]). We say that T is Lipschitz (pseudo) about $(x_0, y_0) \in GPHT$ with modulus μ is there exist parameters $\alpha \geq 0$ and $\beta \geq 0$ such that for each $u, v \in U[x_0, \beta]$

$$E(T(u) \cap U[y_0, \alpha], T(v)) \leq \mu \|u - v\|. \quad (5)$$

Remark 1.

- (a) Let $v_n \subset \mathbb{B}_1$. Then, $\lim_{n \rightarrow \infty} v_n = x_*$ (linearly), if for $e_n = \|x_n - x_*\|$.
 (b) Divided difference $F[\cdot, \cdot] : \mathbb{B}_1 \times \mathbb{B}_2 \rightarrow L(\mathbb{B}_1, \mathbb{B}_2)$ satisfies

$$F[u, v] = F(u) - F(v) \text{ for each } u, v \in \mathbb{B}_1 \text{ with } u \neq v. \quad (6)$$

If F is differentiable, we get $F[x, x] = F'(x)$ [8].

Definition 2. This definition does not uniquely specify the divided difference. But a popular choice is [4,11,12]

$$F[u, v] = \int_0^1 F'(u + \tau(v - u)) d\tau. \quad (7)$$

We need the auxiliary result on fixed points.

Proposition 1 ([1,2]). Consider f to be a set-valued operator. Suppose that the following hypotheses hold for $\sigma_0 \in \mathbb{B}_1$, $\rho \geq 0$ and $p \in [0, 1)$

- (i) $\text{DIST}(\sigma_0, f(\sigma_0)) \leq (1 - p)\rho$.
 (ii) $E(f(u) \cap U[\sigma_0, \rho], f(v)) \leq p\|u - v\|$ for each $u, v \in U[\sigma_0, \rho]$.

Then, f has a fixed point x in $U[\sigma_0, \rho]$.

Moreover, if f is a single valued function, then x is a unique fixed point for f in $U[\sigma_0, \rho]$.

The following hypotheses H shall be used. Suppose:

(H₁) Operators $q_j : W \rightarrow W$ are q_j center-Lipshitz and

$$q_j(x_*) = x_*, \text{ where } q_j \in [0, 1).$$

(H₂) $\|F[x_*, x] - F[q_1(x), q_2(x)]\| \leq g(\|x_* - q_1(x)\|, \|x_* - q_2(x)\|)$ for all $x \in W$, and some functions $g : [0, \infty) \times [0, \infty) \rightarrow [0, \infty)$, which is continuous and non-decreasing.

(H₃) The operator Q^{-1} is Lipschitz (pseudo) about $(-F(x_*), x_*)$.

(H₄) $\|F[u, v]\| \leq \delta$ and

$$\mu \left[\delta + g(a_1\alpha, (1 + a_2)\alpha) \right] < 1,$$

for all $u, v \in W$.

Remark 2. In [1], the following hypotheses (H)' were used.

(H₁)' Operators $q_j : W \rightarrow W$ are $\bar{q}_j$ center-Lipshitz and

$$q_j(x_*) = x_*, \text{ where } \bar{q}_j \in [0, 1).$$

(H₂)' $\|F[x, y] - F[u, v]\| \leq \bar{g}(\|x - u\|, \|x - v\|)$ for all $x \in W$, and some function $\bar{g} : [0, \infty) \times [0, \infty) \rightarrow [0, \infty)$ which is continuous and non-decreasing.

(H₃)' = (H₃).

(H₄)' $\|F[u, v]\| \leq \delta$ and

$$\mu \left[\delta + \bar{g}(\bar{a}_1\alpha, (1 + \bar{a}_2)\alpha) \right] < 1,$$

for all $u, v \in W$.

Then, clearly

$$a_j \leq \bar{a}_j, \quad (8)$$

$$g \leq \bar{g}, \quad (9)$$

$$(H_2)' \Rightarrow (H_2), \quad (10)$$

and

$$(H_4)' \Rightarrow (H_4). \quad (11)$$

In view of items (8)–(11), the results that follow improve the corresponding ones in [1]. Examples where items (8)–(11) are strict can be found in [13].

