mathematics

Article

Tripled Fixed Points and Existence Study to a Tripled Impulsive Fractional Differential System via Measures of Noncompactness

Sina Etemad [1,†], Mohammed M. Matar [2,†], Maria Alessandra Ragusa [3,4,*,†] and Shahram Rezapour [1,5,*,†]

1 Department of Mathematics, Azarbaijan Shahid Madani University, Tabriz 3751-71379, Iran; sina.etemad@azaruniv.ac.ir
2 Department of Mathematics, Al-Azhar University-Gaza, Gaza P.O. Box 1277, Palestine; m.matar@alazhar.edu.ps
3 Dipartimento di Matematica e Informatica, Universita di Catania, Viale Andrea Doria 6, 95125 Catania, Italy
4 RUDN University, 6 Miklukho-Maklay St., 117198 Moscow, Russia
5 Department of Medical Research, China Medical University Hospital, China Medical University, Taichung 40402, Taiwan
* Correspondence: mariaalessandra.ragusa@unict.it (M.A.R.); sh.rezapour@azaruniv.ac.ir (S.R.)
† These authors contributed equally to this work.

Abstract: In this paper, a tripled fractional differential system is introduced as three associated impulsive equations. The existence investigation of the solution is based on contraction principle and measures of noncompactness in terms of tripled fixed point and modulus of continuity. Our results are valid for both Kuratowski and Hausdorff measures of noncompactness. As an application, we apply the obtained results to a control problem.

Keywords: tripled system; impulsive conditions; caputo derivative; tripled fixed point; measures of noncompactness

check for updates

Citation: Etemad, S.; Matar, M.M.; Ragusa, M.A.; Rezapour, S. Tripled Fixed Points and Existence Study to a Tripled Impulsive Fractional Differential System via Measures of Noncompactness. *Mathematics* **2022**, *10*, 25. https://doi.org/10.3390/math10010025

Academic Editor: Ivanka Stamova

Received: 19 November 2021
Accepted: 18 December 2021
Published: 22 December 2021

1. Introduction

Fractional differential equations are considered as the most appropriate models for many applicable phenomena (see in [1,2] and references therein). This opens the research gate to study analytic solutions and their behaviors in a theoretic sense such as existence, uniqueness, stability, controllability, etc. [3–21].

The investigations of existence problems of fractional differential equations have diverse topics ranging from the shape of initial and boundary conditions including impulsive conditions, throughout various types of the used fractional derivatives, and reaching to different forms fixed point theorems.

The fixed point theorems are essential resources for solving many existence problems of solutions of differential and integral equations. In the meantime, the standard Banach principle for contractions can be used not only for establishing the existence of a solution, but also to investigate the uniqueness of this solution. The main assumption for using Banach's fixed point theorem [22] is the contraction principle applied to the operator equation. Another famous theorem is the Schauder's fixed point theorem [23], which mainly utilizes the relative compactness of the image of the solution operator. The application of these two important theorems can be observed in different fractional modeling problems such as the investigation of existence-uniqueness criteria for the generalized Navier system [24], the nonsingular 4D-memristor-based circuit model [25], the SARS-CoV-2 virus model [26], the hearing loss model caused by mump virus [27], the Ebola model [28], the Langevin problem [29], the fractional BVP of the hexasilinane graph [30], etc.

In connection with bounded relatively compact subsets, measures of noncompactness are considered very applicable tools to investigate existence problems by imposing weaker conditions [31–33]. Therefore, there exists a correlation between the measure of noncompactness and the Schauder theorem. Darbo [34] used this idea to prove a generalization of

Schauder fixed point theorem, and then many researchers presented extended results of Darbo's fixed point criterion [35].

Coupled systems are introduced as two associated differential equations that may be solved simultaneously [36–38]. The main notion of the existence of coupled fixed points to be a solution of coupled systems are considered recently by many researchers [39–43]. In [31], the authors considered coupled fractional systems using the idea of the measure of compactness.

Recently, many researchers introduced a tripled system and a tripled fixed point [44–47]. In [45], the authors used tripled fixed points and the measure of noncompactness to investigate the existence of a solution in relation to a functional tripled system via fractional operators.

We design and discuss in this article a tripled impulsive nonlinear system formulated as

$$\begin{cases} {}^{c}D_{a}^{\kappa_m} x_m(t) = f_m(t, x(t)), t \in J', \\ x_m(a) = \Phi_m x, x'_m(a) = \Theta_m x, \\ \Delta x_m\big|_{t=t_k} = I_{m,k}(x(t_k)), \Delta x'_m\big|_{t=t_k} = \bar{I}_{m,k}(x(t_k)), \end{cases} \tag{1}$$

where $J = [a, b]$, $J' = J - \{t_1, t_2, ..., t_p\}$, $a = t_0 < t_1 < \cdots < t_p < t_{p+1} = b$, ${}^{c}D_{a}^{\kappa_m}$, $m = 1, 2, 3$, are the Caputo fractional derivatives such that $\kappa_m \in (1, 2]$, $f_m : J \times \mathbb{R}^3 \to \mathbb{R}$, $x(t) = (x_1(t), x_2(t), x_3(t))$, $I_{m,k}$, $\bar{I}_{m,k} : \mathbb{R}^3 \to \mathbb{R}$, $k = 1, 2, ..., p$, are given functions, Φ_m, Θ_m are given operators, $\Delta x_m\big|_{t=t_k} = x_m(t_k^+) - x_m(t_k^-)$, $\Delta x'_m\big|_{t=t_k} = x'_m(t_k^+) - x'_m(t_k^-)$, and

$$x_m(t_k^+) = \lim_{\hbar \to 0^+} x_m(t_k + \hbar), \quad x_m(t_k^-) = \lim_{\hbar \to 0^-} x_m(t_k + \hbar),$$

represent the right and left limits of the function $x_m(t)$ at the given points $t = t_k$, respectively. The main contribution and novelty of the present manuscript are that we here discuss the existence notion with the help of a combination of extended fixed point theorems for tripled fixed points in relation to a tripled impulsive system for the first time. Darbo's criterion and measure of noncompactness are used for tripled fixed points. Note that our findings will be valid for both Kuratowski and Hausdorff measures [33–35].

The structure of the present research is organized as follows. In Section 2, some useful preliminaries and lemmas are recalled to facilitate the proof of main theorems. In Section 3, we prove and verify the existence results via different fixed point theorems. To conclude, we introduce an example to examine the results.

2. Basic Notions

For convenience, we present firstly some preliminaries concerning with fractional calculus. For more details on this topic, see the monograph [48].

Definition 1 ([48])**.** *A real-valued function f is said to be κth-fractional integrable at t in the sense of Riemann–Liouville (RL) if the integral*

$$\int_a^t \frac{(t-s)^{\kappa-1} f(s)}{\Gamma(\kappa)} ds,$$

exists for $\kappa > 0$. If f is κth-fractional RL-integrable for every $t \in J$, then it is said to be κth-fractional RL-integrable. The κth-fractional RL-integral of f is denoted by $I_a^{\kappa} f(t)$, and conventionally, we let $I_a^0 f = f$, and $I_a^{\kappa} f(a) = 0$.

It is obvious that if f is κth-fractional integrable, then it is βth-fractional integrable for any $0 < \beta \leq \kappa$. We also notice that if f is κth-fractional integrable on $[a, b]$, then it satisfies that $I_0^{\kappa} f(t_k^+) = I_0^{\kappa} f(t_k^-)$, $k = 1, 2, ..., p$. However, if f is continuous, then it is κth-fractional integrable.

Definition 2 ([48])**.** *A real-valued function f is said to has a Caputo derivative on J of order $\kappa \in (n-1, n)$, $n \in \mathbb{N}$, if the nth derivative $f^{(n)}$ of f is $(n-\kappa)$th-fractional integrable on J. It is denoted by ${}^cD_a^\kappa f$, and then we write ${}^cD_a^\kappa f = I_a^{n-\kappa} f^{(n)}$.*

The continuity of the nth derivative $f^{(n)}$ ensures the continuity of ${}^cD_a^\kappa f$. We notice that the nth derivative of any function of the form of $c_i(t-a)^i, i \in \{0, \ldots, n-1\}$ is zero, then the Caputo derivative on J of order $\kappa \in (n-1, n]$ for such functions is zero.

Lemma 1 ([48])**.** *Assume that f has a Caputo derivative on $[a, b]$ of order $\kappa \in (n-1, n]$; then, D_a^κ $I_a^\kappa f = f$ and*

$$I_a^\kappa \ {}^cD_a^\kappa f(t) = f(t) + \sum_{i=0}^{n-1} c_i (t-a)^i, \tag{2}$$

in which $c_i \in \mathbb{R}$ and $t \geq a$.