Next, we show that weaker and actually needed hypotheses (H) can replace (H)'.

3. Convergence for SLS

We shall show the main local convergence result for SLS under hypotheses (H).

Theorem 1. Under hypotheses (H), choose $c \in (c_0, 1)$, where

$$c_0 = \frac{\mu f(a_1\alpha, (1+a_2)\alpha)}{1 - \mu\delta}.$$

Then, there exists $\epsilon > 0$, such that for any $x_0 \in U[x_*, \epsilon] - \{x_*\}$, sequence $\{x_n\}$ generated by SLS converges to x_* so that for $e_n = \|x_n - x_*\|$:

$$e_{n+1} \leq ce_n. \quad (12)$$

Some notations and discussion follow to help with the proof of Theorem 1.

Define set-valued operators $G : \mathbb{B}_1 \rightarrow \mathbb{B}_2$ and $\psi_n : \mathbb{B}_1 \rightarrow \mathbb{B}_2$ as

$$G(\cdot) := F(x_*) + Q(\cdot), \quad \psi_n(\cdot) := G^{-1}(P_n(\cdot)),$$

where $P_n : \mathbb{B}_1 \rightarrow \mathbb{B}_2$ is given by

$$P_n(x) := F(x_*) - F(x_n) - F[q_1(x_n), q_2(x_n)](x - x_n).$$

Notice that iterate x_1 is such that $\psi_0(x_1) = x_1$ if and only if

$$0 \in F(x_0) + F[q_1(x_n), q_2(x_n)](x_1 - x_0) + Q(x_1).$$

Lemma 1. Suppose hypotheses (H) hold. Then, there exists $\epsilon > 0$ such that ψ_0 has a fixed point in $U[x_*, \epsilon]$ so that

$$e_1 \leq ce_0. \quad (13)$$

Proof. Choose $\epsilon < c_2$, where

$$c_2 = \min\{\alpha, c_1\},$$

and

$$c_1 = \frac{\beta}{\delta + 2g(a_1\alpha, (1+a_2)\alpha)}.$$

Using (H₃), we get

$$E(G^{-1}(u) \cap U[x_*, \alpha], G^{-1}(v)) \leq \mu\|u - v\|, \quad (14)$$

for each $u, v \in U[0, \beta]$.

By the definition of $P_n, (H_1), (H_2)$ and (H_4) , we can write in turn for $x \in U[x_*, \epsilon]$:

$$\begin{aligned} \|P_0(x)\| &= \|F(x_*) - F(x_0) - F[q_1(x_0), q_2(x_0)](x_1 - x_0)\| \\ &= \|F[x_*, x_0]((x_* - x) + (x - x_0)) - F[q_1(x_0), q_2(x_0)](x_1 - x_0)\| \\ &\leq \|F[x_*, x_0]\| \|x_* - x\| + \|F[x_*, x_0] - F[q_1(x_0), q_2(x_0)]\| \|x - x_0\| \\ &\leq \delta \|x_* - x\| + g(\|x_* - q_1(x_0)\|, \|x_* - q_2(x_0)\|) \|x - x_0\|. \end{aligned} \quad (15)$$

By (7), (14), (15) and for a proposition $\sigma_0 = x_*$, $p = \mu\delta$, $\rho = \rho_0 = ce_0$, we have in turn that

$$\begin{aligned} DIST(x_*, \psi_0(x_*)) &\leq E(G^{-1}(0) \cap U[x_*, \epsilon], \psi_0(x_1)) \\ &\leq \mu g(a_1\alpha, (1 + a_2)\alpha) e_0 \leq c(1 - p)e_0. \end{aligned} \quad (16)$$

(By the choice of c and (16)).

Hence, hypothesis (i) of the Proposition 1 is true. Next, by the choice of ϵ and (15) we conclude that for each $x \in U[x_*, \epsilon]$, we obtain

$$P_0(x) \in U[0, \beta].$$

Hence for each $u, v \in U[x_*, r]$, we get in turn that

$$\begin{aligned} E(\psi_0(u) \cap U[x_*, \rho_0], \psi_0(v)) &\leq E(\psi_0(u) \cap U[x_*, \epsilon], \psi_0(v)) \\ &\leq \mu \|P_0(u) - P_0(v)\| \leq \mu \|F[q_1(x_0), q_2(x_0)]\| \|u - v\| \\ &\leq p \|v - u\|, \end{aligned}$$

showing (ii) of Proposition 1. Hence, we conclude that $\psi_0(x_1) = x_1 \in U[x_*, \rho_0]$, so that (13) holds. $\square$

Next, we present the proof of Theorem 1.

Proof of Theorem 1. Set $\rho := \rho_n = ce_n$. Then, mathematical induction and the proof and application of Lemma 1 imply $\psi_n(x_{n+1}) = x_{n+1} \in U[x_*, \rho_n]$, so (12) holds. $\square$

Remark 3. Our results are connected to the one in [1,4,5,7,8,10–12] so they can extend them as follows. Consider hypothesis (see (H_2))

$$\|F[x_*, x]\| - F[q_1(x), q_2(x)] \leq \mu(\|x_* - q_1(x)\|^{\delta_0} + \|x_* - q_2(x)\|^{\delta_0}), \quad (17)$$

for $\mu > 0, \delta_0 \in [0, 1]$ and each $x \in W$. Then, we can choose

$$g(s_1, s_2) = \mu(s_1^{\delta_0} + s_2^{\delta_0}).$$

In the case of $(H_2)'$, we first introduce hypothesis

$$\|F[x, y]\| - F[u, v] \leq \bar{\mu}(\|x - u\|^{\delta_0} + \|x - v\|^{\delta_0}). \quad (18)$$

Then, we can choose

$$\bar{g}(s_1, s_2) = \bar{\mu}(s_1^{\delta_0} + s_2^{\delta_0}).$$

Clearly, in this case we have

$$\mu \leq \bar{\mu}, \quad (19)$$

so the same benefits are obtained in this special case too in particular with the function g as previously defined, we have:

Proposition 2. Suppose hypotheses (H) hold. Then, for each

$$c > \frac{\mu\kappa(a_1^{\delta_0} + (1 + a_2)^{\delta_0})}{1 - \mu 6\delta},$$

there exists $\epsilon > 0$ so that sequence generated by SLS converges to x_* and

$$e_{n+1} = ce_n^{\delta_0+1}.$$

Proof. Simply follow the steps of the proof of Theorem 1, but choose the convergence radius $\epsilon < \epsilon_1$ so that where

$$\epsilon_1 = \min\left\{\alpha, \frac{1}{\sqrt[\delta_0]{c}}, \frac{\beta}{2\delta}, c_1\right\},$$

and

$$c_1 = \left(\frac{\beta}{4\kappa(a_1^{\delta_0} + (1 + a_2)^{\delta_0})}\right)^{\frac{1}{\delta_0+1}}.$$

□

Notice that if $\delta_0 = 1$ quadratic convergence is attained.

4. Inexact Scheme

A combination of a derivative and a divided difference has been used in a scheme to solve equations containing a non-differentiable term [4]. We extend the results of Section 3 in an analogous way to solve generalized equation

$$0 \in F(x) + H(x) + Q(x), \quad (20)$$

where H is differentiable for $x = x_*$ but not necessarily so in W . The corresponding to (20) inexact defined by

$$0 \in F(x_n) + H(x_n) + \left(F'(x_n) + F[q_1(x_n), q_2(x_n)]\right)(x_{n+1} - x_n) + Q(x_{n+1}). \quad (21)$$

As in Section 3, we use hypotheses (H)¹. Suppose:

$$(H_1)^1 = (H_1).$$

$$(H_2)^1 = (H_2), \text{ (for } F \text{ being } H).$$

$$(H_3)^1 \left(F'(x_*)(-x_*) + H(\cdot) + Q(\cdot)\right)^{-1} \text{ is Lipschitz (pseudo) about } \left(-F(x_*), x_*\right) \text{ (see definition 1).}$$

$$(H_4)^1 \|F'(x) - F'(x_*)\| \leq d_0(\|x - x_*\|) \text{ for each } x \in W \text{ and some function } d_0 : [0, \infty) \rightarrow [0, \infty) \text{ and which is continuous and non-decreasing.}$$