Definition 3 ([44])**.** *A triplet $(\varkappa_1, \varkappa_2, \varkappa_3) \in Y$ for $Y := X^3$ is termed as a tripled fixed point of a mapping $\digamma : Y \to X$ whenever $\digamma(\varkappa_1, \varkappa_2, \varkappa_3) = \varkappa_1$, $\digamma(\varkappa_2, \varkappa_1, \varkappa_3) = \varkappa_2$, and $\digamma(\varkappa_3, \varkappa_2, \varkappa_1) = \varkappa_3$.*

Here, in our fundamental theorem, X is a Banach space. In general, Definition 3 can be applied for any space X which has some primitive algebraic structures such as partially ordered space [49].

Define $\Psi : X^3 \to X^3$ so that

$$\Psi(\varkappa_1, \varkappa_2, \varkappa_3) = (\digamma(\varkappa_1, \varkappa_2, \varkappa_3), \digamma(\varkappa_2, \varkappa_1, \varkappa_3), \digamma(\varkappa_3, \varkappa_2, \varkappa_1)).$$

Then, $(\varkappa_1, \varkappa_2, \varkappa_3)$ is a tripled fixed point of $\digamma$ iff $(\varkappa_1, \varkappa_2, \varkappa_3)$ is a fixed point of Ψ, i.e., $\Psi(\varkappa_1, \varkappa_2, \varkappa_3) = (\varkappa_1, \varkappa_2, \varkappa_3)$.

Next, we recall some preliminaries about a measure of noncompactness.

Definition 4 ([34])**.** *Let X be a Banach space and $\mathcal{B}_X$ the collection of bounded sets in X. A measure $\mu : \mathcal{B}_X \to [0, \infty)$ is termed as the measure of noncompactness if for any $V, V_1, V_2 \in \mathcal{B}_X$, it fulfills the following:*

- **(M1)** *(Regularity)*$\ker \mu = \{V \in \mathcal{B}_X : \mu(V) = 0\}$ *is nonempty subset of the category of relatively compact sets in $\mathcal{B}_X$;*
- **(M2)** *(Monotonicity) $V_1 \subset V_2$ implies $\mu(V_1) \leq \mu(V_2)$;*
- **(M3)** *(Invariance) $\mu(Conv\ V) = \mu(\overline{V}) = \mu(V)$, where $Conv\ V$, and $\overline{V}$ are, respectively, the closed convex hull and closure of V;*
- **(M4)** *(Semi-homogeneity)$\mu(cV) \leq |c|\mu(V)$, for $c \in \mathbb{R}$;*
- **(M5)** *(Sublinearity) $\mu(V_1 + V_2) \leq \mu(v_1) + \mu(V_2)$;*
- **(M6)** *$\mu(cV_1 + (1-c)V_2) \leq c\mu(V_1) + (1-c)\mu(V_2)$;*
- **(M7)** *If (V_J) is a decreasing sequence of subsets in $\mathcal{B}_X$ with $\lim_{J\to\infty} \mu(V_J) = 0$, $\bigcap_{J=1}^\infty V_J$ is nonempty.*

For more properties and details, the reader may refer to the works in [34,35]. Kuratowski and Hausdorff measures of noncompactness are two famous measures of this type which are defined, respectively, as

$$\alpha(V) = \inf\{\mathfrak{R} > 0 : V \text{ can be covered by a finite number of sets of diameter} \leq \mathfrak{R}\},$$

and

$$\chi(V) = \inf\{\mathfrak{R} > 0 : V \text{ can be covered by a finite number of balls of radius} \leq \mathfrak{R}\},$$

which are equivalent to regular measures, since $\chi(V) \le \alpha(V) \le 2\chi(V)$. In fact all regular measures are equivalent [34]. The diameter $diam\ V = \sup\{\|a_1 - a_2\|$, for all $a_1, a_2 \in V\}$ and the norm $\|V\| = \{\|x\| : x \in V\}$ of a set A are nonregular measures with kernels of singleton sets and $\{0\}$, respectively.

In the space of continuous mappings given on J, the modulus of continuity of $x \in C(J)$ is a function $\omega(x,\cdot) : [0,\infty) \to [0,\infty)$ such that

$$\omega(x,\mathfrak{R}) = \sup\{|x(s) - x(t)| : s,t \in J, |s-t| \le \mathfrak{R}\},$$

and the modulus of continuity of a set $V \subseteq C(J)$ is defined as

$$\omega(V,\mathfrak{R}) = \sup\{\omega(x,\mathfrak{R}) : x \in V\}.$$

Define a measure of noncompactness ω_0 as

$$\omega_0(V) = \lim_{\mathfrak{R}\to 0} \omega(V,\mathfrak{R}),$$

where it satisfies that $\omega_0(V) = 2\chi(V)$ [34].

Schauder fixed point theorem is one of the well-known applications on existence problems but it focuses on compact operators.

Theorem 1 ([23])**.** *Let $\Omega \neq \emptyset$ be a convex closed set with boundedness property in a Banach space X. Then $\exists\, x \in \Omega$ for every continuous compact mapping $\digamma : \Omega \to \Omega$ so that $x = \digamma x$.*

Therefore, if Ω satisfies the hypotheses of Theorem 1, and $\digamma$ is continuous whose image embedded in Ω and the set $\digamma\Omega$ is equicontinuous, then by Arzela Ascoli theorem $\digamma$ is compact, i.e., $\digamma\Omega$ is relatively compact. This means that $\digamma\Omega \in \ker\mu$, or $\mu(\digamma\Omega) = 0$, where μ is an arbitrary regular measure.

A useful extension of Darbo's fixed point criterion is given in the next step.

Theorem 2 ([35])**.** *Let $\Omega \neq \emptyset$ be a convex closed set with boundedness property in a Banach space X and $\digamma : \Omega \to \Omega$ be continuous which satisfies*

$$\mu(\digamma(V)) \le \sigma(\mu(V)), \tag{3}$$

$\forall\, V \subseteq \Omega$, in which $\sigma : [0,\infty) \to [0,\infty)$ is increasing with $\lim_{J\to\infty} \sigma^J(t) = 0$, $\forall\, t \in [0,\infty)$, and μ is an arbitrary measure of noncompactness. Then, $\exists\, x \in \Omega$ so that $x = \digamma x$.

Assuming $\sigma(t) = kt$ for $t \ge 0$, and $k < 1$,

$$\lim_{J\to\infty} \sigma^J(t) = \lim_{J\to\infty} k^J t = 0.$$

Moreover, the condition (3) becomes $\mu(\digamma(V)) \le k\mu(V)$ which is called Darbo's condition or μ-contraction and the theorem will be the same original Darbo's fixed point result [34].

The next result concerns with an integral solution of the corresponding linear system of (1).

Lemma 2. *Let f_m be κ_mth-fractional integrable with $I_0^{\kappa} f_m(t_k^+) = I_0^{\kappa} f_m(t_k^-)$, $0 < \kappa \leq \kappa_m$, $k = 1, 2, ..., p$, and also x_m be differentiable of the Caputo type on J of order κ_m where $\kappa_m \in (1, 2]$, $m = 1, 2, 3$. In this case, the solution of the impulsive fractional differential system*

$$\begin{cases} {}^cD_0^{\kappa_m} x_m(t) = f_m(t), t \in J', \\ x_m(a) = a_m, x_m'(a) = b_m, \\ \Delta x_m|_{t=t_k} = I_{m,k}(x(t_k)), \Delta x_m'|_{t=t_k} = \bar{I}_{m,k}(x(t_k)), \end{cases} \tag{4}$$

is equivalent to

$$x_m(t) = \begin{cases} a_m + b_m t + \int_0^t \frac{(t-s)^{\kappa_m - 1}}{\Gamma(\kappa_m)} f_m(s) ds, t \in J_0, \\ a_m + b_m t + \int_0^t \frac{(t-s)^{\kappa_m - 1}}{\Gamma(\kappa_m)} f_m(s) ds \\ + \sum_{i=1}^{k} I_{m,i}(x(t_i)) + \sum_{i=1}^{k} (t - t_i) \bar{I}_{m,i}(x(t_i)), t \in J_k. \end{cases} \tag{5}$$

Proof. The given conditions imply that $I_a^{\kappa_m} D_a^{\kappa_m} x_m$ exists and satisfies the identity (2). Taking the fractional integral $I_a^{\kappa_m}$ to both sides of the differential Equation (4), and using Lemma 1, we obtain

$$x_m(t) = a_{k,m} + b_{k,m} t + I_a^{\kappa_m} f_m(t), t \in J_k, \tag{6}$$

and

$$x_m'(t) = b_{k,m} + I_a^{\kappa_m - 1} f_m(t), t \in J_k,$$

where $I_a^{\kappa_m} f_m(t) = \sum_{i=1}^{k} \int_{t_{i-1}}^{t_i} \frac{(t-s)^{\kappa_m - 1}}{\Gamma(\kappa_m)} f_m(s) ds + \int_{t_k}^{t} \frac{(t-s)^{\kappa_m - 1}}{\Gamma(\kappa_m)} f_m(s) ds$, and $a_{k,m}$, $b_{k,m}$, $k = 1, 2, ..., p$, $m = 1, 2, 3$, are constants to be determined. Applying the boundary conditions in (4), if $t \in J_0$, we have $a_{0,m} = a_m$, and $b_{0,m} = b_m$. Then, the solution (6) becomes

$$x_m(t) = a_m + b_m t + I_a^{\kappa_m} f_m(t).$$

Next, if $t \in J_k$, $k = 1, 2, ..., p$, then

$$\begin{aligned} a_{k,m} &= a_{k-1,m} - (b_{k,m} - b_{k-1,m}) t_k + I_{m,k}(x(t_k)), \\ b_{k,m} &= b_{k-1,m} + \bar{I}_{m,k}(x(t_k)). \end{aligned}$$