Remark 4. (a) The hypotheses (H)² were used in [1]

$$(H_1)^2 = (H_1)'$$

$$(H_2)^2 = (H_2)' \text{ (for } F \text{ being } H).$$

$$(H_3)^2 = (H_3)'$$

$$(H_4)^2 \|F'(y) - F'(x)\| \leq d(\|y - x\|) \text{ for each } x, y \in W \text{ and some function } d : [0, \infty) \rightarrow [0, \infty), \text{ which is continuous and non-decreasing.}$$

Notice that

$$d_0 \leq d, \quad (22)$$

so the hypotheses (H)¹ can be used of (H)² in the proofs in [1] to extend those results too. In particular, we have:

Theorem 2. Suppose: hypotheses $(H)^1$ hold and

$$R = \mu\left(d_0(\alpha) + g(a_1\alpha, (1 + a_2)\alpha)\right) < 1.$$

Then, for each $c^2 \in (R, 1)$, there exists $c^2 > 0$ so that sequence $\{x_n\}$ generated by ISLS is in $U[x_*, c^2]$, $\lim_{n \rightarrow \infty} x_n = x_*$ and $e_{n+1} \leq c^1 e_n$.

Proof. Simply follow the steps of the proof in Theorem 1, but define

$$\begin{aligned} G^1(x) &:= F(x_*) + H(x) + F'(x_*)(x - x_*) + Q(x), \\ \psi_n^1(x) &:= (G^1)^{-1}(P_n^1(x)), \end{aligned}$$

where

$$\begin{aligned} P_n^1(x) &:= F(x_n) + H(x) + F'(x_*)(x - x_*) - F(x_n) - H(x_n) \\ &= -\left(F'(x_n) + H[q_1(x_n), q_2(x_n)]\right)(x - x_n). \end{aligned}$$

□

5. Convergence of Higher Order Schemes

We shall demonstrate how to extend the ideas of the previous section in the case of some special schemes of high convergence order. The same idea can be used on other schemes.

In particular, we revisit the fourth convergence order scheme defined by

$$\begin{aligned} y_n &= x_n - F[z_n, x_n]^{-1}F(x_n), \quad z_n = x_n + F(x_n) \\ x_{n+1} &= y_n - A_n^{-1}\left(F[y_n, x_n] + F[y_n, z_n] - F[z_n, x_n]\right)F[y_n, x_n]F(y_n), \end{aligned} \quad (23)$$

used to solve equation

$$F(x) = 0, \quad (24)$$

where $A_n = F[y_n, x_n] + 2F[y_n, z_n] - 2F[z_n, x_n]$ and $F : \Omega \subset \mathbb{B}_1 \rightarrow \mathbb{B}_1$ for some Ω open and $\Omega \neq \emptyset$. The fourth order of convergence of scheme (23) was established in [8] in the special case when $\mathbb{B}_1 = \mathbb{B}_2 = \mathbb{R}^i$ using Taylor series requiring the existence of up to the fifth derivative (not on scheme (23)), which restrict the applicability of scheme (23). Let us consider a motivational example. Therefore, we assume the following function G on $\mathbb{B}_1 = \mathbb{B}_2 = \mathbb{R}$, $\Omega = [-\frac{1}{2}, \frac{3}{2}]$ as:

$$G(t) = \begin{cases} t^3 \ln t^2 + t^5 - t^4, & t \neq 0 \\ 0, & t = 0 \end{cases}. \quad (25)$$

We yield

$$\begin{aligned} G'(t) &= 3t^2 \ln t^2 + 5t^4 - 4t^3 + 2t^2, \\ G''(t) &= 6t \ln t^2 + 20t^3 - 12t^2 + 10t, \\ G'''(t) &= 6 \ln t^2 + 60t^2 - 12t + 22. \end{aligned}$$

So, we identify that $G'''(t)$ is not bounded in Ω . Therefore, results requiring the existence of $G'''(t)$ or higher cannot apply for studying the convergence of (23).