Solving these recursions leads to

$$b_{k,m} = b_m + \sum_{i=1}^{k} \bar{I}_{m,i}(x(t_i)),$$

and

$$a_{k,m} = a_m - \sum_{i=1}^{k} t_i \bar{I}_{m,i}(x(t_i)) + \sum_{i=1}^{k} I_{m,i}(x(t_i)).$$

Substituting these constants in (6), we obtain

$$\begin{aligned} x_m(t) &= a_m + b_m t + I_a^{\kappa_m} f_m(t) \\ &\quad + \sum_{i=1}^{k} (t - t_i) \bar{I}_{m,i}(x(t_i)) + \sum_{i=1}^{k} I_{m,i}(x(t_i)). \end{aligned}$$

This is equivalent to the solution (5). On another side, as x_m has $\kappa_m th$-Caputo derivative, using Lemma 1, it is easy to deduce (4). This finishes the proof. $\square$

If x has a continuous second derivative and f is continuous on J, then the result of Lemma 2 are valid, as $I_a^{\kappa_m} f_m$ and $D_a^{\kappa_m} x_m$ are continuous.

3. Results on the Existence Criterion

In this place, we discuss the existence and uniqueness problems for the impulsive tripled system (1).

A Banach space $C(J)$ of all real-valued continuous mappings is endowed with the supremum norm. Consider the space $PC(J)$ defined by

$$PC(J) = \Big\{\zeta : J \to \mathbb{R} | \zeta \in C(J^{'}), \zeta(t_k^+) \text{ and } \zeta(t_k^-) \text{ exist with } \zeta(t_k^-) = \zeta(t_k)\Big\},$$

endowed with the norm $\|\zeta\| = \sup\{|\zeta(t)|, t \in J\}$, and $k = 1, 2, ..., p$. Let $X = PC(J)$, and $Y := X^3$ be the usual tripled product which becomes a Banach space with $\|x\| = \max\{\|x_1\|, \|x_2\|, \|x_3\|\}$ for any $x = (x_1, x_2, x_3) \in Y$. Define the operators $\Psi : Y \to Y$, and $\Psi_m : Y \to X, m = 1, 2, 3$ such that

$$\Psi(x_1, x_2, x_3) = (\Psi_1(x_1, x_2, x_3), \Psi_2(x_2, x_1, x_3), \Psi_3(x_3, x_2, x_1))$$

and Ψ_m satisfying $\Psi_1(x_1, x_2, x_3) = x_1$, $\Psi_2(x_2, x_1, x_3) = x_2$, and $\Psi_3(x_3, x_2, x_1) = x_3$. Without loss of generality we use a common notation $x = (x_1, x_2, x_3)$ for three cases of domain Ψ_m. Using Lemma 2, Ψ_m satisfies the corresponding integral solution (5) of the system (1) given by

$$\Psi_m x(t) = \begin{cases} \Phi_m x + t\Theta_m x + \displaystyle\int_a^t \frac{(t-s)^{\kappa_m - 1}}{\Gamma(\kappa_m)} f_m(s, x(s)) ds, t \in J_0, \\ \Phi_m x + t\Theta_m x + \displaystyle\int_a^t \frac{(t-s)^{\kappa_m - 1}}{\Gamma(\kappa_m)} f_m(s, x(s)) ds \\ + \displaystyle\sum_{i=1}^{k} I_{m,i}(x(t_i)) + \sum_{i=1}^{k} (t - t_i) \bar{I}_{m,i}(x(t_i)), t \in J_k, k = 1, 2, ..., p. \end{cases} \tag{7}$$

We need the following assumptions:

Hypothesis 1 (H1). *f_m is a Carathéodory function, that is $(s, x) \longmapsto f_m(s, x)$ is continuous in x and strongly measurable in s. There is a nondecreasing $\kappa_m th$-fractional integrable function $\psi_{f_m} : J \to [0, \infty)$, and*

$$|f_m(t, x) - f_m(t, y)| \le \psi_{f_m}(t)\|x - y\|,$$

$\forall\, t \in J$ and $x, y \in \mathbb{R}^3$, $m = 1, 2, 3$. Moreover, let $\sup_{t \in J} I_a^{\kappa_m} \psi_{f_m}(t) \le L_{\psi_m}$, for $m = 1, 2, 3$, and $L_\psi = \max\{L_{\psi_1}, L_{\psi_2}, L_{\psi_3}\}$.

Hypothesis 2 (H2). *$I_{m,i}$ and $\bar{I}_{m,i}$ are continuous functions that maps zero vector into zero value, and there exist constants $L_{I_{m,i}}, L_{\bar{I}_{m,i}} > 0$, provided that $\forall\, x, y \in \mathbb{R}^3, i = 1, 2, ..., p$,*

$$|I_{m,i}(x) - I_{m,i}(y)| \le L_{I_{m,i}}\|x - y\|, \; |\bar{I}_{m,i}(x) - \bar{I}_{m,i}(y)| \le L_{\bar{I}_{m,i}}\|x - y\|.$$

Moreover, let $L_{I_i} = \max\{L_{I_{1,i}}, L_{I_{2,i}}, L_{I_{3,i}}\}$, and $L_{\bar{I}_i} = \max\{L_{\bar{I}_{1,i}}, L_{\bar{I}_{2,i}}, L_{\bar{I}_{3,i}}\}$.

Hypothesis 3 (H3). *For continuous maps $\Phi_m, \Theta_m : \mathbb{R}^3 \to \mathbb{R}$, $\exists\, L_{\Phi_m}, L_{\Theta_m} > 0$ such that for all $x, y \in \mathbb{R}^3, m = 1, 2, 3$,*

$$\begin{cases} |\Phi_m(x) - \Phi_m(y)| \leq L_{\Phi_m}\|x - y\|, \\ |\Theta_m(x) - \Theta_m(y)| \leq L_{\Theta_m}\|x - y\|. \end{cases}$$

Moreover, let $L_\Phi = \max\{L_{\Phi_1}, L_{\Phi_2}, L_{\Phi_3}\}$, and $L_\Theta = \max\{L_{\Theta_1}, L_{\Theta_2}, L_{\Theta_3}\}$.

Theorem 3. *Let (H1)–(H3) be held. Then, there is a unique solution for the impulsive tripled system (1) whenever*

$$L_\Phi + L_\Theta b + L_\psi + \sum_{i=1}^{p} L_{I_i} + \sum_{i=1}^{p}(t_{i+1} - t_i)L_{\bar{I}_i} < 1. \tag{8}$$

Proof. Take $X^3 =: Y$. Let $\|\Phi_m 0\| = N_{\Phi_m} \geq 0$, $\|\Theta_m 0\| = N_{\Theta_m} \geq 0$, and $\max_{t\in J}\|f_m(t, 0)\| = N_{f_m} \geq 0$. Define a subset $\mathfrak{B}_r = \{x \in Y : \|x\| \leq r\}$ of Y via

$$r \geq \frac{N_\Phi + N_\Theta b + N_f N_0}{1 - L_\Phi + L_\Theta b + L_\psi + \sum\limits_{i=1}^{p} L_{I_i} + \sum\limits_{i=1}^{p}(t_{i+1} - t_i)L_{\bar{I}_i}}, \tag{9}$$

where $N_\Phi = \max\{N_{\Phi_1}, N_{\Phi_2}, N_{\Phi_3}\}$, $N_\Theta = \max\{N_{\Theta_1}, N_{\Theta_2}, N_{\Theta_3}\}$, and

$$N_0 = \max\left\{\frac{(b-a)^{\kappa_m}}{\Gamma(\kappa_m + 1)}, m = 1, 2, 3\right\}.$$

In this case, for $t \in J_0$,

$$\begin{aligned} |\Psi_m x(t)| &\leq \|\Phi_m x\| + t\|\Theta_m x\| + \int_a^t \frac{(t-s)^{\kappa_m - 1}}{\Gamma(\kappa_m)}\|f_m(s, x(s))\|ds \\ &\leq \Big(L_{\Phi_m} + L_{\Theta_m} t + I_a^{\kappa_m}\psi_{f_m}(t)\Big)\|x\| \\ &\quad + \left(N_{\Phi_m} + N_{\Theta_m} t + \frac{N_{f_m}(t-a)^{\kappa_m}}{\Gamma(\kappa_m + 1)}\right) \\ &\leq \big(L_{\Phi_m} + L_{\Theta_m} t_{k+1} + L_{\psi_m}\big)\|x\| \\ &\quad + \left(N_{\Phi_m} + N_{\Theta_m} t_{k+1} + \frac{N_{f_m}(t_{k+1}-a)^{\kappa_m}}{\Gamma(\kappa_m + 1)}\right). \end{aligned}$$

Similarly, for $t \in J_k$ we write $(k = 1, 2, ..., p)$,

$$\begin{aligned} |\Psi_m x(t)| &\leq \|\Phi_m x\| + t\|\Theta_m x\| + \int_a^t \frac{(t-s)^{\kappa_m - 1}}{\Gamma(\kappa_m)}\|f_m(s, x(s))\|ds \\ &\quad + \sum_{i=1}^{k}\|I_{m,i}(x(t_i))\| + \sum_{i=1}^{k}(t - t_i)\|\bar{I}_{m,i}(x(t_i))\| \\ &\leq \left(L_{\Phi_m} + L_{\Theta_m} t + I_a^{\kappa_m}\psi_{f_m}(t) + \sum_{i=1}^{k} L_{I_{m,i}} + \sum_{i=1}^{k}(t - t_i)L_{\bar{I}_{m,i}}\right)\|x\| \\ &\quad + \left(N_{\Phi_m} + N_{\Theta_m} t + \frac{N_{f_m}(t-a)^{\kappa_m}}{\Gamma(\kappa_m + 1)}\right) \end{aligned}$$

$$\leq \left(L_{\Phi_m} + L_{\Theta_m} t_{k+1} + L_{\psi_m} + \sum_{i=1}^{k} L_{I_{m,i}} + \sum_{i=1}^{k} (t_{i+1} - t_i) L_{\bar{I}_{m,i}}\right) \|x\| + \left(N_{\Phi_m} + N_{\Theta_m} t_{k+1} + \frac{N_{f_m}(t_{k+1} - a)^{\kappa_m}}{\Gamma(\kappa_m + 1)}\right).$$

As $\sum_{i=1}^{k} L_{I_{m,i}} + \sum_{i=1}^{k} (t - t_i) L_{\bar{I}_{m,i}} \geq 0$, then for $x \in \mathfrak{B}_r$, we obtain

$$\|\Psi x\| \leq \left(L_\Phi + L_\Theta b + L_\psi + \sum_{i=1}^{p} L_{I_i} + \sum_{i=1}^{p} (t_{i+1} - t_i) L_{\bar{I}_i}\right) r + N_\Phi + N_\Theta b + N_f N_0.$$

In virtue of (9), we deduce that $\Psi x \in \mathfrak{B}_r$. Next, the contractivity of Ψ is checked. Let $x, y \in Y$, we have

$$\begin{aligned} |\Psi_m x(t) - \Psi_m y(t)| &\leq \|\Phi_m x - \Phi_m y\| + t\|\Theta_m x - \Theta_m y\| \\ &\quad + \int_a^t \frac{(t-s)^{\kappa_m - 1}}{\Gamma(\kappa_m)} \|f_m(s, x(s)) - f_m(s, y(s))\| ds \\ &\leq \left(L_{\Phi_m} + L_{\Theta_m} t + I_a^{\kappa_m} \psi_{f_m}(t)\right) \|x - y\| \\ &\leq \left(L_{\Phi_m} + L_{\Theta_m} t_1 + L_{\psi_m}\right) \|x - y\|, t \in J_0, \end{aligned}$$

and for $t \in J_k$, we get

$$\begin{aligned} |\Psi_m x(t) - \Psi_m y(t)| &\leq \|\Phi_m x - \Phi_m y\| + t\|\Theta_m x - \Theta_m y\| \\ &\quad + \int_a^t \frac{(t-s)^{\kappa_m - 1}}{\Gamma(\kappa_m)} \|f_m(s, x(s)) - f_m(s, y(s))\| ds \\ &\quad + \sum_{i=1}^{k} \|I_{m,i}(x(t_i)) - I_{m,i}(y(t_i))\| \\ &\quad + \sum_{i=1}^{k} (t - t_i) \|\bar{I}_{m,i}(x(t_i)) - \bar{I}_{m,i}(y(t_i))\| \\ &\leq \left(L_{\Phi_m} + L_{\Theta_m} t + I_a^{\kappa_m} \psi_{f_m}(t) + \sum_{i=1}^{k} L_{I_{m,i}} + \sum_{i=1}^{k} (t - t_i) L_{\bar{I}_{m,i}}\right) \|x - y\| \\ &\leq \left(L_{\Phi_m} + L_{\Theta_m} t_{k+1} + L_{\psi_m} + \sum_{i=1}^{k} L_{I_{m,i}} + \sum_{i=1}^{k} (t_{i+1} - t_i) L_{\bar{I}_{m,i}}\right) \|x - y\|. \end{aligned}$$

In virtue of condition (8), and the estimate

$$\|\Psi x - \Psi y\| \leq \left(L_\Phi + L_\Theta b + L_\psi + \sum_{i=1}^{p} L_{I_i} + \sum_{i=1}^{p} (t_{i+1} - t_i) L_{\bar{I}_i}\right) \|x - y\|,$$

it is figured out that Ψ is a contraction. Therefore, by the Banach contraction criterion, a unique fixed point and so a unique solution is found for Ψ, and the impulsive tripled system (1), respectively. The uniqueness proof is ended here. □

The next result is an investigation of the existence of a tripled fixed point to the operator $\Psi : X^3 \to X^3$ that leads to a solution for the mentioned impulsive system (1) using the Schauder Theorem 1. These hypotheses are needed to establish the result and are given as follows:

(A1) $f_m : J \times \mathbb{R}^3 \to \mathbb{R}$ is Carathéodory, and there exist nondecreasing κth-fractional integrable function $v : J \to [0, \infty)$ and nondecreasing continuous $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$ satisfying $\varphi(s) < s$ for all $s > 0$ equipped with

$$|f_m(t,x) - f_m(t,y)| \leq v_f(t)\varphi(\|x - y\|), t \in J,$$

$\forall\, x, y \in \mathbb{R}^3$, κ is chosen such that $\sup_{t\in J} I_a^\kappa v_f(t) \leq C_v$. Moreover, let $\sup_{t\in J}|f_m(t,0)| \leq C_f$ for any $m = 1,\ 2,\ 3$.

(A2) $I_{m,i}$ and $\bar{I}_{m,i}$ are continuous functions, and $\exists\, C_I, C_{\bar{I}} > 0$, such that for all $x, y \in \mathbb{R}^3$, $i = 1,\ 2,\ ...,\ p$, we have

$$|I_{m,i}(x) - I_{m,i}(y)| \leq C_I\|x - y\|,\ |\bar{I}_{m,i}(x) - \bar{I}_{m,i}(y)| \leq C_{\bar{I}}\|x - y\|.$$

Moreover, let $I_{m,i}(0) = \bar{I}_{m,i}(0) = 0$.

(A3) Φ_m and Θ_m are continuous operators for $m = 1,\ 2,\ 3$, and there exist constants C_Φ, $C_\Theta > 0$ s.t. $\forall\, x, y \in X^3$,

$$\begin{cases} \|\Phi_m(x) - \Phi_m(y)\| \leq C_\Phi\|x - y\|, \\ \|\Theta_m(x) - \Theta_m(y)\| \leq C_\Theta\|x - y\|. \end{cases}$$

Moreover, let $\|\Phi_m 0\| \leq C_{\Phi_0}$, and $\|\Theta_m 0\| \leq C_{\Theta_0}$.

(A4) The inequality

$$C_{\Phi_0} + bC_{\Theta_0} + \frac{C_v(b-a)^\kappa}{\Gamma(\kappa+1)} + (C_\Phi + C_\Theta b + pC_I + C_{\bar{I}}(b-a))r + C_v\varphi(r) \leq r,$$

has a solution $r_0 > 0$.

The function φ that is satisfying the condition in (A1) is equivalent to the corresponding condition in Theorem 2 [35]. It is obvious by definition of $\Psi = (\Psi_1, \Psi_2, \Psi_3)$ that, the value of its norm on J_0 is less than or equal to the corresponding norm value on J_k. Therefore, without loss of generality, we apply the norms on J_k, in the next result $(k = 1, 2, ..., p)$.