In order to extend the applicability of scheme (23), we introduce hypotheses on only the divided difference $F[\cdot, \cdot]$ and $F'(x_*)$.

We first develop some non-negative functions and parameters. Set $D = [0, \infty)$ and consider $\theta_0 \geq 0$, $\theta \geq 0$. Suppose function:

- (i) $\zeta_0(t) - 1$ has a least zero $\tau_0 \in D_0 - \{0\}$, for some function $\zeta_0 : D \rightarrow D$, which is continuous and non-decreasing. Set $D_0 = [0, \tau_0]$.

- (ii) $\Gamma_1(t) - 1$ has a least zero $\tau_1 \in D - \{0\}$, where $\Gamma_1 : D_0 \rightarrow D$, $\Gamma_1(t) = \frac{\zeta(t)}{1-\zeta_0(t)}$ and function $\zeta : D_0 \rightarrow D$ is continuous and non-decreasing.
- (iii) $\zeta_3(t) - 1$ has a least zero $\tau_2 \in D_0 - \{0\}$, where $\zeta_3(t) = \zeta_1(t) + 2\zeta_2(t) + 2\zeta_0(t)$, where functions ζ_1 and ζ_2 are the functions as ζ .
- (iv) $\zeta_1(t) - 1$ has a least zero $\tau_3 \in D_0 - \{0\}$. Set $\tau_4 = \min\{\tau_2, \tau_3\}$ and $D_1 = [0, \tau_4]$.
- (v) $\Gamma_2(t) - 1$ has a least zero $\tau_5 \in D_1 - \{0\}$, where $\Gamma_2 : D_1 \rightarrow D$ is defined by

$$\Gamma_2(t) = \left[\frac{\zeta_3(t) + 2\zeta_2(t) + 2\zeta(t)}{1 - \zeta_3(t)} + \frac{(\zeta_2(t) + \zeta_0(t))\theta}{(1 - \zeta_1(t))(1 - \zeta_3(t))} \right] \Gamma_1(t),$$

where functions ζ_3 and ζ_3 are the functions as ζ_2 .

Next, we shall show that parameter

$$\tau_* = \min\{\tau_1, \tau_5\}, \quad (26)$$

is a convergence radius for scheme (23). Set $D_2 = [0, \tau_*]$.

By the definition of τ_* it follows that for each $t \in D_2$

$$0 \leq \zeta_0(t) < 1, \quad 0 \leq \zeta_1(t) < 1, \quad (27)$$

$$0 \leq \zeta_3(t) < 1, \quad (28)$$

$$0 \leq \Gamma_1(t) < 1, \quad (29)$$

and

$$0 \leq \Gamma_2(t) < 1. \quad (30)$$

The hypotheses (C) are needed with “ ζ ” functions as previously given and x_* being a simple solution of Equation (24). Suppose:

(C₁) For each $x \in \Omega$

$$\left\| F'(x_*)^{-1} \left(F[x + F(x), x] - F'(x_*) \right) \right\| \leq \zeta_0(\|x - x_*\|)$$

and

$$\left\| I + F[x, x_*] \right\| \leq \theta_0.$$

Set $\Omega_1 = U[x_*, \tau_0] \cap \Omega$.

(C₂) For each $x, y \in \Omega_0$

$$\left\| F'(x_*)^{-1} \left(F[x + F(x), x] - F[x, x_*] \right) \right\| \leq \zeta(\|x - x_*\|),$$

$$\left\| F'(x_*)^{-1} \left(F[y, x] - F'(x_*) \right) \right\| \leq \zeta_1(\|x - x_*\|),$$

$$\left\| F'(x_*)^{-1} \left(F[y, x] - F[y, x_*] \right) \right\| \leq \zeta_2(\|x - x_*\|),$$

and

$$\left\| F'(x_*)^{-1} F[x, x_*] \right\| \leq \theta.$$

(C₃) $U[x_*, \bar{\tau}] \subset \Omega$, where $\bar{\tau} = \max\{\tau_*, \theta_0\tau_*\}$.

Next, we present the local convergence analysis of scheme (23) using the hypotheses (C).

Theorem 3. Under hypotheses (C) further choose $x_0 \in U[x_*, \tau_*] - \{x_*\}$. Then, $\lim_{n \rightarrow \infty} x_n = x_*$, where sequence $\{x_n\}$ is generated by scheme (23).

Proof. The following assertions shall be shown based on mathematical induction:

$$\{x_n\} \subset U[x_*, \tau_*], \quad (31)$$

$$\|y_n - x_*\| \leq \Gamma_1(e_n) \leq e_n < \tau_* \quad (32)$$

and

$$e_{n+1} \leq \Gamma_2(e_n) \leq e_n. \quad (33)$$

By hypotheses (C_1) , (26), (27) and $x_0 \in U[x_*, \tau_*] - \{x_*\}$, we have in turn that

$$\left\| F'(x_*)^{-1} \left(F[x + F(x), x] - F'(x_*) \right) \right\| \leq \zeta_0(\|x - x_*\|) \leq \zeta_0(\tau_*) < 1.$$

So the Banach Lemma for linear invertible operators [8] gives $F[x + F(x), x]^{-1} \in L(\mathbb{B}_1, \mathbb{B}_2)$ and

$$\left\| F[x + F(x), x]^{-1} F'(x_*) \right\| \leq \frac{1}{1 - \zeta_0(\|x - x_*\|)}, \quad (34)$$

where we also used

$$\begin{aligned} \|x + F(x) - x_*\| &\leq \|(I + F[x, x_*])(x - x_*)\| \\ &\leq \|I + F[x, x_*]\| \|x - x_*\| \leq \theta_0 \tau_* \leq \bar{\tau}. \end{aligned}$$

Notice that for $x = x_0$ iterate y_0 is well defined by the first substep of scheme (23) from which we can also write

$$y_0 - x_* = \left(F[z_0, x_0]^{-1} F'(x_*) \right) \left(F'(x_*)^{-1} (F[z_0, x_0] - F[x_0, x_*]) (x_0 - x_*) \right). \quad (35)$$

In view of (C_2) , (26), (29), (34) (for $x = x_0$) and (35), we get in turn that

$$\begin{aligned} \|y - x_*\| &\leq \frac{\zeta(\|x_0 - x_*\|) \|x_0 - x_*\|}{1 - \zeta(\|x_0 - x_*\|)} \leq \Gamma_1(\|x_0 - x_*\|) \|x_0 - x_*\| \\ &\leq \|x_0 - x_*\| \leq \tau_*, \end{aligned} \quad (36)$$

showing (32) for $n = 0$ and $y_0 \in U[x_*, \tau_*]$. Next, we shall show that A_0 and $[y_0, x_0]^{-1} \in L(\mathbb{B}_1, \mathbb{B}_1)$. Using (26), (28) and (C_2) , we obtain in turn

$$\begin{aligned} \left\| F'(x_*)^{-1} (A_0 - F'(x_*)) \right\| &\leq \left\| F'(x_*)^{-1} (F[y_0, x_0] - F'(x_*)) \right\| \\ &\quad + 2 \left\| F'(x_*)^{-1} (F[y_0, z_0] - F'(x_*)) \right\| \\ &\quad + 2 \left\| F'(x_*)^{-1} (F[z_0, x_0] - F'(x_*)) \right\| \\ &\leq \zeta_1(\|x_0 - x_*\|) + 2\zeta_2(\|x_0 - x_*\|) + 2\zeta_0(\|x_0 - x_*\|) \\ &\leq \zeta_3(\|x_0 - x_*\|) \leq \zeta_3(\tau_*) < 1, \end{aligned}$$

so

$$\left\| A_0^{-1} F'(x_*) \right\| \leq \frac{1}{1 - \zeta_3(\|x_0 - x_*\|)}. \quad (37)$$