Theorem 4. *Let (A1)–(A4) be held. Then, the impulsive tripled system (1) involves at least one solution.*

Proof. Let $Y = X^3$, and (x_n) be a sequence in Y such that $x_n \to x$ in Y. Let $\mathfrak{R} > 0$, and choose $\delta = \mathfrak{R}\Big(C_\Phi + C_\Theta b + C_f + pC_I + C_{\bar{I}}(b-a)\Big)^{-1}$ such that $\|x_n - x\| < \delta$. Using assumptions (A1)–(A3), we deuce that

$$\begin{aligned} |\Psi_m x_n(t) - \Psi_m x(t)| \leq\ & \|\Phi_m x_n - \Phi_m x\| + t\|\Theta_m x_n - \Theta_m x\| \\ & + \int_a^t \frac{(t-s)^{\kappa_m - 1}}{\Gamma(\kappa_m)}\|f_m(s, x_n(s)) - f_m(s, x(s))\|ds \\ & + \sum_{i=1}^k \|I_{m,i}(x_n(t_i)) - I_{m,i}(x(t_i))\| \\ & + \sum_{i=1}^k (t - t_i)\|\bar{I}_{m,i}(x_n(t_i)) - \bar{I}_{m,i}(x(t_i))\| \end{aligned}$$

$$\leq \left(C_\Phi + C_\Theta t + I_a^\varsigma v_f(t) + kC_I + C_{\bar{I}} \sum_{i=1}^{k} (t - t_i) \right) \|x_n - x\|.$$

Hence

$$\|\Psi x_n - \Psi x\| \leq (C_\Phi + C_\Theta b + C_v + pC_I + C_{\bar{I}}(b-a))\|x_n - x\| < \mathfrak{R}.$$

This shows that Ψ is a uniformly continuous operator on Y.

Let $B_{r_0} = \{\zeta \in X : \|\zeta\| \leq r_0\} \neq \emptyset$, then B_{r_0} is a closed, and convex set in X with boundedness property. Further, the subset $B^3_{r_0} = B_{r_0} \times B_{r_0} \times B_{r_0}$ inherits the properties of B_{r_0} but in Y. We estimate

$$\begin{aligned} |\Psi_m x(t)| &\leq \|\Phi_m x\| + t\|\Theta_m x\| + \int_a^t \frac{(t-s)^{\kappa_m - 1}}{\Gamma(\kappa_m)} \|f_m(s, x(s))\| ds \\ &\quad + \sum_{i=1}^{k} \|I_{m,i}(x(t_i))\| + \sum_{i=1}^{k} (t - t_i) \|\bar{I}_{m,i}(x(t_i))\| \\ &\leq \left(C_\Phi + C_\Theta t + kC_I + C_{\bar{I}} \sum_{i=1}^{k} (t - t_i) \right) \|x\| \\ &\quad + I_a^\kappa v_f(t) \varphi(\|x\|) + \|\Phi_m 0\| + t\|\Theta_m 0\| + \frac{C_v (t-a)^\kappa}{\Gamma(\kappa+1)}. \end{aligned}$$

Therefore, if $x_m \in B_{r_0}$, $m = 1, 2, 3$, we deduce by (A4) that

$$\begin{aligned} \|\Psi x\| &\leq (C_\Phi + C_\Theta b + pC_I + C_{\bar{I}}(b-a))r_0 + C_v \varphi(r_0) \\ &\quad + \frac{C_v (b-a)^\kappa}{\Gamma(\kappa+1)} + C_{\Phi_0} + bC_{\Theta_0} \\ &\leq r_0. \end{aligned}$$

This shows that Ψ maps $B^3_{r_0}$ into $B^3_{r_0}$.

Let A_m, $m = 1, 2, 3$ be a nonempty subset in B_{r_0}; then A_m inherits the boundedness and convexity properties from B_{r_0}. Let $\zeta \in \Psi_m(A_1 \times A_2 \times A_3)$, then there exists $y = (x_1, x_2, x_3) \in A_1 \times A_2 \times A_3$ such that $\Psi_m y = \zeta$, and $x_m \in A_m$, $m = 1, 2, 3$. The definition of Ψ_m implies that

$$\zeta(t) = \Psi_m y(t) = \begin{cases} \Phi_m y + t\Theta_m y + \int_a^t \frac{(t-s)^{\kappa_m - 1}}{\Gamma(\kappa_m)} f_m(s, y(s)) ds, t \in J_0, \\ \Phi_m y + t\Theta_m y + \int_a^t \frac{(t-s)^{\kappa_m - 1}}{\Gamma(\kappa_m)} f_m(s, y(s)) ds \\ + \sum\limits_{i=1}^{k} I_{m,i}(y(t_i)) + \sum\limits_{i=1}^{k} (t - t_i) \bar{I}_{m,i}(y(t_i)), t \in J_k, k = 1, 2, ..., p. \end{cases}$$

For any $\mathfrak{R} > 0$, let $\tau_1, \tau_2 \in J$ be such that $|\tau_2 - \tau_1| \leq \mathfrak{R}$, then, we have

$$\begin{aligned} &|\zeta(\tau_2) - \zeta(\tau_1)| \\ \leq\ &|\tau_2 - \tau_1| \|\Theta_m y\| + \varphi(\|y\|) \int_a^{\tau_1} \left| \frac{(\tau_2 - s)^{\kappa_m - 1}}{\Gamma(\kappa_m)} - \frac{(\tau_1 - s)^{\kappa_m - 1}}{\Gamma(\kappa_m)} \right| v_f(s) ds \\ &+ \int_a^{\tau_1} \left| \frac{(\tau_2 - s)^{\kappa_m - 1}}{\Gamma(\kappa_m)} - \frac{(\tau_1 - s)^{\kappa_m - 1}}{\Gamma(\kappa_m)} \right| |f(s, 0)| ds \end{aligned}$$

$$
\begin{aligned}
& +\varphi(\|y\|)\int_{\tau_1}^{\tau_2}\left|\frac{(\tau_2-s)^{\kappa_m-1}}{\Gamma(\kappa_m)}\right|v_f(s)ds+\int_{\tau_1}^{\tau_2}\left|\frac{(\tau_2-s)^{\kappa_m-1}}{\Gamma(\kappa_m)}\right||f(s,0)|ds \\
& +|\tau_2-\tau_1|\sum_{i=1}^{k}\|\bar{I}_{m,i}(x(t_i))\|, a.e.on J, \\
\leq\; & |\tau_2-\tau_1|\|\Theta_m y\| \\
& +\varphi(\|y\|)v_f(\tau_1)\left(\left|\frac{(\tau_2-a)^{\kappa_m}}{\Gamma(\kappa_m+1)}-\frac{(\tau_1-a)^{\kappa_m}}{\Gamma(\kappa_m+1)}\right|+\frac{|\tau_2-\tau_1|^{\kappa_m}}{\Gamma(\kappa_m+1)}\right) \\
& +C_f\left(\left|\frac{(\tau_2-a)^{\kappa_m}}{\Gamma(\kappa_m+1)}-\frac{(\tau_1-a)^{\kappa_m}}{\Gamma(\kappa_m+1)}\right|+\frac{|\tau_2-\tau_1|^{\kappa_m}}{\Gamma(\kappa_m+1)}\right) \\
& +\varphi(\|y\|)v_f(\tau_2)\frac{|\tau_2-\tau_1|^{\kappa_m}}{\Gamma(\kappa_m+1)}+C_f\frac{|\tau_2-\tau_1|^{\kappa_m}}{\Gamma(\kappa_m+1)}+p(\tau_2-\tau_1)C_{\bar{I}}.
\end{aligned}
$$

Without loss of generality, we take $\tau_1 > a$, and $\mathfrak{R} < \tau_1 - a$. Then,

$$
\begin{aligned}
\omega(\zeta,\mathfrak{R}) &= \sup\{|\zeta(\tau_2)-\zeta(\tau_1)| : \tau_1,\tau_2\in J, |\tau_2-\tau_1|\leq\mathfrak{R}\} \\
&\leq \mathfrak{R}\big(C_\Theta\|y\|+C_{\Theta_0}\big)+\varphi(\|y\|)v_f(\tau_1) \\
&\quad\times\left(\frac{|\tau_1-a|^{\kappa_m}\left|O\left(\frac{\mathfrak{R}}{\tau_1-a}\right)\right|}{\Gamma(\kappa_m+1)}+\frac{\mathfrak{R}^{\kappa_m}}{\Gamma(\kappa_m+1)}\right) \\
&\quad+C_f\left(\frac{|\tau_1-a|^{\kappa_m}\left|O\left(\frac{\mathfrak{R}}{\tau_1-a}\right)\right|}{\Gamma(\kappa_m+1)}+\frac{\mathfrak{R}^{\kappa_m}}{\Gamma(\kappa_m+1)}\right) \\
&\quad+\varphi(\|y\|)v_f(\tau_2)\frac{\mathfrak{R}^{\kappa_m}}{\Gamma(\kappa_m+1)}+\frac{C_f\mathfrak{R}^{\kappa_m}}{\Gamma(\kappa_m+1)}+\mathfrak{R}pC_{\bar{I}},
\end{aligned}
$$

where $O\left(\frac{\mathfrak{R}}{\tau_1-a}\right)$ is the big O function [50] that converges to 0 as $\mathfrak{R}$ converges to zero. It follows that

$$
\begin{aligned}
\omega(\Psi_m(A_1\times A_2\times A_3),\mathfrak{R}) &= \sup\{\omega(x,\zeta) : \zeta\in\Psi_m(A_1\times A_2\times A_3)\} \\
&\leq \mathfrak{R}\big(C_{\Theta_0}+C_\Theta r_0\big)+\varphi(r_0)v_f(b) \\
&\quad\times\left(\frac{|\tau_1-a|^{\kappa_m}\left|O\left(\frac{\mathfrak{R}}{\tau_1-a}\right)\right|}{\Gamma(\kappa_m+1)}+\frac{\mathfrak{R}^{\kappa_m}}{\Gamma(\kappa_m+1)}\right) \\
&\quad+C_f\left(\frac{|\tau_1-a|^{\kappa_m}\left|O\left(\frac{\mathfrak{R}}{\tau_1-a}\right)\right|}{\Gamma(\kappa_m+1)}+\frac{\mathfrak{R}^{\kappa_m}}{\Gamma(\kappa_m+1)}\right) \\
&\quad+\varphi(r_0)v_f(b)\frac{\mathfrak{R}^{\kappa_m}}{\Gamma(\kappa_m+1)}+\frac{C_f\mathfrak{R}^{\kappa_m}}{\Gamma(\kappa_m+1)}+\mathfrak{R}pC_{\bar{I}}.
\end{aligned}
$$