Similarly by using (27) and (C_2) , we get

$$\left\| F[y_0, x_0]^{-1} F'(x_*) \right\| \leq \frac{1}{1 - \zeta_1(\|x_0 - x_*\|)}. \quad (38)$$

Hence, iterate x_1 is well defined by the second substep of scheme (23) from which we can also write

$$x_1 - x_* = y_0 - x_* - A_0^{-1} F(y_0) - A_0^{-1} (F[y_0, z_0] - F[z_0, x_0]) F[y_0, x_0]^{-1} F(y_0). \quad (39)$$

By (26), (30), (C₂) and (37)–(39), we obtain in turn

$$\begin{aligned} \|x_1 - x_*\| &\leq \left[\frac{3\zeta_2(\|x_0 - x_*\|) + 2\zeta(\|x_0 - x_*\|)}{1 - \zeta_3(\|x_0 - x_*\|)} \right. \\ &\quad \left. + \frac{\theta(\zeta_2(\|x_0 - x_*\|) + \zeta_0(\|x_0 - x_*\|))}{(1 - \zeta_3(\|x_0 - x_*\|))(1 - \zeta_1(\|x_0 - x_*\|))} \right] \|y_0 - x_*\| \\ &\leq \bar{\zeta}_2(\|x_0 - x_*\|) \|x_0 - x_*\| \leq \|x_0 - x_*\|, \end{aligned} \quad (40)$$

showing (33) for $n = 0$ and $\zeta \in U[x_*, \tau_*]$. Simply exchange x_0, z_0, y_0, x_1 by x_m, z_m, y_m, x_{m+1} , respectively, in the previous calculations to terminate the induction for the items (31)–(33). It then follows from the estimation

$$\|x_{m+1} - x_*\| \leq l \|x_m - x_*\| < \tau_*, \quad (41)$$

where $l = \Gamma_2(\|x_0 - x_*\|) \in [0, 1)$, we deduce $x_{m+1} \in U[x_*, \tau_*]$ and $\lim_{m \rightarrow \infty} x_m = x_*$. $\square$

Next, concerning the uniqueness of the solution, we give a result not necessarily relying on the hypotheses (C).

Proposition 3. Suppose: equation $F(x) = 0$ has a simple solution $x_* \in \Omega$.

$$\|x_{m+1} - x_*\| \leq l \|x_m - x_*\| < \tau_*,$$

For all $x \in \Omega$

$$\left\| F'(x_*)^{-1} (F[x, x_*] - F'(x_*)) \right\| \leq \zeta_4(\|x_m - x_*\|), \quad (42)$$

and function $\zeta_4(t) - 1$ has a smallest positive zero τ_6 , where $\zeta_4 : M \rightarrow M$ is a continuous and non-decreasing function. Set $\Omega_1 = S[x_*, \bar{\rho}] \cap \Omega$ for $\tau < \tau_6$. Then, the only solution of equation $F(x) = 0$ in the region Ω_1 is x_* .

Proof. Set $U = F[x^*, x_*]$ for some $x^* \in \Omega_1$ with $F(x^*) = 0$. Then, using (42), we get

$$\left\| F'(x_*)^{-1} (U - F'(x_*)) \right\| \leq \zeta_4(\|x - x_*\|) \leq \zeta_4(\bar{\tau}) < 1,$$

so $U^{-1} \in L(X, X)$ and $x^* = x_*$ follows from $U(x^* - x_*) = F(x^*) - F(x_*)$. $\square$

Remark 5. Let us consider choices

$$F[x, y] = \frac{1}{2} (F'(x) + F'(y))$$

or

$$F[x, y] = \int_0^1 F'(x + \theta(y - x)) d\theta$$

or the standard definition of the divided difference when $X = \mathbb{R}^i$ [12].