Therefore, passing the limit as $\mathfrak{R}$ approaches to zero, we deduce then $\omega_0(\Psi(A_1\times A_2\times A_3))$ approaches to zero for any $A_1\times A_2\times A_3\subseteq B^3_{r_0}$. Applying Schauder fixed point Theorem 1, the impulsive tripled system (1) has a solution in $B^3_{r_0}\in\ker\omega_0$. $\square$

The equivalent relations between the regular measures of noncompactness permit to use Kuratowski α and Hausdorff χ measures of noncompactness in the previous theorem.

It is still possible to use a general measure μ together with general conditions to obtain the existence result.

(A5) Assume that
$$\mu(f_m(t,A)) \leq \rho_m(t)\psi(\mu(A)),$$
for any bounded subset $A \subseteq X$, where $\rho_m : J \to \mathbb{R}_+$ be so that
$$\max_{t\in J}\{I_a^\kappa\rho_1(t), I_a^\kappa\rho_2(t), I_a^\kappa\rho_3(t)\} \leq L_\rho,$$
and $\psi : [0,\infty) \to [0,\infty)$ is increasing with $\lim_{n\to\infty}\psi^n(t) = 0$, $\forall\, t \in [0,\infty)$ and $\psi(ks) = k\psi(s)$, $k \geq 0$.

(A6) There exist constants L_{Φ_m}, L_{Θ_m} such that
$$\begin{aligned}\mu(\Phi_m A) &\leq L_{\Phi_m}\psi(\mu(A)),\\ \mu(\Theta_m A) &\leq L_{\Theta_m}\psi(\mu(A)).\end{aligned}$$
Moreover, let $L_\Phi = \max\{L_{\Phi_1}, L_{\Phi_2}, L_{\Phi_3}\}$, and $L_\Theta = \max\{L_{\Theta_1}, L_{\Theta_2}, L_{\Theta_3}\}$.

(A7) There exist constants $L_{I_{m,i}}, L_{\bar{I}_{m,i}}$ such that
$$\mu\{I_{m,i}(x(t_i)), x \in A\} \leq L_{I_{m,i}}\psi(\mu(A)),\ \ \mu\{\bar{I}_{m,i}(x(t_i)), x \in A\} \leq L_{\bar{I}_{m,i}}\psi(\mu(A)).$$
Furthermore, let $L_{I_i} = \max\left\{L_{I_{1,i}}, L_{I_{2,i}}, L_{I_{3,i}}\right\}$, and $L_{\bar{I}_i} = \max\left\{L_{\bar{I}_{1,i}}, L_{\bar{I}_{2,i}}, L_{\bar{I}_{3,i}}\right\}$.

(A8) Assume that
$$L_\Phi + bL_\Theta + \sum_{i=1}^{p} L_{I_i} + (b-a)\sum_{i=1}^{p} L_{\bar{I}_i} + L_\rho < 1.$$

The further lemma is needed for our goal.

Lemma 3 ([33]). *Regard $f : J \times X \to \mathbb{R}$ satisfying (A5), and let $K{:}J \times J \to \mathbb{R}$ be a bounded continuous mapping. If A is an equicontinuous set of functions, then*
$$\mu\left(\left\{\int_I K(t,s)f(s,x(s))ds : x \in A\right\}\right) \leq \int_I |K(t,s)|\rho(s)\psi(\mu(\{x(s) : s \in A\})ds,$$
for any subset I of J and any $t \in I$.

Theorem 5. *Let μ be any measure of noncompactness. If (A1)–(A8) are fulfilled, then the impulsive tripled system (1) involves a solution.*

Proof. Let $Y = X^3$. The conditions (A1)–(A4) imply that the operator $\Psi : B^3_{r_0} \to B^3_{r_0}$ is continuous operator on the closed convex bounded subset $B^3_{r_0}$ of Y. Moreover, $\overline{\Psi B^3_{r_0}}$ is compact subset in $B^3_{r_0}$. If $\mu(B^3_{r_0}) = 0$, then by (M2), we have $\mu(A) = 0$ for any $A \subseteq B^3_{r_0}$, hence we have the result as in Theorem 4. Otherwise, let $\mu(A) > 0$, then, by (M4) and (M5), we have
$$\begin{aligned}\mu((\Psi_m A)(t)) \ \leq\ & \mu\Big(\Big\{\Phi_m x + t\Theta_m x + \int_a^t \frac{(t-s)^{\kappa_m-1}}{\Gamma(\kappa_m)} f_m(s,x(s))ds\\ &+ \sum_{i=1}^{k} I_{m,i}(x(t_i)) + \sum_{i=1}^{k}(t-t_i)\bar{I}_{m,i}(x(t_i)) :\\ &x \in A, t \in J_k, k = 0,1,2,\ldots,p\Big\}\Big)\end{aligned}$$

$$
\begin{aligned}
&\leq \mu(\{\Phi_m x : x \in A\}) + t\mu(\{\Theta_m x : x \in A\}) + \\
&\quad + \sum_{i=1}^{k} \mu(\{I_{m,i}(x(t_i)) : x \in A\}) \\
&\quad + \sum_{i=1}^{k} (t - t_i)\mu(\{\bar{I}_{m,i}(x(t_i)) : x \in A\}) \\
&\quad + \mu\left(\left\{\int_a^t \frac{(t-s)^{\kappa_m - 1}}{\Gamma(\kappa_m)} f_m(s, x(s))ds : x \in A\right\}\right).
\end{aligned}
$$

Using Lemma 3,

$$
\begin{aligned}
\mu((\Psi_m A)(t)) &\leq \mu(\{\Phi_m x : x \in A\}) + t\mu(\{\Theta_m x : x \in A\}) + \\
&\quad + \sum_{i=1}^{k} \mu(\{I_{m,i} A : x \in A\}) \\
&\quad + \sum_{i=1}^{k} (t - t_i)\mu(\{\bar{I}_{m,i}(x(t_i)) : x \in A\}) \\
&\quad + \int_a^t \frac{(t-s)^{\kappa_m - 1}}{\Gamma(\kappa_m)} \mu(\{f_m(s, x(s)) : x \in A\})ds.
\end{aligned}
$$

In view of assumptions (A5)–(A7), we deduce

$$
\begin{aligned}
\mu((\Psi_m A)(t)) &\leq L_{\Phi_m}\psi(\mu(A)) + tL_{\Theta_m}\psi(\mu(A)) + \\
&\quad + \psi(\mu(A)) \sum_{i=1}^{k} L_{I_{m,i}} + \psi(\mu(A)) \sum_{i=1}^{k} L_{\bar{I}_{m,,i}}(t - t_i) \\
&\quad + \int_a^t \frac{(t-s)^{\kappa_m - 1}}{\Gamma(\kappa_m)} \mu(\{f_m(s, x(s)) : x \in A\})ds \\
&\leq \left(L_{\Phi_m} + tL_{\Theta_m} + \sum_{i=1}^{k} L_{I_{m,i}} + \sum_{i=1}^{k} L_{\bar{I}_{m,,i}}(t - t_i) + I_a^{\kappa}\rho_m(t)\right)\psi(\mu(A)).
\end{aligned}
$$

Taking the maximum over $m = 1,\ 2,\ 3$, we have

$$
\begin{aligned}
\mu((\Psi A)(t)) &= \sup\{\mu((\Psi_1 A)(t)), \mu((\Psi_2 A)(t)), \mu((\Psi_3 A)(t))\} \\
&\leq \left(L_\Phi + tL_\Theta + \sum_{i=1}^{k} L_{I_i} + \sum_{i=1}^{k} L_{\bar{I}_i}(t - t_i) + L_\rho\right)\psi(\mu(A)).
\end{aligned}
$$

Taking the supremum over J, we have

$$
\begin{aligned}
\mu((\Psi A)) &\leq \left(L_\Phi + bL_\Theta + \sum_{i=1}^{p} L_{I_i} + (b-a)\sum_{i=1}^{p} L_{\bar{I}_i} + L_\rho\right)\psi(\mu(A)) \\
&= \phi(\gamma(A)), \qquad (10)
\end{aligned}
$$

where $\phi(s) = \left(L_\Phi + bL_\Theta + \sum_{i=1}^{p} L_{I_i} + (b-a)\sum_{i=1}^{p} L_{\bar{I}_i} + I_a^\kappa \rho(b)\right)\psi(s)$ which satisfies $\phi^n(s) = \left(L_\Phi + bL_\Theta + \sum_{i=1}^{p} L_{I_i} + (b-a)\sum_{i=1}^{p} L_{\bar{I}_i} + I_a^\kappa \rho(b)\right)^n \psi^n(s)$ by using (A5) and (A8). In accordance with (A8), we get the required results by applying Theorem 2. This finishes the proof. □

4. Application

We give a general example to examine the obtained results.

Example 1. *Consider the following impulsive tripled system:*

$$\begin{cases} {}^cD_0^{\frac{6}{5}} x_1(t) = \dfrac{1}{\sqrt{100+t^2}}\left(\dfrac{|x_1(t)|}{1+|x_1(t)|} + \dfrac{|x_2(t)|}{1+|x_2(t)|} + \dfrac{|x_3(t)|}{1+|x_3(t)|}\right), \\ {}^cD_0^{\frac{3}{2}} x_2(t) = \dfrac{t|x_1(t)|}{10} + \dfrac{t|x_2(t)|}{10(t^3+1)} + \dfrac{t|x_3(t)|}{10(1+t)}, \\ {}^cD_0^{\frac{9}{8}} x_3(t) = \dfrac{|x_1(t)|}{\sqrt{100+t^2}} + \dfrac{|x_2(t)|}{10+t} + \dfrac{|x_3(t)|}{10\sqrt{1+t^2}}, \end{cases} \tag{11}$$

where $t \in [0,1]$, subject to boundary conditions

$$\begin{cases} 10mx_m(0) = \displaystyle\int_0^1 |\sin(x_m(s))|ds, 10mx_m'(0) = \dfrac{|x_m(1)|}{1+\left(x_0'(0)+x_1'(0)+x_3'(0)\right)^2}, \\ m\Delta x_m|_{t=0.5} = \dfrac{0.1|x_m(0.5)|}{1+|x_m(0.5)|}, m\Delta x_m'|_{t=0.5} = \dfrac{0.1|x_m(0.5)|}{1+2|x_m'(0.5)|}. \end{cases}$$

In view of the above data, we have ($m = 1,\ 2,\ 3$)

$$\Phi_m(x) = \frac{1}{10m}\int_0^1 |\sin(x_m(s))|ds,$$

$$\Theta_m(x) = \frac{1}{10m}\frac{|x_m(1)|}{1+\left(x_0'(0)+x_1'(0)+x_3'(0)\right)^2},$$

$$I_{m,1}(x) = \frac{1}{m}\frac{0.1|x_m(0.5)|}{1+|x_m(0.5)|},$$

and

$$\bar{I}_{m,1}(x) = \frac{1}{m}\frac{0.1|x_m(0.5)|}{1+2|x_m'(0.5)|},$$

where $\kappa_1 = \dfrac{6}{5}, \kappa_2 = \dfrac{3}{2}, \kappa_3 = \dfrac{9}{8}, t_0 = 0, t_1 = 0.5$, and $t_2 = 1$. Furthermore, we notice that $\psi_{f_1}(t) = \dfrac{1}{\sqrt{100+t^2}}$, $\psi_{f_2}(t) = \frac{t}{10}$, and $\psi_{f_3}(t) = \dfrac{1}{10}$. Therefore, by the assumptions (H1)–(H3), we can find that $L_\Phi = L_\Theta = L_\psi = L_{I_i} = L_{\bar{I}_i} \le \dfrac{1}{10}$. Hence

$$L_\Phi + L_\Theta b + L_\psi + \sum_{i=1}^{p} L_{I_i} + \sum_{i=1}^{p}(t_{i+1}-t_i)L_{\bar{I}_i} = 0.45 < 1.$$

Thus, all hypotheses of Theorem 3 are satisfied, then there exists a unique solution to the impulsive tripled system (11).

On other hand, by referring to hypotheses (A1)–(A4), similar calculations can be performed and so we find $C_{\Phi_0} = C_{\Theta_0} = 0, a = 0, p = 1 = b, \kappa = 1.5$ and $C_\Phi = C_\Theta = C_I = C_{\bar{I}} = 0.1$,

$C_v \leq 0.1$. *Therefore, we solve the inequality given in (A4) and we find that* $r_0 \approx 0.23$ *satisfies this inequality. Therefore using Theorem 4, there exists a solution to the supposed impulsive tripled system* (11).

5. Conclusions

In this paper, we designed a tripled system consisting of impulsive fractional equations involving the generalized boundary conditions with some given operators. By introducing two types of measure of noncompactness (Kuratowski and Hausdorff), we investigated necessary hypotheses and conditions implying the existence of solutions with the help of the tripled fixed point and modulus of continuity. Furthermore, the Banach principle was applied to confirm the uniqueness property. In this research, we showed that our results are valid for both Kuratowski and Hausdorff measures of noncompactness. To confirm this correctness, we designed an example of the control problem. For future works, we can generalize our results to such an impulsive problem with nonsingular operators.

Author Contributions: Conceptualization, S.E. and M.M.M.; Formal analysis, S.E., M.M.M., M.A.R. and S.R.; Funding acquisition, M.A.R.; Methodology, S.E., M.M.M., M.A.R. and S.R.; Software, S.E. All authors have read and agreed to the published version of the manuscript.

Funding: This research received no external funding.

Institutional Review Board Statement: Not applicable.

Informed Consent Statement: Not applicable.

Data Availability Statement: Data sharing not applicable to this article as no datasets were generated or analyzed during the current study.

Acknowledgments: The first and fourth authors would like to thank Azarbaijan Shahid Madani University. This paper has been supported by the RUDN University Strategic Academic Leadership Program. Furthermore, the authors would like to thank dear reviewers for their constructive and valuable remarks and comments to improve the quality of the paper.

Conflicts of Interest: The authors declare no conflict of interest.

References

1. Klafter, J.; Lim, S.C.; Metzler, R. *Fractional Dynamics in Physics*; World Scientific: Singapore, 2011.
2. Samko, S.; Kilbas, A.A.; Marichev, O. *Fractional Integrals and Derivatives, Theory and Applications*; Gordon and Breach: New York, NY, USA, 1993.
3. Matar, M.M.; Trujillo, J.J. Existence of local solutions for differential equations with arbitrary fractional order. *Arab. J. Math.* **2016**, *5*, 215–224. [CrossRef]
4. Shah, K.A.; Zada, A. Controllability and stability analysis of an oscillating system with two delays. *Math. Methods Appl. Sci.* **2021**, *44*, 14733–14765. [CrossRef]
5. Boutiara, A.; Etemad, S.; Hussain, A.; Rezapour, S. The generalized U-H and U-H stability and existence analysis of a coupled hybrid system of integro-differential IVPs involving φ-Caputo fractional operators. *Adv. Differ. Equ.* **2021**, *2021*, 95. [CrossRef]
6. Ragusa, M.A.; Razani, A.; Safari, F. Existence of radial solutions for a $p(x)$-Laplacian Dirichlet problem. *Adv. Differ. Equ.* **2021**, *2021*, 215. [CrossRef]
7. Thabet, S.T.M.; Etemad, S.; Rezapour, S. On a new structure of the pantograph inclusion problem in the Caputo conformable setting. *Bound. Value Probl.* **2020**, *2020*, 171. [CrossRef]
8. Matar, M.M.; Abu Skhail, E.S. On stability of nonautonomous perturbed semilinear fractional differential systems of order $\alpha \in (1,2)$. *J. Math.* **2018**, *2018*, 1723481. [CrossRef]
9. Ramdoss, M.; Pachaiyappan, D.; Park, C.; Lee, J.R. Stability of a generalized n-variable mixed-type functional equation in fuzzy modular spaces. *J. Inequal. Appl.* **2021**, *2021*, 61. [CrossRef]
10. Rezapour, S.; Ntouyas, S.K.; Iqbal, M.Q.; Hussain, A.; Etemad, S.; Tariboon, J. An analytical survey on the solutions of the generalized double-Order φ-integrodifferential equation. *J. Funct. Spaces* **2021**, *2021*, 6667757. [CrossRef]
11. Matar, M.M. On controllability of linear and nonlinear fractional integrodifferential systems. *Fract. Differ. Calc.* **2019**, *9*, 19–32. [CrossRef]
12. Abbas, M.I.; Ragusa, M.A. On the hybrid fractional differential equations with fractional proportional derivatives of a function with respect to a certain function. *Symmetry* **2021**, *13*, 264. [CrossRef]
13. Amara, A.; Etemad, S.; Rezapour, S. Topological degree theory and Caputo-Hadamard fractional boundary value problems. *Adv. Differ. Equ.* **2020**, *2020*, 369. [CrossRef]