Moreover, suppose

$$\left\| F'(x_*)^{-1} (F'(x) - F'(x_*)) \right\| \leq \phi_0(\|x - x_*\|)$$

and

$$\left\| F'(x_*)^{-1} (F'(x) - F'(y)) \right\| \leq \phi(\|x - x_*\|)$$

where function $\phi_0 : M \rightarrow M$, $\phi : M \rightarrow M$ are continuous and non-decreasing. Then, under the first or second choice above it can easily be seen that the hypotheses (C) require

$$\begin{aligned}\zeta_0(t) &= \frac{1}{2}(\phi_0(\theta_0 t) + \phi_0(t)), \\ \zeta(t) &= \frac{1}{2}(\phi(\theta t) + \phi_0(t)), \\ \zeta_1(t) &= \frac{1}{2}[\phi_0(t) + \phi_0(t)], \\ \zeta_2(t) &= \zeta(t).\end{aligned}$$

6. Numerical Examples

Here, we present the computational results based on the suggested theoretical results in this paper. We choose a applied science problem for the computational results, which is illustrated in Example 1. The results are listed in Table 1. Additionally, we obtain the COC approximated by means of

$$\lambda = \frac{\ln \frac{\|x_{n+1} - x_*\|}{\|x_n - x_*\|}}{\ln \frac{\|x_n - x_*\|}{\|x_{n-1} - x_*\|}}, \quad \text{for } n = 1, 2, \dots \quad (43)$$

or ACOC [13] by:

$$\lambda^* = \frac{\ln \frac{\|x_{n+1} - x_n\|}{\|x_n - x_{n-1}\|}}{\ln \frac{\|x_n - x_{n-1}\|}{\|x_{n-1} - x_{n-2}\|}}, \quad \text{for } n = 2, 3, \dots \quad (44)$$

The computations are performed with the package *Mathematica* 11 with multiple precision arithmetic.

Example 1. Let $\mathbb{B}_1 = \mathbb{B}_2 = \mathbb{R}^3$ and $\Omega = \mathbb{S}(0, 1)$. Assume F on Ω with $v = (x, y, z)^T$ as

$$F(u) = F(u_1, u_2, u_3) = \left(e^{u_1} - 1, \frac{e-1}{2}u_2^2 + u_2, u_3 \right)^T, \quad (45)$$

where, $u = (u_1, u_2, u_3)^T$. Then, we obtain

$$F'(u) = \begin{bmatrix} e^{u_1} & 0 & 0 \\ 0 & (e-1)u_2 + 1 & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

the Fréchet-derivative. Hence, for $x_* = (0, 0, 0)^T$, $F'(x_*) = F'(x_*)^{-1} = \text{diag}\{1, 1, 1\}$, we have

$$\phi_0(t) = (e-1)t, \quad \phi(t) = e^{\frac{1}{e-1}t} \quad \theta_0 = \frac{1}{2}(3 + e^{\frac{1}{e-1}}), \quad \text{and} \quad \theta = \frac{1}{2}(1 + e^{\frac{1}{e-1}}).$$

So, we obtain convergence radii that are mentioned in Table 1.

Table 1. Radii for Example 1.

τ_0	τ_1	τ_2	τ_3	τ_5	τ_*
0.342865	0.199053	0.0849919	0.581976	0.06169217	0.0849919

Example 2. Define scalar function

$$h(t) = \zeta_0 t + \zeta_1 + \zeta_2 \sin \zeta_3 t, \quad x_0 = 0$$

where $\xi_j, j = 0, 1, 2, 3$ are scalars. Choose $\varphi_0(t) = Kt$ and $\varphi(t) = Lt$. Then, clearly for ξ_3 large and ξ_2 small, $\frac{K_0}{L}$ can be small (arbitrarily). Notice that $\frac{K}{L} \rightarrow 0$. In early studies [1], $K = L$. Hence, our results constitute a significant improvement without additional hypotheses, since in practice the computation of L requires that of K as a special case.

7. Conclusions

The region of applicability for SLS and ISLS has been extended to solve generalized Equations (1) and (21), respectively, and under weaker or the same hypotheses. Moreover, tighter error distances are realized as well as a more precise knowledge of the whereabouts of the solution x_* . Due to its generality of our idea can provide the same benefits when applied to other methods.

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