14. Abbas, M.I.; Ragusa, M.A. Solvability of Langevin equations with two Hadamard fractional derivatives via Mittag-Leffler functions. *Appl. Anal.* **2021**. [CrossRef]
15. Hammad, H.A.; De la Sen, M. Exciting fixed point results under a new control function with supportive application in fuzzy cone metric spaces. *Mathematics* **2021**, *9*, 2267. [CrossRef]
16. Phuong, N.D.; Sakar, F.M.; Etemad, S.; Rezapour, S. A novel fractional structure of a multi-order quantum multi-integro-differential problem. *Adv. Differ. Equ.* **2020**, *2020*, 633. [CrossRef]
17. Shokri, A. The symmetric two-step P-stable nonlinear predictor-corrector methods for the numerical solution of second order initial value problems. *Bul. Iran. Math. Soc.* **2015**, *41*, 201–215.
18. Hammad, H.A.; Agarwal, P.; Guirao, J.L.G. Applications to boundary value problems and homotopy theory via tripled fixed point techniques in partially metric spaces. *Mathematics* **2021**, *9*, 2012. [CrossRef]
19. Ahmad, H.M.; Elbarkouky, R.A.; Omar, O.A.M.; Ragusa, M.A. Models for COVID-19 daily confirmed cases in different countries. *Mathematics* **2021**, *9*, 659. [CrossRef]
20. Akdemir, A.O.; Karaoglan, A.; Ragusa, M.A.; Set, E. Fractional integral inequalities via Atangana-Baleanu operators for convex and concave functions. *J. Funct. Spaces* **2021**, *2021*, 1055434. [CrossRef]
21. Aydi, H.; Rakić, D.; Aghajani, A.; Došenović, T.; Noorani, M.S.M.; Qawaqneh, H. On fixed point results in G_b-metric spaces. *Mathematics* **2019**, *7*, 617. [CrossRef]
22. Kreyszig, E. *Introductory Functional Analysis with Applications*; John Wiley: Hoboken, NJ, USA, 1978.
23. Granas, A.; Dugundji, J. *Fixed Point Theory*; Springer: New York, NY, USA, 2005.
24. Rezapour, S.; Tellab, B.; Deressa, C.T.; Etemad, S.; Nonlaopon, K. H-U-type stability and numerical solutions for a nonlinear model of the coupled systems of Navier BVPs via the generalized differential transform method. *Fractal Fract.* **2021**, *5*, 166. [CrossRef]
25. Deressa, C.T.; Etemad, S.; Rezapour, S. On a new four-dimensional model of memristor-based chaotic circuit in the context of nonsingular Atangana-Baleanu-Caputo operators. *Adv. Differ. Equ.* **2021**, *2021*, 444. [CrossRef]
26. Khan, T.; Ullah, R.; Zaman, G.; Alzabut, J. A mathematical model for the dynamics of SARS-CoV-2 virus using the Caputo-Fabrizio operator. *AIMS Math. Biosci. Eng.* **2021**, *18*, 6095–6116. [CrossRef] [PubMed]
27. Mohammadi, H.; Kumar, S.; Rezapour, S.; Etemad, S. A theoretical study of the Caputo-Fabrizio fractional modeling for hearing loss due to Mumps virus with optimal control. *Chaos Solitons Fractals* **2021**, *144*, 110668. [CrossRef]
28. Farman, M.; Akgul, A.; Abdeljawad, T.; Naik, P.A.; Bukhari, N.; Ahmad, A. Modeling and analysis of fractional order Ebola virus model with Mittag-Leffler kernel. *Alex. Eng. J.* **2021**, 61, 2062–2073. [CrossRef]
29. Rezapour, S.; Ahmad, B.; Etemad, S. On the new fractional configurations of integro-differential Langevin boundary value problems. *Alex. Eng. J.* 2021, *60*, 4865–4873. [CrossRef]
30. Turab, A.; Mitrovic, Z.D.; Savic, A. Existence of solutions for a class of nonlinear boundary value problems on the hexasilinane graph. *Adv. Differ. Equ.* **2021**, *2021*, 494. [CrossRef]
31. Aghajani, A.; Sabzali, N. Existence of coupled fixed points via measure of noncompactness and applications. *J. Nonlinear Convex Anal.* **2014**, *14*, 941–952.
32. He, J.W.; Liang, Y.; Ahmad, B.; Zhou, Y. Nonlocal fractional evolution inclusions of order $\alpha \in (1,2)$. *Mathematics* **2019**, *7*, 209. [CrossRef]
33. Szufla, S. On the application of measure of noncompactness to existence theorems. *Rendiconti del Seminario Matematico della Università di Padova Tome* **1986**, *75*, 1–14.
34. Banaś, J.; Goebel, K. *Measures of Noncompactness in Banach Spaces*; Marcel Dekker: New York, NY, USA, 1980; Volume 60.
35. Banaś, J.; Jleli, M.; Mursaleen, M.; Samet, B.; Vetro, C. *Advances in Nonlinear Analysis via the Concept of Measure of Noncompactness*; Springer Nature: Singapore, 2017.
36. Salem, A.; Almaghamsi, L. Existence solution for coupled system of Langevin fractional differential equations of Caputo type with Riemann-Stieltjes integral boundary conditions. *Symmetry* **2021**, *13*, 2123. [CrossRef]
37. Shokri, A. An explicit trigonometrically fitted ten-step method with phase-lag of order infinity for the numerical solution of the radial Schrödinger equation. *Appl. Comput. Math.* **2015**, *14*, 63–74.
38. Trofimov, V.; Loginova, M. Conservative finite-difference schemes for two nonlinear Schrodinger equations describing frequency tripling in a medium with cubic nonlinearity: Competition of invariants. *Mathematics* **2021**, *9*, 2716. [CrossRef]
39. Ahmad, B.; Nieto, J.J. Existence results for a coupled system of nonlinear fractional differential equations with three-point boundary conditions. *Comput. Math. Appl.* **2009**, *58*, 1838–1843. [CrossRef]
40. Matar, M.M.; Abo Amra, I.E. Existence of mild solutions for non-periodic coupled fractional differential equations. *J. Fract. Calc. Appl.* **2020**, *11*, 41–53.
41. Samadi, A.; Nuchpong, C.; Ntouyas, S.K.; Tariboon, J. A study of coupled systems of ψ-Hilfer type sequential fractional differential equations with integro-multipoint boundary conditions. *Fractal Fract.* **2021**, *5*, 162. [CrossRef]
42. Alam, M.; Zada, A.; Riaz, U. On a coupled impulsive fractional integrodifferential system with Hadamard derivatives. *Qual. Theory Dyn. Syst.* **2022**, *21*, 8. [CrossRef]
43. Humaira; Hammad, H.A.; Sarwar, M.; De la Sen, M. Existence theorem for a unique solution to a coupled system of impulsive fractional differential equations in complex-valued fuzzy metric spaces. *Adv. Differ. Equ.* **2021**, *2021*, 242. [CrossRef]
44. Berinde, V.; Borcut, M. Tripled fixed point theorems for contractive type mappings in partially ordered metric spaces. *Nonlinear Anal. Theory Methods Appl.* **2011**, *74*, 4889–4897. [CrossRef]

45. Karakaya, V.; Bouzara, N.E.H.; Dogan, K.; Atalan, Y. Existence of tripled fixed points for a class of condensing operators in Banach spaces. *Sci. World J.* **2014**, *2014*, 541862. [CrossRef]
46. Hammad, H.A.; De la Sen, M. A technique of tripled coincidence points for solving a system of nonlinear integral equations in POCML spaces. *J. Inequal. Appl.* **2020**, *2020*, 211. [CrossRef]
47. Hammad, H.A.; De la Sen, M. A tripled fixed point technique for solving a tripled-system of integral equations and Markov process in CCbMS. *Adv. Differ. Equ.* **2020**, *2020*, 567. [CrossRef]
48. Kilbas, A.A.; Srivastava, H.M.; Trujillo, J.J. *Theory and Applications of Fractional Differential Equations*; North-Holland Mathematics Studies, 204; Elsevier: Amsterdam, The Netherlands, 2006.
49. Mohiuddine, S.A.; Alotaibi, A. Some results on a tripled fixed point for nonlinear contractions in partially ordered G-metric spaces. *Fixed Point Theory Appl.* **2012**, *2012*, 179. [CrossRef]
50. Knuth D. Teach calculus with big O notation. *Notices Am. Math. Soc.* **1998**, *45*, 687.